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STUDENTS' CONCEPTIONS OF GEOMETRY IN A DYNAMIC

GEOMETRY SOFTWARE ENVIRONMENT

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CHAPTER I
THE RESEARCH OBJECTIVE

When students study plane geometry, they work within a domain whose tools have remained essentially unchanged for more than 2,000 years. Known as "Euclidean geometry," the subject bears the name of Euclid (ca. B.C.E. 300), the Greek mathematician credited with developing an axiomatic approach to systematize the field.

Euclidean geometry's tools consist of paper, pencil, straightedge, compass, blackboard, and chalk. Geometric constructions built with these items possess two notable characteristics:

1. *They are static*. Any illustration drawn on paper or blackboard remains fixed in place and cannot be altered without some erasing.

2. *They are particular*. Any constructed square represents a specific square with a particular side length. As defined, squares can have *any* side length, but no single, stationary picture captures the generality of this definition.

Now at the beginning of the 21st century, a fresh medium for building geometric constructions stands alongside these familiar tools of the trade. A breed of software programs known collectively as "dynamic geometry" (DG) has established itself in schools, teaching journals, and university mathematics departments as an attractive alternative to straightedge and compass (Olive, 1998). The Geometer's Sketchpad (Jackiw, 1995) and Cabri Geometry (Texas Instruments, 1994), two of the earliest and most popular DG software packages, reached the mathematics community in the early 1990s.

The characteristics of DG software contrast to the capabilities of traditional geometric tools[1]:

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[1] Readers unfamiliar with the features and capabilities of dynamic geometry will find a description of the software in Appendix A.

1. *Geometric objects can be moved and reshaped interactively*. By clicking and "dragging" with the computer mouse, the software user can animate static images, thereby making them "dynamic" in nature. Segments can stretch and shrink, angles can change measurement, objects can rotate and translate across the screen.

2. *A single onscreen image represents a whole class of geometric objects*. By constructing built-in constraints, a DG user can build a square that will change its size and orientation when dragged, but still retain the invariant features common to all squares—four equal sides and four 90-degree angles.

Mathematics educators and teachers have embraced dynamic geometry in part because interactivity and motion seem, on an intuitive level, like sound educational features of software (Hoyles & Noss, 1994; King & Schattschneider, 1997b). A square that can be resized with a simple click and drag of a mouse holds definite appeal for a generation accustomed to the static, hands-off nature of textbook illustrations.

Yet DG software is more than a copy of Euclidean geometry with interactive, eye-catching graphics. The tools, definitions, exploration techniques, and visual representations associated with dynamic geometry contribute to a learning environment fundamentally removed from its straightedge-and-compass counterpart (Laborde, 1998). How students come to understand geometry in this setting remains an open question in the mathematics education literature (Goldenberg, 1998).

Dynamic geometry software allows for interactivity via the computer mouse. Does the malleability of DG objects make certain aspects of geometry more transparent than might otherwise be the case in a static representation? Or does the interactivity foster geometric misconceptions unique to the medium?

Characterizing the properties of an object as it grows, shrinks, and rotates may yield descriptions that would not arise from viewing its static counterpart. What language do students use to describe the motion they observe?

Dynamic geometry's construction tools do not function identically to a traditional straightedge and compass. Are there instances where a particular DG tool or technique blocks a student from completing a construction task (Laborde, 1993)? Or do Sketchpad's tools promote novel constructions that might not surface in a static setting?

Educational research aimed at studying these issues serves several functions. From an academic perspective, the investigation of how a particular technological innovation (in this case, dynamic geometry) affects geometric conceptions affords a fascinating opportunity to test—and perhaps modify—existing theories of geometric development (Hölzl, 1996).

On a practical level, research into DG learning can benefit three populations in the education community: teachers, curriculum developers, and software designers. Teachers faced with limited time and a crowded computer lab of students might use research results to anticipate DG-related pitfalls and to recognize the germ of fruitful ideas in the language and construction actions of their students. Curriculum developers accustomed to writing texts for paper-and-pencil geometry might find inspiration for new activities targeted to the needs of dynamic geometry learners. And software developers engaged in designing future incarnations of dynamic geometry programs might benefit from the knowledge of how well current DG tools support students' geometric explorations.

Motivated by the factors above, I participated in the design and execution of a student interview study focused on the learning of geometry in a dynamic geometry software environment. The study was conducted in collaboration with Senior Scientist E. Paul Goldenberg at Education Development Center, Inc. during the summer of 1997.

In the central portion of our interviews, middle-school students (working individually or in pairs) received pre-built "mystery"/black-box constructions on The Geometer's Sketchpad. These constructions included such common geometric objects as squares, rectangles, isosceles triangles, and perpendicular bisector lines. Interviewees were asked to explore these objects by dragging each of their parts with the computer mouse.

As they experimented, students described what they observed on screen and explained how they thought each object might have been constructed. Beginning then with a fresh blank screen, students attempted to reconstruct the identical objects from scratch. Throughout the interview, videotape recorded the precise mouse movements and menu selections of the students, as well as their accompanying commentary. The detailed nature of these tapes makes them an ideal source for analyzing the geometric conceptions of students in a dynamic geometry setting.

Research Questions Investigated

In brief, interviewees faced three responsibilities for each black-box task in our study:

1. Exploring how each object and its component parts moved in response to mouse dragging,

2. Describing orally the motions observed and other noteworthy behaviors,

3. Constructing an identical object using Sketchpad's tools.

The research areas addressed by my dissertation can be summarized by three keywords corresponding to the division above: *motion*, *language*, and *construction*. Each of these descriptors serves as the focus for one of three

analysis chapters in this dissertation. With these overarching themes as a guide, I developed the following set of research questions and subquestions to structure my viewing of the videotapes:

1. How do students explore and interpret the behavior of pre-built constructions in a dynamic geometry setting?

    a. What mouse actions do students perform when examining onscreen geometric objects?

    b. How do students characterize invariant geometric properties like parallelism, equality of lengths, and perpendicularity?

    c. Are students able to differentiate between geometric behavior that is an artifact of Sketchpad's design and behavior that is inherent to geometry itself?

2. How do students use Sketchpad's tools when moving from a pre-made construction to building one that displays identical behavior?

Organizational Structure

This dissertation begins in Chapter II by addressing the need behind dynamic geometry research. It introduces four areas of concern, all pointing to the benefits of conducting DG-based interviews.

Chapter III surveys the field's related literature. In doing so, it steps back to highlight findings from static geometry. As an older and more established field, static geometry research offers support to its cousin, dynamic geometry. Chapter III also introduces a theoretical framework for framing my own analyses.

Chapter IV describes the genesis of our dynamic geometry interviews and supplies details of the study's design. It also addresses the limitations and delimitations of the work.

Chapter V takes a closer look at our black-box mystery constructions. It describes in a step-by-step manner how each construction was built, and explains the motivation behind the various tasks.

Chapter VI is the first of three analysis chapters and focuses on the issue of motion. Through the use of interview excerpts, it establishes three thematic consistencies in the manner that students interpreted onscreen movement.

Chapter VII devotes itself to the study of interviewees' language. It examines interviewees' kinesthetic descriptions of parallelism as well as the metaphors they employed to capture onscreen behavior.

Chapter VIII describes interviewees' attempts at rebuilding a DG square from scratch. It details interviewees' ability to subvert Sketchpad's construction tools and build geometric models that succeed on their own terms.

Chapter IX, the concluding chapter, summarizes the themes of this dissertation and examines the implications of its findings for teachers and curriculum developers. It also offers suggestions and related research questions for mathematics educators pursuing the study of dynamic geometry.

CHAPTER II
NEED FOR THE STUDY

Each spring, the National Council of Teachers of Mathematics (NCTM) publishes a yearbook that collects articles on a timely theme in mathematics education. In 1987, the yearbook topic was geometry (Lindquist & Shulte, 1987). Of the 20 collected articles, only two examine computer-based geometry learning, and neither entertains thoughts of a motion-based, geometric software environment. Today, fourteen years later, the situation has changed considerably.

Activities incorporating dynamic geometry appear on a nearly monthly basis in the NCTM classroom journal, the *Mathematics Teacher*. In 1997, the Mathematical Association of America published *Geometry Turned On* (King & Schattschneider, 1997a), an entire volume devoted to applications of the software. Several secondary curricula include DG explorations in their texts (Scher, 2002; Education Development Center, 2000; Gay, 1998; Serra, 1997). And the new *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), the guiding beliefs document of the NCTM, recommends the use of the software to promote mathematical investigations.

Given this flurry of new curricular ideas, it is important to remember that making a successful transition to a DG-centered course involves more than switching from one textbook to another. Enthusiasm for the software will mean little if students fail to connect with dynamic geometry because of unforeseen learning considerations. The discussion below highlights four issues pointing to the need for a close study of students' interactions with DG software:

1. Motion's effects on learning,

2. An increased reliance on visual interpretations,

3. The role of mathematical invariants,

4. Differences between dynamic and Euclidean geometry.

Motion's Affects on Learning

In his 1947 treatise, *Vision in Motion*, artist and photographer Moholy-Nagy notes that in an increasingly technological world, people's ability to attach meaning to images of motion cannot be taken for granted:

A rendering of vision in motion is given in photographs of fireworks, in the diagram of the continuous flight of a skywriting plane…as well as in industrial time and motion studies. People have to be educated in deciphering and understanding them, just as they are taught to read and write. In days to come, when more science and technology will be introduced in the daily routine, there will be more need for space-time rendering and their punctilious interpretation. (p. 121)

Roughly fifty years later, mathematics education researchers such as Dreyfus (1994) and Tsuyuki (1998) offer similar cautionary notes regarding students' ability to interpret moving images on the computer screen. Says Dreyfus:

In every case, visual [computer] representations need to be carefully constructed and their cognitive properties for the student need to be investigated in detail. The adaptation and correction of features of these visual representations on the basis of student reaction to them is an integral part of the development, and in some cases has been reported in the literature. (p. 119)

Goldenberg (1998) lists some specific types of student "reactions" worth tracking with dynamic geometry software:

Although research still leaves many unanswered questions about how students use static visualizations, DG raises the ante and requires us to understand how students glean geometric ideas from pictures that involve motion, often quite complex. How do students develop a sense of where to look, what objects to track, what questions to ask, what experiments to perform? (p. 364)

The three needs-related sections that follow each address issues arising from dynamic geometry's motion capabilities.

<u>An Increased Reliance on Visual Interpretations</u>

In a traditional geometry textbook, a problem might read: "Figure 1 shows an arbitrary quadrilateral $ABCD$. The midpoints of its four sides are E, F, G, and H. Prove that the quadrilateral $EFGH$ is a parallelogram."
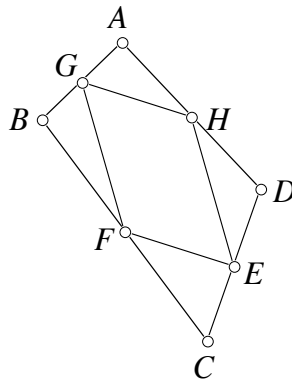
Figure 1. An arbitrary quadrilateral *ABCD* with its midpoints connected.

The above problem tells students exactly what they must prove. It could have asked them to form their own hypothesis, but geometry texts dissuade their readers from making conjectures based on the limited evidence provided by a single picture. To stress this point, they will sometimes distort illustrations to discourage unfounded visual assumptions. "Don't be too quick to trust what you see" is the message imparted to students.

With dynamic geometry, students are no longer limited to single textbook illustrations that may or may not be accurately drawn. Using The Geometer's Sketchpad's tools, students can build their own interactive model of Figure 1 and freely change the locations of *ABCD*'s vertices. By doing so, they will see a wealth of precise, highly accurate illustrations, three of which are displayed in Figure 2. In each case, the midpoint quadrilateral sitting inside *ABCD* appears to have opposite sides that are equal in length and parallel—the very qualities of a parallelogram.
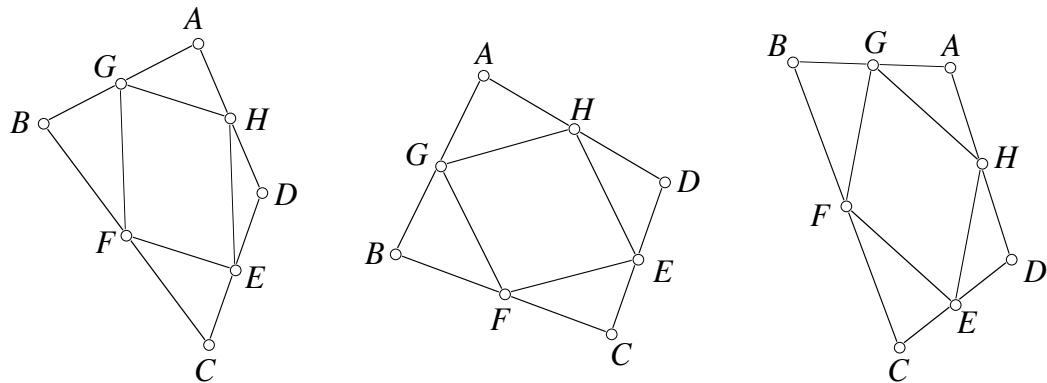

Figure 2. Three individual snapshots of Figure 1's construction.

No amount of visual evidence in mathematics can ever serve as a replacement for an airtight deductive proof because there will always remain untested cases. But the sheer quantity of visual data collected in a dynamic geometry environment does allow students to make strong conjectures that later can be verified or rejected through formal proof.

As such, dynamic geometry curricula push students to formulate their own working hypotheses based on their visual interpretations of onscreen behavior. Bennett says that Sketchpad "…encourag[es] a process of discovery that more closely reflects the way mathematics is invented: a mathematician first visualizes and analyzes a problem, making conjectures before attempting a proof" (1998, p. vii).

This shift to a discovery-based approach places a higher premium on the art of visual observation than older geometry curricula (Laborde & Laborde, 1995). Students exploring the parallelogram in Figure 2, for example, must be able to focus on some geometric property of *EFGH* as its shape shifts on the computer screen. Can they sense that opposite sides of *EFGH* stay parallel? Or perhaps that each pair of opposite sides is equal in length? Research does not yet have answers to these types of questions.

The Role of Mathematical Invariants

Within each branch of mathematics sits a collection of domain–specific terminology and techniques. Yet cutting across topics are concepts with universal mathematical applicability. One such unifying theme is the presence of *invariants*: those features of a mathematical system that stay fixed while others change (Cuoco et al., 1996). Consider the steps for finding the x value satisfying $3x + 7 = 25$:

i. $3x + 7 = 25$

ii. $3x = 18$ (subtract 7 from both sides)

iii. $x = 6$ (divide by 3 on both sides)

While equations (i) to (iii) appear different, they each share the same solution set. One can subtract seven or divide by three on both sides of the equation because these operations are guaranteed to leave invariant the value of x that satisfies the original equation.

The manipulation of algebraic equations is a time-honored example of invariants familiar to all middle-school mathematics teachers. In the field of geometry, however, DG software introduces subtle notions of invariants not found in its paper-and-pencil counterpart. Take, for example, an isosceles triangle. In Euclidean geometry, any triangle with exactly two sides equal in length is isosceles. But in a dynamic geometry world, this measurement test no longer suffices. One must drag the vertices of the triangle to check whether it remains

isosceles when perturbed. If the triangle changes its size and shape but retains two equal lengths, the triangle was *constructed* and fits the DG definition of isosceles (Finzer & Bennett, 1995; Glass & Deckert, 2001).

Should, however, the triangle deform into an arbitrary triangle with no equal lengths, it is a *drawing* and not considered isosceles (see Appendix A for a complete description of the drawing-versus-construction distinction and how to construct an isosceles triangle). Stated succinctly, a DG triangle is isosceles only when the equality of its two sides remains *invariant* under dragging.

The presence of invariants in dynamic geometry software marks a fundamental shift in what it means to be an isosceles triangle or, for that matter, an equilateral triangle, square, rectangle, or parallelogram. Many teachers have likely not considered the implication of invariants on geometric definitions, geometric constructions, and student learning. Are students able to recognize geometric shapes by the invariant properties they retain under dragging? Can they build their own shapes with the necessary invariant features? Just as students in a static geometry environment might recognize an isosceles triangle but be unable to build one with a straightedge and compass, so might they have difficulty constructing one with Sketchpad's unique tool set (Laborde, 1993).

Differences Between Dynamic and Euclidean Geometry

At a time when we become surprised if something on our computer screen *doesn't* move, it might seem obvious, even natural, that the static figures from Euclidean geometry should give way to dynamic ones. But "obvious" would be an inappropriate word to describe the foundations of dynamic geometry. Despite the similarity of straightedge-and-compass geometry to its software counterpart, the crafting of Sketchpad and Cabri was not a straightforward matter of transporting Euclid's axioms to the computer. Cabri author Jean-Marie Laborde explains that some departures from Euclidean axioms were inevitable (Scher, 2000):

> The general principle was to make the distance [between Cabri and Euclid] as small as possible, but at the very beginning, I was not aware that it would remain finally at some distance…People weren't happy at all [with the] expression 'Cabri geometry.' But we decided nevertheless to introduce that concept to make definite the point that what comes from the screen is not Euclidean geometry, it's not projective geometry…It has to be different. (p. 42)

Nicholas Jackiw, the designer and programmer of Sketchpad, echoes these comments in a November 1994 posting to the online Swarthmore Geometry Forum:

'Why hasn't anyone done this before?' is the most common initial reaction to seeing something like Sketchpad. But then...one realizes something strange is going on behind the curtain—something that may seem intuitive, but which is by no means obvious, and by no means predetermined by the geometry and mathematics we understood before the advent of these programs.

As Jackiw implies, many of Sketchpad's features were not foregone conclusions of Euclidean geometry's structure. How objects moved onscreen and the construction techniques used to create them were more an issue of industrial design than mathematical necessity.

Consider, for example, illustration (i) in Figure 3. On segment *AB* lies a randomly placed point *C*. As point *B* is dragged to the right, lengthening the segment, should point *C* keep its same distance from point *B* (illustration (ii)) or maintain the original ratio of *AC* to *CB* (illustration iii)? Euclidean geometry has no answers to these questions because it concerns only stationary objects. In this particular case, both Sketchpad and Cabri adopt the latter option of keeping ratios constant, but there are other instances where their choices are different.
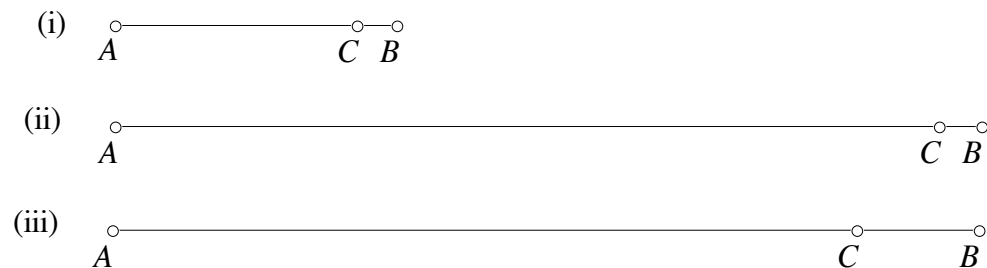
(i)

A   C B

(ii)

A   C B

(iii)

A   C   B

Figure 3. Two possible behaviors for point *C* when point *B* is dragged.

While a skilled teacher can filter out these software-dependent features, a neophyte might have more trouble. Nemirovsky et al. (1998) liken the situation to x-ray interpretation:

Many of us have participated in a conversation with a physician analyzing an x-ray photograph from a part of the body. Often the physician points to a small gray area on the image and says 'See this?' to help us focus on a little spot that otherwise might be absolutely unremarkable to us. The patterns of significance expressed by the physician can be very counterintuitive to a lay person; what looks salient may be an irrelevant optical artifact of the x-ray device. (p. 8)

With Sketchpad, students must play a role similar to that of a physician as they separate software "artifacts" from genuine geometric observations. Are they capable of doing so?

CHAPTER III
RELATED RESEARCH AND THEORY

Chapter II established the need for studying students' conceptions of geometry in a dynamic geometry software environment. In the nearly ten years since the introduction of Sketchpad and Cabri, educational research has begun to make inroads into this territory. This chapter examines some of the trends, research, and theories that inform this dissertation. In doing so, it steps back to include work from the field of static geometry research. Broadening the scope of the discussion frames my work in an historical context and acknowledges the continued importance of established theories.

The literature review below is sorted by the three themes of this dissertation—motion, language, and construction.

Motion

The multimedia capabilities of today's computers may lead some to assume that interest in motion geometry is a new phenomenon. Yet a search of the mathematics education literature reveals a foreshadowing of dynamic geometry in both film and hands-on devices. Dynamic geometry software fits into a tradition that has harnessed the technology of the time to achieve non-static representations. Syer, in 1945, describes the ability of film to create "continuous" geometric images. His advocacy of the moving picture reads much as a modern-day justification for dynamic geometry software:

> In addition to true-life demonstrations of solid geometry, it would be interesting to make greater use of the peculiar advantage of moving pictures over ordinary models. In plane geometry films, we used figures that changed shapes, position, and color without distracting pauses or outside aid. This continuous and swift succession of illustrations is fast enough to keep up with a spoken description, or even as fast as the thought processes that are developing the idea. Thus no time is lost erasing pictures from the blackboard, changing lantern slides, or holding up illustrations, because the illustrations and thought move simultaneously. (p. 344)

Burnes in 1954 uses the very word "dynamic" when describing the benefits of malleable mathematical devices:

Effective results from the laboratory method are related to three simple principles: (a) the device or material should be simple, and whenever possible, flexible or dynamic as opposed to a static piece… (p. 142)

A theory known as the "prototype phenomenon," explained below, helps to explain the appeal of motion as it relates to the learning of geometry.

The Prototype Phenomenon

In studies conducted with static, paper-and-pencil geometry, researchers note a trend regarding the identification of geometric objects. When shown a collection of triangles on paper, students are more likely to identify a triangle as isosceles (one with exactly two sides equal in length) when its base is sitting horizontally on the paper (see Figure 4). Similarly, students are better able to spot a right triangle when it sits in an "upright" position, with the sides forming its right angle parallel to the edges of the paper (Clements & Battista, 1992; Yerushalmy & Chazan, 1993). Again, see Figure 4.
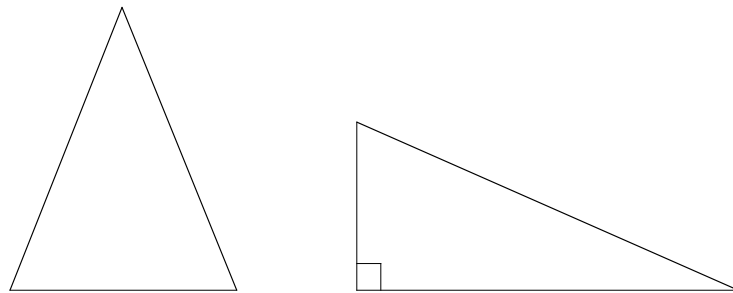
Figure 4. Two prototypic triangle positions.

Hasegawa (1997) refers to such findings as the "prototype phenomenon": through everyday experiences and their work in school, students develop prototypic mental images of geometric shapes, which often are not robust (see also Mequita, 1998; Schifter, 1999). A single textbook illustration of an isosceles triangle might, for example, cause students to over-generalize and assume that only triangles sitting on their base qualify as isosceles.

The purpose of the Geometric Supposer (Schwartz & Yerushalmy, 1991), a software precursor to dynamic geometry, was to address this issue by generating multiple static images of geometric shapes, all in random sizes and orientations (such as the "tilted" isosceles triangles in Figure 5).

Sketchpad takes this notion one step further, allowing students to drag shapes into whatever positions they choose, and watch the objects shift in a continuous, fluid manner. On an intuitive level, it is plausible to hypothesize that Sketchpad's

motion capabilities free students from overgeneralizing the particulars of a static image. Chapter VI of this dissertation examines the strength of this claim through the analysis of videotape data.
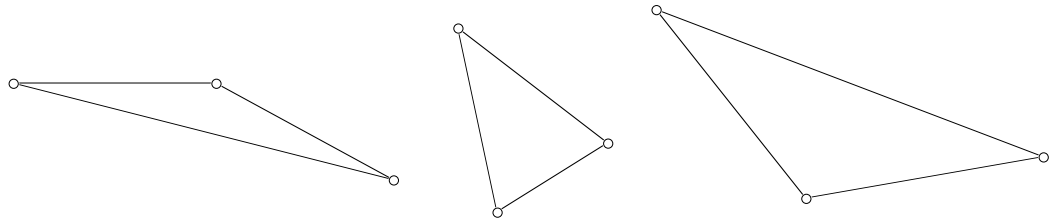

Figure 5. Several isosceles triangles in "tilted" positions.

Language

The work of Dutch educators Pierre and Dina van Hiele stands as perhaps the most celebrated research into the learning of static geometry. Introduced in the late 1950s and still influential today, their model of geometric knowledge maps students' development through a five-level hierarchy of skills (Clements & Battista, 1992). In brief, these stages are:

a. Stage 1: Visual

Students recognize objects based on overall appearance, not their geometric properties. A rectangle might be identified by its similarity in shape to a door frame.

b. Stage 2: Descriptive/Analytic

Students describe geometric objects based on their properties rather than appearance. An object is a square, for example, because it contains four right angles and four equal sides.

c. Stage 3: Abstract/Relational

Students use the properties of geometric objects to classify them hierarchically. A square, for example, is a special case of a rectangle. See Kim (1994, 2000) for an excellent example of Stage 3 thinking.

d. Stage 4: Formal Description

Students can construct proofs within an axiomatic system.

e. Stage 5: Rigor/Mathematical

Students can apply formal reasoning to compare different axiomatic systems.

According to the van Hiele model, learners progress through these stages in order without skipping any levels in the process (Clements & Battista, 1992; van

Hiele, 1986). The model suggests that attainment of higher levels depends more upon the method of instruction rather than one's age. The van Hiele structure has served as the basis not only for static-geometry research (see, for example, Fuys, Geddes, & Tischler, 1988), but also as the foundation for at least two dynamic geometry curricula (Battista, 1998; Choi, 1996).

Stages one through three of the van Hiele model all describe students' use of language when viewing static geometric shapes. Do any of these three stages remain relevant in the world of dynamic geometry? Yes. Just as a student at van Hiele stage one might describe a rectangle as a door, so did our interviewees invoke metaphors in their oral accounts. Yet rather than classify objects by their shape (as with the rectangle), our students described objects in ways that related to their onscreen movement. Much more will be said on this issue in Chapter VII, but below is an account of "situated description," a theory that suggests why such results might be expected.

<div align="center">Situated Description</div>

While students can study geometry with either straightedge and compass or computer software, the geometric conceptions developed in these two mediums may not be the same. Mathematics education research reveals that the setting or "situation" in which learning occurs affects the way knowledge is acquired and interpreted (Fennema & Franke, 1992). The term "situated knowledge" signifies that learning is not independent of its setting.

Hölzl (1996) proposes a theory of "situated description" whereby students who use mathematical software develop language mirroring the types of interactions they experience with the software. These computer-centered descriptions often include active verbs, especially those expressing movement.

Construction

Chapter II introduced the distinction between a dynamic geometry drawing and construction. Finzer and Bennett (1995) refine this dichotomy further,

presenting a four-stage structure for describing and assessing students' shape-building efforts :

a. Stage 1: Drawing

Students build a square by sketching an arbitrary quadrilateral onscreen and then tinkering with its sides until it satisfies the square definition. Dragging any part of the so-called square, however, causes it to deform back into a random quadrilateral.

b. Stage 2: Underconstraint

Here, students might be able to guarantee that the quadrilateral's angles remain ninety degrees when tugged, but its side lengths do not all remain equal.

c. Stage 3: Overconstraint

Now, the object students build is indeed a constructed square, but only one particular square: dragging any of its vertices does not cause it to grow or shrink.

d. Stage 4: Appropriate Constraints

Finally, at this level, the square maintains its shape while dragged and also has the flexibility to change its size.

In theory at least, Finzer and Bennett's categories seem to provide a complete classification scheme for analyzing students' construction efforts. In practice, we found that these stages, while helpful, were not as unambiguous and distinct as we thought. Chapter VIII re-examines Finzer and Bennett's theory from the point of view of two interviewees building a dynamic geometry square.

CHAPTER IV
THE RESEARCH STUDY

Origins of the Study

The 1980s saw the introduction of a powerful piece of educational technology: the graphing calculator. Unlike traditional calculators that displayed purely numerical results, the graphing calculator allowed students to enter an equation and see its corresponding graph in the calculator's viewing window. The ability to create graphs, choose axis scaling, and vary an equation's parameters captured the interest of educators, positioning the calculators for a wide introduction into secondary mathematics curricula.

Tempering this enthusiasm were some voices of concern. Senior Scientist E. Paul Goldenberg of Education Development Center, Inc. (EDC) questioned whether the unique tools and visual representations of the graphing calculator might alter the way students understood functions and their graphs. Goldenberg's research team set out to develop a series of interview tasks to elicit students' conceptions of functions in a graphing-calculator environment (Goldenberg, 1988).

Videotaped interviews revealed that technology did affect students' understanding of the subject. A simple change in the calculator's viewing window dimensions, for example, influenced whether students regarded two

parallel lines as horizontal translations of each other or vertical translations. Such observations gleaned from an analysis of the interviews led to a collection of recommendations aimed at teachers and curriculum developers (Goldenberg, 1988, 1991; Harvey, 1991).

With the coming of dynamic geometry software in the 1990s, Goldenberg recognized a situation mirroring the introduction of the graphing calculator. Once again, a new technology was entering schools without a thorough understanding of how it might affect students' conceptions of the corresponding subject matter (in this case, geometry).

At the May 1993 meeting of the NATO Advanced Research Workshop, Goldenberg (1995) presented the paper "Ruminations About Dynamic Imagery (and a Strong Plea for Research)," in which he urged DG researchers to consider the same types of epistemological questions he had explored with graphing calculators: How did the tools and visual representations unique to DG software influence students' geometric conceptions?

One year later, Goldenberg won a grant from the National Science Foundation (NSF) to study these issues both with middle-school and college students (Goldenberg, 1994). In the spring of 1997, I joined his Epistemology of Dynamic Geometry project as a Research Associate. At the time, Goldenberg had interviewed undergraduates but not middle schoolers. We soon developed DG interview items for this younger age group.

Development of the Interview Questions

The timing of our interview study coincided with the conclusion of another geometry-related effort at EDC—*Connected Geometry* (Education Development Center, 2000). Developed and piloted over a five-year period with funding from the NSF, *Connected Geometry* is a geometry text for middle and high school-level mathematics. Paul Goldenberg and I were both members of the writing team and had authored the dynamic geometry portion of the text. Since *Connected Geometry* had been field tested in classrooms and reviewed by mathematicians and mathematics educators, we felt that its DG section would serve as appropriate and sound source material for our interviews.

When adapting *Connected Geometry* for our study, we made one significant change to its format. The textbook, as written, told students explicitly to build such objects as rectangles and isosceles triangles. Since one purpose of our interviews was to probe students' conceptions of onscreen geometric figures, we felt that telling students what to build would create a missed opportunity. Rather, we decided to build objects such as a rectangle *prior* to the interview, and then, during the session, ask the interviewees to explore and describe these "mystery constructions" by dragging their parts. Only then would we ask them to build the identical objects from scratch.

To test the soundness of this approach, we piloted our interview with two middle-school students whose parents were EDC employees. The analysis of the sessions led to slight modifications in the DG construction tasks (see Chapter V and Recommendation Nine in Chapter IX for details). The final interview protocol was then approved by other EDC mathematics educators who served as consultants to the project.

The Students Interviewed

Our interview study targeted middle-school students with no prior dynamic geometry experience. Interviewees participated on a voluntary basis and were recruited through local schools in Newton, Massachusetts.

In total, we interviewed three sixth graders and five seventh graders from two Newton middle schools (six boys and two girls). Students from regular-track mathematics classes accounted for six of the interviewees, with the remaining two drawn from the upper-track program.

The eight students were interviewed in the summer of 1997. Each student participated in two interview sessions held on separate days, with individual sessions running approximately two hours. The participants received twenty dollars for each day, and signed an EDC consent form stating they were free to view any of the tapes made during the sessions.

The Interview Setting and Procedures

The interview study was conducted in an EDC office equipped with a Macintosh computer loaded with The Geometer's Sketchpad, version 3.0. A videotape recording of the sessions (as opposed to audiotape) made sense, as we wished to capture the actions occurring on the computer screen as interviewees discussed their work. If a student were to say, "Something strange happens to my line when I move point B over here," the videotape would allow us to pinpoint the exact behavior under scrutiny.

We were fortunate to have the resources to videotape the sessions from two different camera angles (see Figure 6). Aside from a camera that recorded the computer screen, another camera videotaped the interviewee. These separate images were transferred to a mixing board sitting outside the interview room, where a technician combined the shots to produce a split-screen composite tape. Thus even when our interviewees remained silent during their investigations, we were able to monitor their facial expressions and gestures as clues to onscreen activity that surprised or puzzled them.
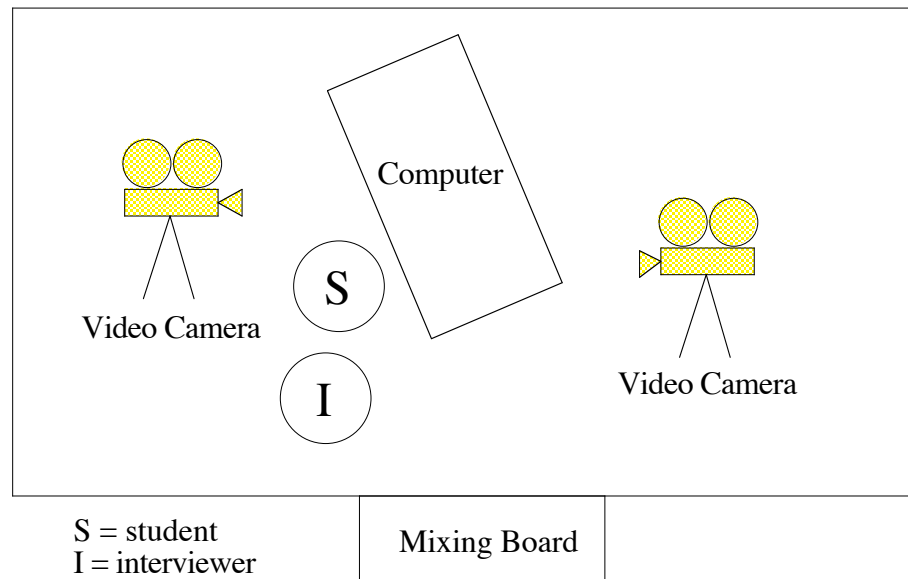
<u>Figure 6.</u> The arrangement of the interview space.

For the first of the two interview sessions, students worked in pairs. By interviewing students together, we hoped they would be more likely to discuss their mouse actions and interpretations of onscreen images. This arrangement, however, tended to result in one student remaining silent or not having sufficient opportunity to use the computer mouse. As a result, many of the second-round interviews were conducted on a one-to-one basis.

The Roles of the Interviewer

Before beginning the first of the two sessions, the interviewer (either Paul Goldenberg or myself) began by providing students with a brief summary of the study's purpose and their role in the research. A paraphrased account of this introduction follows:

> We've invited you here to learn about a new kind of geometry software called The Geometer's Sketchpad. As you explore the software, we're going to be videotaping your actions on the computer screen. This study is not intended to be a test of your ability; we're not going to be marking you 'correct' or 'incorrect' in your work. Instead, we're looking to see how bright students like yourself use the software, and what lessons we can learn that will help people to design better curriculum and software. Since we won't always know what's going on inside your head, we're going to be asking you a lot of questions about what you're doing and what you're thinking.

As the students progressed through the interview tasks (described below), the interviewer sat by their side and asked them to explain their actions and observations. The setting was informal, with students able to analyze and build geometric constructions without time limits, and the researcher free to change the direction of an interview if a particular item caused problems. By not imposing a strict protocol on the interviews, we kept ourselves open to noticing unanticipated student ideas and thinking. During the sessions, the interviewer functioned in two roles:

1. When a student was uncertain whether Sketchpad contained a particular feature or forgot where it was located, the interviewer offered assistance. Throughout the session, the interviewer reminded students that the interview was not a test of how well they had memorized the software commands; rather, it was intended to uncover how they thought about the objects on screen.

2. As students worked through the construction challenges, the interviewer would periodically ask questions like, "What are you trying to do? Describe to me what you're seeing. Can you explain why that line behaves the way it does? How might you test your theory?" The interviewer would also restate or rephrase some of the students' observations to spotlight comments that would benefit from their further attention. Confrey (1993) terms this method "close listening":

> Close listening involves an act of decentering by an adult or possibly a peer, in order to imagine what the view of the child might be like. It includes repeated requests for a child to explain what the problem is that she is addressing, what she sees herself doing, and how she feels about her progress. It requires one to ask for elaboration from the child about what, where, how, and why. (p. 311)

The Interview Questions

The interview was divided into three parts, with Parts I and II designed to give students a concise introduction to the tools and techniques of Sketchpad. These parts occupied roughly the first two-hour block of the interview. Returning for a second day of taping, students began Part III, where the focus shifted to construction challenges requiring applications of the techniques from Parts I and II. This portion of the interview also spanned two hours.

Below is a brief summary of all three interview parts. Appendix B contains the interview questions from Parts I and II. Chapter V contains a detailed account of the items found in Part III.

Part I: Getting Comfortable with Drawing Tools

Students experimented with Sketchpad's point, segment, line, ray, and circle tools to create simple drawings. Appendix B lists specific tasks, but we did not rush into these, choosing instead to give students time for unstructured "doodling." To emphasize that figures, once drawn, could be translated, rotated, stretched, or shrunk, we asked interviewees to drag objects and describe the resulting effects. Several drawing tasks (1(f), 1(i), and 1(j) in Appendix B) emphasized that order of construction frequently mattered: for a point to travel only along a segment, the segment needed to be drawn first, followed by the point on top of it.

Part II: Building Geometric Constructions

Students explored Sketchpad's construction menu items by building midpoints, parallel lines, and perpendicular lines. Appendix B also describes a guided construction of a spinning "windmill" that provided an engaging application of perpendicular bisectors.

Part III: Analyzing and Reconstructing Pre-Made Sketches

Now, with a basic understanding of Sketchpad's drawing and construction features, students moved to the central portion of the interview: interpreting and reconstructing pre-made sketches. Prior to the sessions, we built constructions—many with hidden parts—and saved them. When students opened our sketches, we asked them to drag each sketch element with their mouse and describe the resulting onscreen actions. They were encouraged to consider the types of motion, the constraints that might be present in the construction, and the geometric relationships of the objects. With these conjectures in hand, students attempted to build the identical sketches from scratch.

While not every student was able to complete the constructions, our intention was not to mark them as either "successful" or "unsuccessful" in their attempts. Rather, we wished to document the experimentation techniques and spoken commentary underlying their work.

These sketches could all be considered "black boxes" (Galindo, 1998; Laborde, 1998) as we built them ourselves and hid some of their parts. In this sense, they served our interviewing purposes well, requiring students to examine novel constructions. It is worth noting that the need to interpret pre-made sketches is not entirely artificial. Whether swapping constructions with a fellow classmate, observing a teacher operate a demonstration sketch, or interacting online with a Java geometry applet, students must be adept at on-the-fly visual analysis. Even returning to one's own sketch after some time away requires a short period of reorientation.

Throughout the remainder of this dissertation, the reader will notice that the points in our black-box constructions are labeled and identified by single letters. This is traditional notation, but not the one we chose for our interviews. Objects in Sketchpad can be labeled by any text one chooses. A point *J*, for example, can be renamed as point *Joan* or point *Michael*. To give our activities a personal touch, we replaced point labels by the names of our participants. In the interest of keeping the notational system consistent throughout this dissertation, however, I have chosen to adopt single-letter notation.

Analysis of the Interview Tapes

In total, our interviews of eight middle-school students yielded 20 hours of videotape along with a complete set of written transcriptions. With this much raw data, I decided to focus on just one section of the interviews—the black-box sketches from Part III. Appendices C and D contain complete information and examples of my analysis method. Appendix C also describes an alternative means of analysis proposed by Schoenfeld (1985) as a point of contrast.

Limitations and Delimitations

Despite the advantages afforded by videotaping, there are certain limitations inherent in this dissertation's design and execution. The interviews include eight students in total, and as such, cannot be generalized to a larger population. Several of these interviews were conducted in pairs, and the interactions of the students may have influenced the outcome (Artzt & Armour-Thomas, 1992).

Interviewees' introduction to the basics of Sketchpad took place in a short period of time—two hours. In a classroom setting, students would likely devote more time to simple drawing tasks before progressing to the relatively sophisticated constructions under analysis here.

My study did not include an assessment of student's geometric knowledge prior to their work with Sketchpad. As such, it was often hard to pinpoint the root cause of their difficulties. When an interviewee struggled with a question, was it due to a conceptual, Sketchpad-related issue or something more simple, like a gap in her geometry vocabulary? In analyzing my interview tapes, I could not always tell the difference. Furthermore, any results of my analysis that focus on Sketchpad-specific features do not apply to other dynamic geometry programs such as Cabri.

I do wish to emphasize that although the results in this dissertation cannot be generalized, there is still value in finding themes among my small data sample. One could conceivably ask three hundred students to go home, construct a Sketchpad square, and then submit their efforts on disk. Their work could be reviewed and marked as either "successful" or "unsuccessful." One could even categorize the nature of the participants' difficulties. But without some form of

interviews, the researcher's ability to say anything about the interviewees' thoughts during their square construction is extremely limited. Detailed interviews with a limited sample size provided me with greater access to the thinking that accompanied my students' mouse actions.

CHAPTER V

THE BLACK-BOX INTERVIEW ITEMS

Consider the challenges inherent in identifying a square. If the square appears on paper, we see a quadrilateral with specific sidelengths and a specific orientation on the page. Should this given image not trigger our recognition, we're stuck. Unless the interviewer is kind enough to show us another picture, we must make do with the one provided.

Now consider the same identification task transplanted to the world of dynamic geometry. The square sitting in front of us on the computer screen seems just like the one on paper; it has a particular length and orientation on screen. Yet this square has something the static square didn't possess—a behavior. We can drag any of its four vertices with the computer mouse and watch how the square responds. The quadrilateral might change its length and orientation in response to our dragging, but certain features remain invariant. It will always retain four equal sides and four, ninety-degree angles.

The new information obtainable through dragging might either be a help or an added challenge in the identification task. From one perspective, perhaps it is easier to spot a square when viewing multiple squares as opposed to just one. But these squares do not exist as individual snapshots on screen. The user sees a continuous image of a square as it grows, shrinks, and rotates. Interpreting this motion (or perhaps filtering it out) could be difficult.

The nature of a dynamic geometry environment also adds a wrinkle to the challenge of recreating a black-box sketch from scratch. Because a DG square possesses a behavior when dragged, reproducing an identical square from scratch is not simply a matter of drawing a quadrilateral with sides that look equal and angles that look to be ninety degrees. The quadrilateral must maintain its "squareness" when dragged. According Pimm (1995):

> With black-box situations, the challenge is thus not just to make a static, visually indistinguishable screen image copy, but also to create a drawing that is *dynamically* identical as well. This is a mathematically new task, one that was not offerable prior to the development of such software environments. (p. 55)

Our Black-Box Items

The uniqueness of black-box identification and construction challenges made them a natural choice for the central portion of our dynamic geometry interviews. Their novelty, however, required us to exercise care in introducing them to our interviewees. We prefaced the tasks with a verbal description of DG's drawing-versus-construction distinction. Our explanation was not formal; we chose a factory metaphor to convey the information in a simple way. Below is a quotation from the interviewer:

> I'm going to draw a square…it looks pretty good [*the interviewer sketches (as opposed to construct*s) *a square*]. You're going to be part of a factory. The customer calls up and orders a square, and you deliver them. And in fact, they don't want just a single square; they want a square that can be any size and never be anything else. They would be really sad if it turned out like this [*grabs a corner of the square and deforms it into an arbitrary quadrilateral*]. If you delivered this thing to them, the customer would grumble and complain. He'd say, 'Look, this isn't a square any more. I want it to stay a square no matter what happens. This is going to get kicked around the living room, the cat's going to sleep on it. It can change size, but it must stay a square. I don't want it ever to break.'

The remainder of this chapter describes our black-box questions and the rationale behind their inclusion. Readers unfamiliar with Sketchpad will benefit from the brief introduction to the software in Appendix A.

Order of Operations

Aside from labeling differences, the two circles in Figure 7 look identical. When interviewees experimented with these objects on screen, however, they discovered some differences.

Points *P* and *R* were both "control points" of their respective circles. Dragging either point shrunk or expanded its circle's size depending on whether the point moved toward or away from the stationary center.

When we built the leftmost circle prior to the interview, we placed Sketchpad's point tool directly onto the circle's circumference to draw point *F*. Doing so confined point *F* to traveling solely along the circumference. It could not be dragged off the circle.

By contrast, point *V* was drawn separately from its circle and then dragged onto it afterward. Interviewees who moved point *V* found that it could travel anywhere on screen, on or off the circle.

To recreate the two circles, interviewees needed to carry out their constructions in a sequential fashion. One could not, for example, draw point *F* and then

overlay a circle onto it. For point *F* to travel only along the circumference, the circle needed to be drawn first. Most Sketchpad constructions in Version 3.0 require the user to pay attention to such sequencing details.
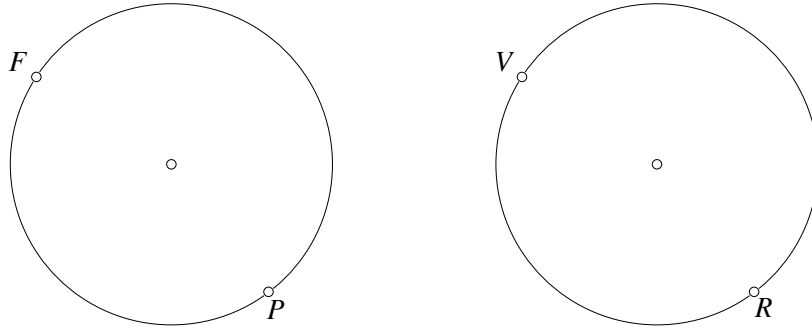

Figure 7. Two constructions that look the same but behave differently.

The Isosceles Triangle

Figure 8 shows the next black-box sketch consisting of two isosceles triangles. In initial appearance, triangles Δ*TNJ* and Δ*EHW* looked identical, with *TJ = TN* and *EW = EH* respectively.
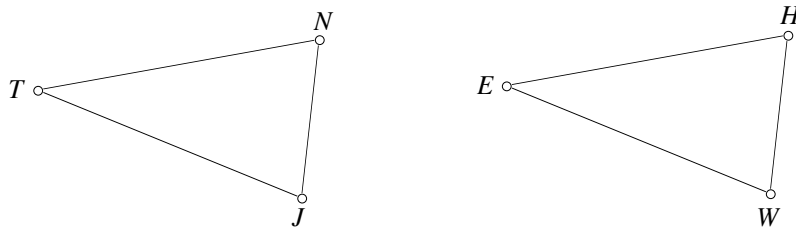

Figure 8. Two isosceles triangles with different behavior.

Triangle *TNJ* concealed a circle with center, *T*, and points *N* and *J* on its circumference. Dragging either *N* or *J* produced a collection of isosceles triangles with sidelengths $\overline{TN}$ and $\overline{TJ}$ remaining unchanged. In Δ*EHW*, there was one construction difference: point *H* behaved like *N* and *J*, but point *W* served as the control point for the circle centered at *E*. Dragging point *W* yielded a collection of isosceles triangles whose sidelengths changed as the hidden circle grew and shrank. Appendix A contains more information about isosceles triangles for the interested reader.

The Square and the Rectangle

Upon opening the next black-box sketch, interviewees saw two quadrilaterals sitting side by side (see Figure 9). Both appeared to be squares, but dragging the objects revealed two distinct quadrilaterals: a square and a rectangle.[2]

Figure 9. A square and a rectangle.

Figure 10 shows our Sketchpad square construction (for other methods, see Pereira (1997) and Appendix E). Two perpendicular segments, *DA* and *DC*, constructed from the circle's center, point *A*, guarantee the side lengths of the quadrilateral will be equal and at right angles to each other.
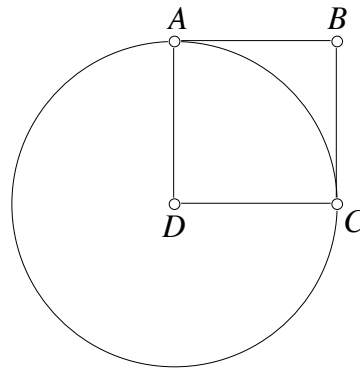
Figure 10. One method for building a Sketchpad square.

Note that although a square is symmetric, the square in Figure 10 does not behave identically when each of its individual vertices are dragged. Dragging point *D* causes the entire square to grow and shrink with point *C* staying fixed. Dragging point *B* translates the square across the computer screen without it changing size or orientation.

---

[2] In our pilot interview, we only presented a square.

The Perpendicular Bisector

The square, rectangle, and triangle problems above share something in common with static geometry identification tasks: participants are asked to recognize self-contained geometric shapes. There are no stray objects—a point or a segment—sitting by themselves on the computer screen.

With dynamic geometry it is possible to arrange a black-box task where interviewees must identify geometric *relationships* as opposed to geometric shapes. Because elements of a sketch can be hidden from sight, two objects in Sketchpad can be related geometrically without touching. In the problem described below, the featured relationships are bisection and perpendicularity.

To prepare the black-box sketch, we drew a segment *AB*, constructed its midpoint *C*, and then constructed line *j* perpendicular to $\overline{AB}$ through point *C* (see Figure 11). If *A* and *B* were points drawn on paper, then *j* would be the line formed by folding point *A* directly onto point *B*.
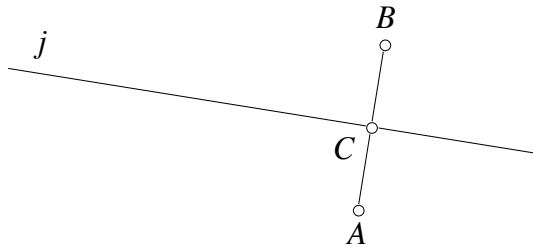

Figure 11. Line *j* is the perpendicular bisector of $\overline{AB}$.

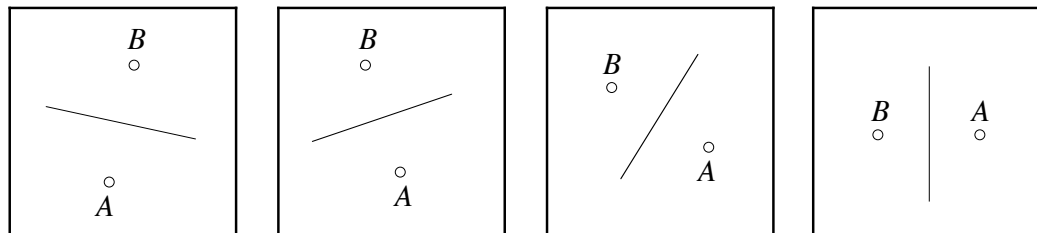To camouflage some of the sketch elements, we hid $\overline{AB}$ (minus its endpoints) and point *C*.[3] See Figure 12.

Figure 12. Four snapshots of the perpendicular bisector construction.

---

[3] In our pilot interview, we did not hide segment *AB* or point *C*.

As shown in Figure 12's four snapshots, when interviewees moved either point *A* or *B*, line *j* repositioned itself to remain perpendicular with the unseen segment *AB*. Dragging line *j* translated the entire construction across the screen without changing any of the spatial relationships.

It is worth mentioning that a perpendicular bisector relationship can be created in an alternative fashion: Starting with a fresh sketch, draw a random point *A* and a line *j*. Then use Sketchpad's "Reflect" feature to reflect point *A* over segment *j*, forming point *B*. The symmetry properties of reflection guarantee that line *j* is the perpendicular bisector of $\overline{AB}$.

Mathematically, the two techniques described above are identical, but their Sketchpad behavior is different. For the reflection method, any movement of point *A* is mirrored in the simultaneous movement of the reflected point *B*. By contrast, dragging point *A* in our black-box sketch moves line *j* without affecting point *B*. A static picture alone, then, does not provide enough information to replicate the original construction. Without exploring the motion relationships built into the sketch, one cannot know how it was built.

Variation on a Theme

The final black-box item presented a variation of the perpendicular bisector construction. Starting from the set-up in Figure 12, we placed a random point *D* along segment *j*. Point *D* was then connected to points *A* and *B* forming two equal-length segments, $\overline{AD}$ and $\overline{BD}$ (Figure 13). Finally, we hid line *j*.
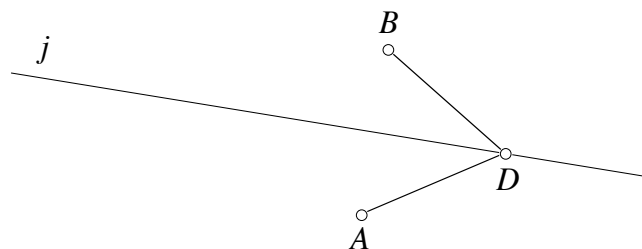

Figure 13. Point *D* travels along perpendicular bisector line *j*.

Now, as shown in Figure 14's four snapshots, students could drag point *D* only along the invisible line *j*, causing the lengths of $\overline{AD}$ and $\overline{BD}$ to change, but always remain equal to each other. Dragging point *A* or *B* produced a behavior specific to Sketchpad: when either point moved, point *D* repositioned itself to keep the measure of angle *ADB* constant.
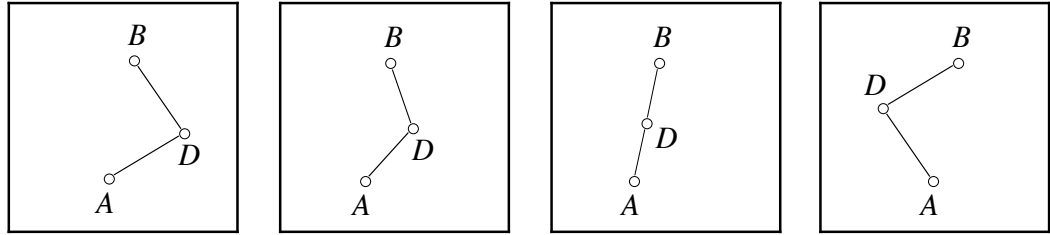
Figure 14. Four snapshots of point *D* moving along the hidden line *j*.

CHAPTER VI
MOTION

In Figure 15 lies a quadrilateral. It is a specific quadrilateral with a length of 3.4 inches, a width of 1.3 inches, and sides parallel to the edges of this paper. None of these particulars matter of course when the quadrilateral applies for membership in the club of rectangles. For those purposes, only the general properties of rectangular shapes bear any importance. Two pairs of parallel sides? Yes. Four right angles? Yes. The quadrilateral is a rectangle.
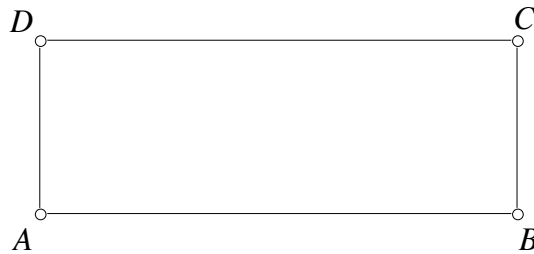

Figure 15. A particular rectangle.

Separating the non-essential particulars of a given rectangle from the universal properties of *all* rectangles is something teachers do automatically. But as described in Chapter III, students, as geometric neophytes, may find a solitary, static illustration too limiting for reaching generalized conclusions.

Well before the coming of the personal computer and dynamic geometry, researchers have suggested a way to broaden the information represented in a single picture: the addition of motion. Art psychologist Rudolf Arnheim writes in 1969:

> The usual illustrations in textbooks and on the blackboard help to make a problem visible, but they also freeze it at one phase of the range to

which the proposition refers. Therefore, they tempt the student to mistake accidental circumstances for essential ones. The solution is not to leave out illustrations but either to produce mobile models, for instance, by means of film animation, or, at least, to use immobile illustrations in such a way that the student realizes which of their dimensions are variables. (p. 182)

In describing the educational benefit of setting geometric shapes into motion, Arnheim concludes, "A static concept has been replaced with a dynamic one. Generality intended is now represented by generality perceived" (1969, p. 180).

Arnheim's argument has remained relevant, and indeed gained backing, 32 years later in today's world of dynamic geometry. The ability of Sketchpad to display not one rectangle, one square, or one isosceles triangle, but a seemingly infinite variety of these objects represents to many educators a compelling example of geometric shapes presented in generalized form (Olive, 2000).

Is this confidence in dynamic geometry well founded? Does DG software allow students to perceive generalities and cast aside particulars? My interview analysis suggest that the answer to the second question is at least sometimes no.

In this chapter, I describe how students continued to search for—and found—particulars amidst Sketchpad's array of seemingly generalized images. Through mouse manipulations of onscreen isosceles triangles and rectangles, interviewees uncovered restrictive movement patterns created not by the dictates of mathematics, but rather by the uniqueness of Sketchpad's design. These restrictions fell into three categories:

1. Certain points moved while others stood still when objects were dragged,

2. Certain points could only be dragged along hidden paths,

3. Certain points, when dragged, restricted the movements of the objects upon which they sat.

Throughout this chapter, I offer examples of the above phenomena and describe how interviewees paid as much, if not more, attention to these particulars than to the general properties of shapes.

What Moves and What Does Not

Stated broadly, a rectangle constructed with Sketchpad can grow and shrink via the dragging of the computer mouse. This is a loose description of Sketchpad's motion feature; one requiring some elaboration. What does it mean to say a rectangle "grows and shrinks"?

Sketchpad provides a definite answer to this question, but first let's formulate a reasonable guess. Since the length and width measurements of a generic rectangle are unimportant, perhaps both quantities should grow and shrink in unison, thereby highlighting their variability. The location of individual points are also immaterial, so perhaps all four vertices should move when any single vertex is dragged. One visual interpretation of these decisions appears in Figure 16.
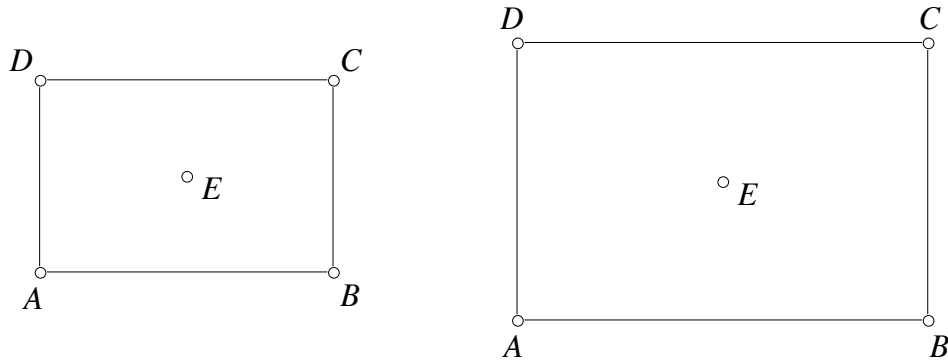

Figure 16. Rectangle *ABCD* expands uniformly about point *E*.

As point *A* (or any other vertex) is dragged outward, the entire rectangle expands uniformly about its center, point *E*. No point or segment on the rectangle stays still. This is indeed one *theoretical* rectangle behavior. It is not the way a typical Sketchpad rectangle reacts under dragging.

Rather, a Sketchpad rectangle behaves differently depending on which vertex gets tugged. The list below charts the reactions upon dragging each vertex of rectangle *ABCD* shown in Figure 15:

1. Drag point *A*. The entire rectangle expands or shrinks uniformly with all of its sides and vertices in motion except point *D*, which stays fixed. Mathematically speaking, point *D* is the center of dilation—a stationary point about which the rectangle grows and contracts. Point *A* can be dragged anywhere on screen.

2. Drag point *B*. As shown in Figure 17, segments *AB* and *DC* grow in response to dragging point *B*. Unlike vertices *A*, *C*, and *D*, which can be dragged in any direction, point *B*'s movement is restricted: it can only move along a stationary hidden line passing through points *A* and *B*. Consequently, dragging point *B* does not alter the lengths of either *AD* or *BC*, nor does it change the tilt of the rectangle.
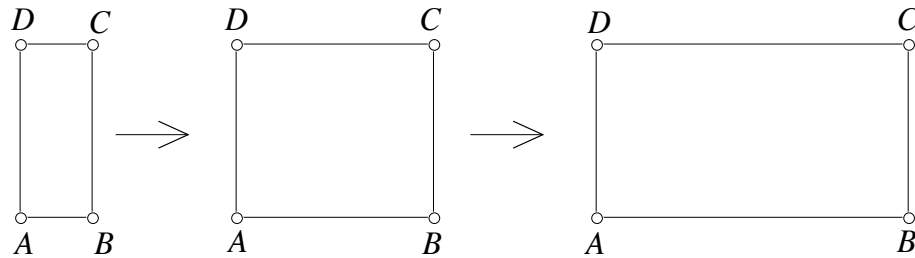
Figure 17. Three snapshots of *ABCD* as point *B* is dragged to the right.

3. Drag point *C*. The entire rectangle is translated across the screen without any change to its dimensions. Point *C* can be dragged anywhere on screen.

4. Drag point *D*. The entire rectangle grows and shrinks about the stationary center of dilation, point *A*. Point *D* can be dragged anywhere on screen.

Note that not one of the movements listed above is identical to the growth pattern depicted in Figure 16. Nor is this type of behavior limited to rectangles: all shapes constructed with Sketchpad exhibit similar movement patterns. Each vertex of an object when dragged will move certain elements of the overall shape while leaving others stationary.

For educators examining a Sketchpad rectangle, the quirky behavior described above is perhaps momentarily surprising. More likely, it passes unnoticed. Yes, point *D* stays fixed when dragging point *A*, but this invariant is of no particular consequence when describing the general properties of rectangles. By contrast, students in our interviews often paid considerable attention to the stationary nature of points. The following three examples illustrate this tendency.

<div align="center">David and Ben</div>

The interview excerpts below find David and Ben[4] examining two pre-built isosceles triangles. Chapter V contains a complete description of these triangles, but a picture of them appears again in Figure 18 for reference. Recall that although these triangles look identical, the behavior of points *J* and *W* is different: whereas dragging point *J* keeps the lengths of $\overline{TN}$ and $\overline{TJ}$ intact, dragging point *W* lengthens or shortens $\overline{EH}$ and $\overline{EW}$ simultaneously.

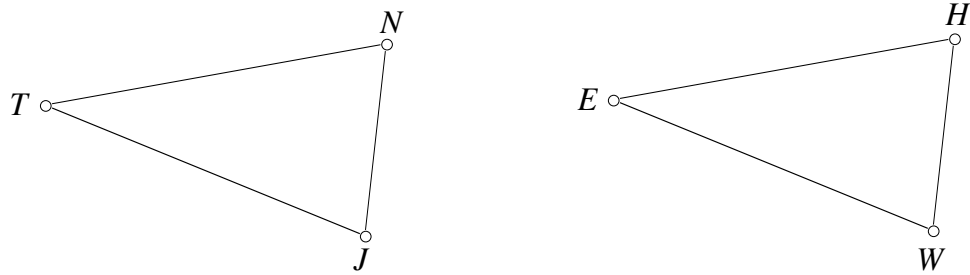[4] The names of all students in this dissertation are pseudonyms.

<u>Figure 18.</u> Two isosceles triangles.

By providing David and Ben with malleable isosceles triangles, we hypothesized that the key invariant property of such triangles—namely, the equality of lengths $\overline{TN}$ and $\overline{TJ}$ as well as $\overline{EH}$ and $\overline{EW}$ and—would catch their attention. Instead, here is what happens when Ben begins to explore Δ*EHW* with the mouse:

> Ben: [*dragging segment EW*] That just moves the triangle [*the entire ΔEHW is translated across the screen without changing size*].
>
> David: [*Ben now drags point E*] That's making *W* your stationary point [*points E and H move while point W stands still*].
>
> Ben: That's moving, like, everything.
>
> David: No, it's not moving point *W*.
>
> Ben: [*dragging point H*] *H* just moves itself.
>
> Ben: [*dragging point W*] Point *W* moves *W* and *H*.

The interviewer now suggests that David and Ben explore the other triangle, *TNJ*:

> David: [*Ben drags point J*] That's just moving *J*.
>
> David: [*Ben drags point T*] That's moving the whole thing…oh, but it's changing the size [of the triangle].
>
> David: [*Ben drags point N*] That's just moving *N*.
>
> Ben: I think *N* and *J* are the same.

David: *N* and *J* have the same movements.

Ben: Yeah.

In the quotations above, David and Ben make no indication that they recognize either triangle to be isosceles. Rather, their discussion focuses on causalities regarding individual points. These causalities are of the type, "Moving point *X* causes point *Y* to move, but leaves point *Z* stationary." Notice, however, that David and Ben do progress beyond a "move/does not move" categorization of points by mentioning that points *N* and *J* "have the same movements" (both *N* and *J* are constrained to a circular path). Asking how a point moves is a

natural follow-up to the simpler question, "Does the point move?" I will discuss this theme further when revisiting David and Ben's progress in an upcoming section.

<p style="text-align:center">Rick and Allen</p>

In the conversation excerpted below, Rick and Allen have experimented with our pre-built rectangle and identified its shape correctly. Their attempt to build a fresh rectangle from scratch has hit a roadblock, so at the interviewer's prompting, they think again about the features they'd like in their construction. Rick returns to our rectangle (see Figure 15) and drags its vertices as he talks:

Rick: I think we want…all these points to have different characteristics. Point *A* moves on point *D* [*Rick drags point A and sees the rectangle rotate and grow around the stationary point D*]…Point *D* moves on point *A* [*Rick drags point D, creating the same behavior*].

Allen: That's also changing the size [of the rectangle].

Rick: Oh it does, yeah…Point *C* moves the rectangle [*Rick drags point C, thereby translating the rectangle*].

Unlike David and Ben, Rick and Allen do know what shape they're trying to build. Yet from their perspective, a rectangle must include not only right angles and parallel sides, but also the specific behaviors associated with each vertex of a Sketchpad rectangle. The interviewer senses this problem and directs the pair back on course:

Int.:   Don't worry for the moment about the characteristics of the points. I think that's possibly distracting. Think about the

shape. What features do you want to preserve in the shape? Because in fact, the customer doesn't really care about the points; the customer cares about the shape.

It is worth mentioning here that the attention of Rick and Allen (as well as David and Ben) to point behavior may be a by-product of our black-box interview style. Since interviewees were told they would need to duplicate our onscreen objects, this directive could be seen as extending to *all* aspects of the construction—including points. More will be said about this issue in the concluding chapter.

<u>Seth and Norman</u>

In comparison to the previous two pairs of interviewees, Seth and Norman succeed in building their construction—a rectangle *ABCD*—without concerning themselves about the behavior of individual vertices. With their rectangle complete and identical to our black-box sketch, they begin to inspect

its four vertices. When dragging points *A* and *D*, they see the length and width of their rectangle uniformly expand and contract. This behavior seems to satisfy them:

Norman: [*drags point A, dilating the rectangle*] Whoa, this is rectangle-e…Now for *D* [*drags point D, again dilating the rectangle*].

Seth: Well, it works.

Int.:   OK, next problem.

The interviewer is ready to move on, but Norman drags point *C* and then point *B*. The behavior of these points mirrors those of our black-box rectangle:

Norman: Whoa! *C*…*C* stays the same [*the rectangle is translated across the screen without changing size*].

Int.:   That's right, but it doesn't ruin the rectangle, right? As long as it doesn't ruin it, the customer won't get upset.

Norman: Why does *C* stay the same? And *B* actually move? [*B's path is restricted, as described on page 50*].

Seth: *C* and *B* are messed up.

Norman: Yeah.

Seth: Points *A* and *D* work great.

From these quotations, it appears that Seth and Norman would like all four vertices to grow and shrink their rectangle. The more restrictive behavior of *C* and *B* leads Seth to pronounce them "messed up."

Uncovering Hidden Paths

The preceding three examples spotlighted students' attention to the mobility and immobility of individual points. Such point behaviors—pure artifacts of Sketchpad's design—did little to aid interviewees in uncovering and recreating the generalized mathematical properties of each mystery shape.

In this section, I present two further examples highlighting students' attention to point behavior. Now, however, their interest lies not merely in determining *whether* a given point moves, but rather, in describing *how* a point moves when dragged.

Consider vertex *N* of isosceles triangle *TNJ* in Figure 18. From this static picture alone, there is no reason to suppose that one couldn't take the computer mouse, click on point *N*, and drag it randomly across the screen. Testing this theory yields a surprise: point *N*'s behavior (as well as point *J*'s) is restricted to an unseen circle with center at point *T* and radius of length *TN*. See Figure 19.
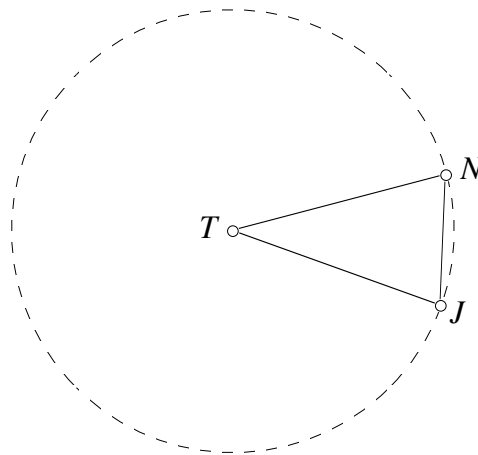

Figure 19. A hidden circle represented by dashed segments.

The circle, tucked away from sight until one chooses Sketchpad's "Show All Hidden" menu option, functions similarly to an architect's blueprints. Just as most

of us remain cheerfully oblivious to the construction features buried behind our walls, so do we not see the circle once the constructor has hidden it. The circle provides the scaffolding needed to build an isosceles triangle without calling undue attention to itself.

Prior to the start of our interviews, we hid the circles that formed the backbone of our isosceles triangles. To our surprise, interviewees soon found them.

<u>David and Ben Revisited</u>

This chapter began by describing David and Ben's exploration of an isosceles triangle. Recall that the pair had dragged individual vertices of their triangle, but had not identified it as isosceles. Sensing that they needed a little help, the interviewer provides a big hint:

Int.: How about the lengths of some of the segments? Is there anything about those that stay the same?

David: These two lengths stay the same [*pointing onscreen to TN and TJ, Figure 18*].

Ben: Yeah it's always a…what is it? An isosceles or whatever?

David and Ben now attempt to build their own isosceles triangle. They draw (as opposed to construct) a triangle that looks isosceles, but they cannot keep two lengths equal when dragging its vertices. At this stage, David and Ben still do not realize that a circle is necessary to assure the equality of lengths.

Ben now returns to our pre-built isosceles triangle, *TNJ*. He begins to drag point *N*. His mouse movements, slow and tentative at first, cause *N* to rock back and forth along a small arc of the hidden circle (Figure 19). Point *N* vacillates in this manner for approximately seven times, never spanning more than half the entire circle's arc.

David and the interviewer continue to talk while Ben stares at point *N*'s movement. Soon Ben interjects, "OK, that's a circle!" Once Ben announces his discovery, he begins to move point *N* rapidly, now twirling it around the entire circle. Ben spins the point again and again, as if to confirm his statement.

David is quick to understand the significance of Ben's observation:

David: Oh, OK, there we go. That helps. So that's the center point of the circle [*gestures at point T*].

The pair now construct their own isosceles triangle, complete with hidden circle, without difficulty.

Ellen and Lynne

Ellen and Lynne pursue their isosceles triangle investigation (Figure 18) in a manner virtually identical to David and Ben:

> Lynne: [*dragging point H in a full circle*] When *H* moves, oh, I think *H* is on a circle or something. So the circle is hidden, [and] when *H* moves, it is attached to the circle and it moves only in a circle.

> Int.: Why do you say that?

> Lynne: Because when you move *E*, you can move it in [towards *W*] and you can move it anywhere, but *H* is restricted, you can't move it into the center, and it seems to go around in a kind of a circular motion.

> Int.: So basically you're saying you think it's a circle because it kind of looks like one.

> Ellen: It's like a protractor [*makes a circular motion with her hands that resembles a compass*].

> Lynne: Yeah.

With some effort, Ellen and Lynne reconstruct both isosceles triangles. At no time, however, do the pair identify the triangles as isosceles or mention the equality of lengths. The interviewer does not press them on this matter, so it cannot be said whether they recognize the general properties of their two triangles. Yet like David and Ben, Ellen and Lynne certainly use their discovery of the hidden circle as an entry point into the construction process.

Commentary

As the examples above illustrate, the detective work of uncovering hidden paths provided an effective, if unexpected, route to rebuilding constructions from scratch. Prior to the interviews, we hypothesized that a student who successfully explored and recreated isosceles triangle *TNJ* might perform the following hypothetical line of reasoning:

> Hmmm…By dragging the vertices of triangle *TNJ* and watching the entire triangle grow and shrink, I see certain key features of its shape remain

invariant. In particular, two of its lengths, *TN* and *TJ*, always stay equal to each other. That must mean I'm dealing with an isosceles triangle. So how can I build one of my own? Well, a circle is a good tool to use when I need to guarantee the equality of two lengths. Let's try that…

There are several assumptions contained in this thought process:

1. Sketchpad's seemingly infinite supply of generalized isosceles triangles would help students to perceive *visible* invariants. Interviewees would pay attention to such visible geometric invariants as the equality of *TN* and *TJ* when viewing Δ*TNJ*.

2. Faced with the need for creating two equal sidelengths, interviewees would puzzle out in their minds the necessary geometric construction.

During our interviews, this chain of events did not occur. Interviewees broke with the above assumptions in the following ways:

1. Interviewees did not (except in one case when prompted) inform us that the lengths of *TN* and *TJ* remained equal or note Δ*TNJ* was isosceles.

2. Interviewees paid particular attention to invariants that were *invisible*. In particular, Point *N* was restrained in its movement to a hidden circle. Rather than deduce on their own the mathematical need for a circle, interviewees uncovered the circle through dragging and observing point *N*'s path.

3. Interviewees did not seem to consider their task as building an isosceles triangle with two equal lengths. Rather, their view of the triangle construction was tied to the hidden circle: build a triangle whose two vertices travel along the circumference of a circle, and whose third vertex serves as the circle's center. This definition has embedded within it the properties that assure the triangle is isosceles and contains two equal lengths, but these traits were never mentioned explicitly.

Reducing the Generality of a Dynamic Illustration

Earlier in this chapter, I described the various ways a Sketchpad rectangle *ABCD* reacts when tugged on each of its vertices. Dragging point *A* produces a seemingly infinite number of rectangles, three of which are shown below in Figure 20:
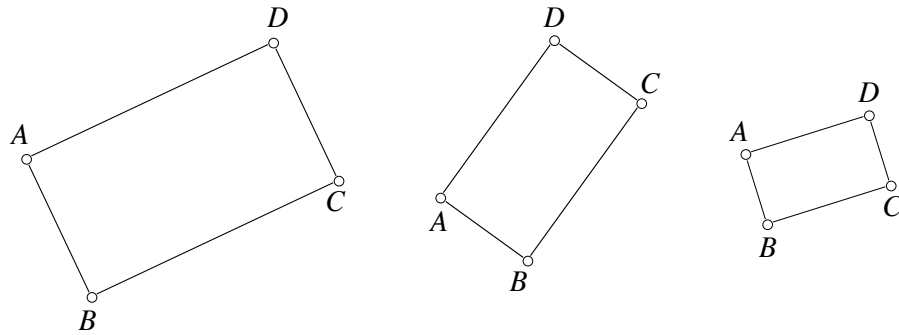
Figure 20. Three general snapshots of rectangle *ABCD*.

The snapshots above illustrate rectangle *ABCD* in a general form. The numerical values of its length and width vary from picture to picture, as does the tilt of the rectangle on the page.

By contrast, dragging point *B* yields a more restricted range of rectangles, illustrated again in Figure 21 below:
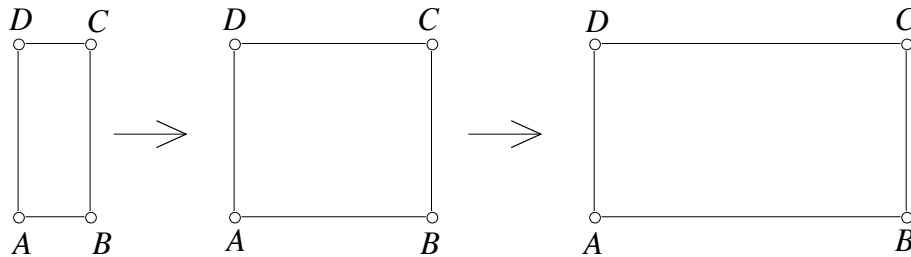

Figure 21. Dragging point *B* restricts the movement of *ABCD*.

As *B* is dragged to the right in a sliding, trombone-like fashion, certain features of the rectangle remain invariant. The lengths of *AD* and *BC* do not change; the distance between parallel lines *DC* and *AB* stays constant; the rectangle does not tilt. The progression of rectangles in Figure 21 displays more generality than a single rectangle image, but less generality than the snapshots in Figure 20. The interview excerpts below describe how interviewees sometimes preferred the reduced generality of Figure 21.

Lynne

Lynne begins her study of rectangle *ABCD* by dragging each vertex individually. She experiments with vertices *A* and *D*—the two points that change the length, width, and tilt of the rectangle—but says she's "not really sure about these yet." She does, however, repay several visits to vertex *B*, examining

both the restricted movement of the point and the rectangle (Figure 21). Lynne comments:

> Lynne: I think that B must be on…I think there might be like a parallel line or something going…going this way *[motioning with her finger in the direction of segment AB]*. Point B can't really adjust; it only stays in the same…never mind.
>
> Int.: What suggested the parallel to you?
>
> Lynne: I think the fact that when you select B it does not move in the way the others do *[pointing to A and D]*. Point B can't adjust like that. It can only adjust this way, keeping the same lengths *[pointing to segments AD and BC, both of which maintain the same lengths when B is dragged]*.

In her final quotation above, Lynne mentions that the lengths of $\overline{AD}$ and $\overline{BC}$ remain constant while dragging point B. Noticing the invariance of these two lengths (and point B's inability to move closer to $\overline{DC}$) might be helping her to see that $\overline{AB}$ and $\overline{DC}$ are parallel. Whereas dragging point A or D changes the tilt and the distances between both pairs of $ABCD$'s parallel sides, dragging point B restricts this generality, perhaps allowing Lynne to focus on just the one pair of parallel sides.

Rick and Allen Revisited

Recall that when we checked the progress of Rick and Allen, they had been unsuccessful in their attempts to build rectangle $ABCD$. When Rick dragged points A and D, the rectangle's length and width expanded and contracted, but he paid attention to the stationary nature of the points instead:

> Rick: Point A moves on point D *[Rick drags point A and sees the rectangle rotate and grow around the stationary point D]*…Point D moves on point A *[Rick drags point D, creating the same behavior]*.

When Rick drags point B (Figure 21), he notices more essential features of the rectangle:

> Rick: I think we want to keep all the angles 90 degrees, and we want to keep the lines parallel…these horizontal lines *[pointing to $\overline{AB}$ and $\overline{DC}$]*.

As with Lynne, Rick seems to focus on the one pair of parallel segments that grows and shrinks when point *B* is dragged.

<div align="center">Commentary</div>

By dragging point *B* and restricting the movement of their Sketchpad rectangles, both Rick and Lynne exhibit behavior resembling the "prototype phenomenon" described in Chapter III. Recall that students, when presented with a static image such as an isosceles triangle, were more successful in identifying the object when it sat with its base horizontal to the paper (Figure 4). Textbooks, whose images are often limited to such "prototypic" positions, had unwittingly narrowed students conceptions of what counted as isosceles.

Here now, in the case of dynamic geometry and rectangle *ABCD*, students are engaging in some narrowing of their own, restricting the types of rectangles they see to make it simpler to identify the quadrilateral's features.

Indeed it might be unrealistic to expect that students can attend to the necessary attributes of a rectangle (or other shape, for that matter) when all of its sides grow, shrink, or rotate simultaneously. Just because we *can* create an idealized rectangle with near-complete variability does not mean, from a pedagogical standpoint, that we should. Perhaps the restrictive rectangle represents a valuable middle ground, where generality is tempered by some specificity.

<div align="center">

CHAPTER VII
LANGUAGE

</div>

Chapter VI examined the influence of motion on the geometric explorations conducted by our interviewees. In this chapter, the focus shifts to issues of language. What verbal descriptions did interviewees use to describe the images they observed on screen? Was their language that of traditional geometry or more informal in nature?

In these language-related issues, motion will continue to play an important role. Chapter III described a theory of "situated description" whereby students who work with computers adopt language reflecting the types of interactions promoted by the software. The Geometer's Sketchpad promotes motion-based interactions with malleable geometric figures. As such, interviewees' choice of words and descriptions were often tied to the motion they viewed onscreen.

Motion-Based Descriptions of Parallelism

Among its other traits, a rectangle possesses two pairs of parallel sides. Without using the word "parallel" explicitly, one might say the sides have the same slope. Or one could note that if the sides were extended indefinitely, the two resulting lines would never intersect.

This section presents two interviewees whose descriptions of a rectangle's parallel sides were neither of the familiar possibilities offered above. Rather, their words show the influence of working within Sketchpad's motion-centered environment.

Interviewee Ben draws a quadrilateral *ABCD* that looks nearly rectangular. He uses no construction tools, so dragging any vertex will deform *ABCD* into an arbitrary quadrilateral. When Ben tugs on his quadrilateral, he realizes that has not constructed a rectangle. As he drags point *A* to various locations (seen in Figure 22's three snapshots), he talks about the behavior he would like to see from *ABCD*:

> Ben: Well, it's not exactly a rectangle, but if you move point *A* out [Figure 22(i)], then *D* has to come out with it. If you move point *A* up [Figure 22(ii)], *B* has to come. If you move point *A* diagonal [Figure 22(iii)], then they [points *B* and *D*] have to go up and to the side.

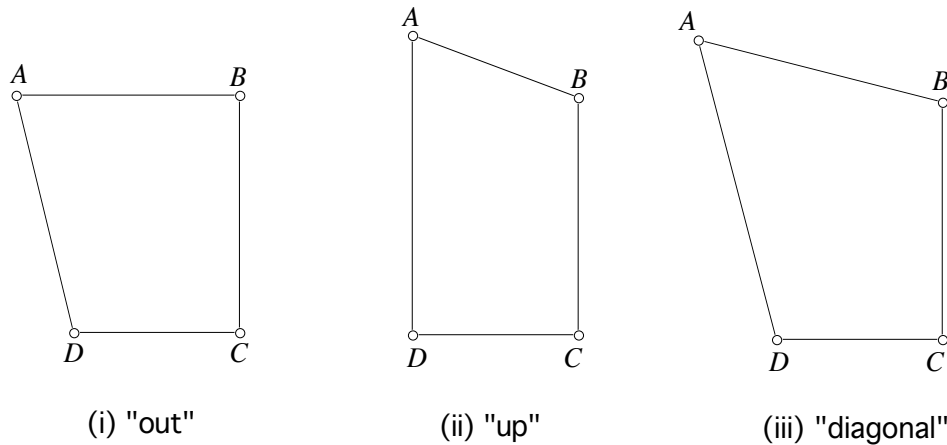(i) "out"   (ii) "up"   (iii) "diagonal"

Figure 22. The results of moving point *A* "out, up, and diagonal."

Ben's quotation adopts a "follow-the-leader" description of how a proper rectangle should behave. When point *A* slides "out" to the left (Figure 22(i)), then point *D* has to "come out with it." When point *A* moves up (Figure 22(ii)), then point *B* must follow suit. Finally, if point *A* moves in a northwest direction (Figure 22(iii)), then points *B* and *D* must move north and west, respectively.

All of the actions described here can be encapsulated into a single, more formal statement: regardless of point *A*'s location, opposite sides of *ABCD* must remain parallel.

Rick, in a separate interview, describes parallelism much the same as Ben. He draws—as opposed to constructs—a rectangle *ABCD*. When he drags point *A* and sees that point *D* does not move with it (Figure 23) he says, "Somehow [points *A* and *D*] need to be attached so that when I pull on point *A*, both *A* and *D* move this way or that."
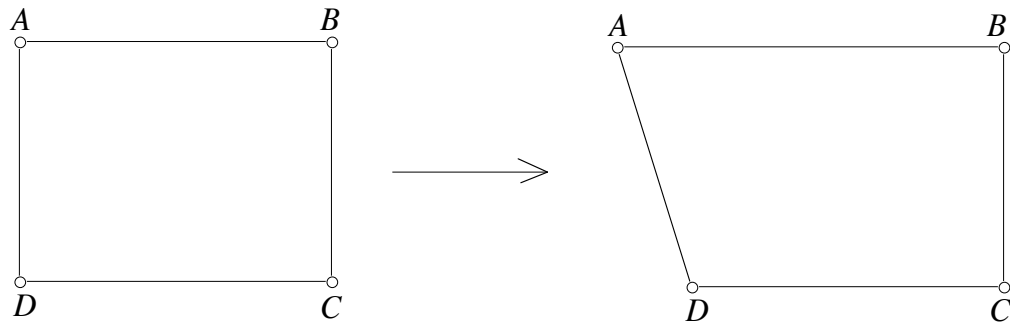

Figure 23. When point *A* is dragged, point *D* does not follow.

Ben and Rick's language, while unexpected, does describe parallelism in a dynamic geometry environment. Their words lack specificity, and it is here that a teacher can press students on the issue: when point *A* moves, where exactly should point *D* move? As point *A* moves, what features of the rectangle should be kept intact? Research results and classroom experience can help teachers develop their own internal database of DG-inspired language and prepare them to capitalize on situations where such language is used.

In the next section, the study of language turns to two DG-related metaphors devised by our interviewees. It is here with metaphors that we will see how language can serve as either a barrier or an entryway into geometric constructions.

Movement-Based Metaphors

Simple and abstract, a static circle or rectangle can serve as the basis for countless metaphors. A circle might trigger an image of a Frisbee; a rectangle might resemble a door. In both instances, it is the object's shape that inspires the association.

A broader role for metaphor emerges in the world of dynamic geometry. Each object created with The Geometer's Sketchpad possesses not only a shape, but a movement associated with the dragging of its parts. Together, shape and movement may suggest metaphors not apparent through a static picture alone.

This section offers two metaphors that arose during interviewees' exploration of our black-box sketches. While both metaphors had the potential to aid in the identification and reconstruction task, only the pair of Ellen and Lynne were able to use their metaphor effectively. What factors contributed to their success? Why did our other interviewee, Norman, encounter more difficulty? The interview excerpts and analyses point to some intriguing possibilities.

Ellen and Lynne

Figure 24 shows four snapshots of our final black-box sketch. Chapter V describes this construction in detail, but in brief, point D sits on a hidden line j that is the perpendicular bisector of segment AB. Lengths AD and BD are always equal.
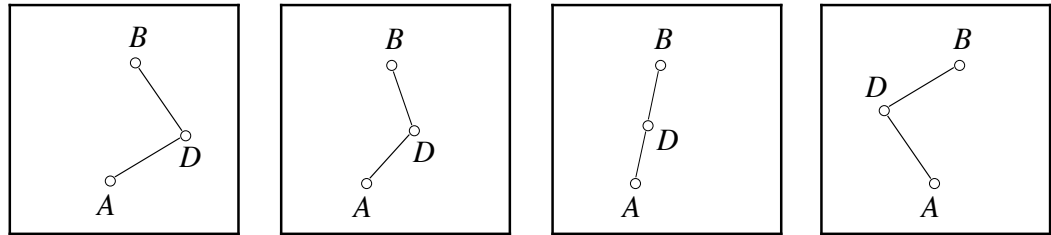

Figure 24. Four snapshots of point D moving along the hidden line j.

Lynne begins her investigation of Figure 24 by dragging points A, B, and D. Her partner, Ellen, comments on the movement of point D:

> Ellen: It's like one of those like…what do you call them? Slingshots. You throw a rubber band and then it…[*moves her hand back and then forward as if tightening and releasing a rubber band*]…When you move D, it does a little slingshot thing…
>
> Int.: What makes it look like a slingshot to you?
>
> Lynne: A slingshot? Well, OK, I was thinking about looking at a slingshot from the top…You know, how it's like a 'Y'? [*uses her hands to trace a big 'Y' in the air*]. That's the two points of the 'Y' [points A and B] from the top and point D is the rubber band.

As a metaphor for capturing the movement of point D, Ellen's choice of a slingshot is an apt one. Figure 25 shows a 'Y'-shaped slingshot with tips at points A and B and a stretched rubber band between them. The center of the rubber band,

point *D* (the contact point for a wad of gum or other projectile), does move like its onscreen Sketchpad counterpart.

Ellen's slingshot metaphor serves as conversational shorthand for what would otherwise be a lengthier geometric description. After listening to Ellen's explanation above, Lynne offers such a geometric take on the situation:

> Lynne: I think I might have an idea…I think that points *A* and *B* are on the same line…they are perpendicular to a different line that point *D* is on.
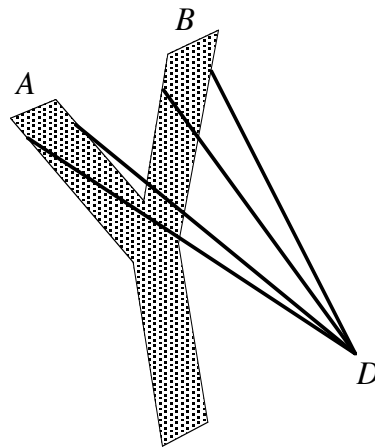

Figure 25. A slingshot model of the construction in Figure 24.

Lynne illustrates her statement by moving Sketchpad's selection arrow between points *A* and *B* to indicate the hidden segment connecting them. She then moves the arrow in the direction of the hidden perpendicular bisector, line *j* (Figure 13).

Lynne is correct; her geometric analysis will allow her to reconstruct the sketch. At this point, it would be natural enough for the interviewer to forgo Ellen's informal metaphor in favor of Lynne's more familiar approach. Instead, he challenges the pair to reconcile their two, complementary views of the situation. He addresses Lynne:

> Int.:   Describe your thing again. You [Ellen] see if you can, while she's describing it, see if you can picture your slingshot. See if the things Lynne describes fits that picture. Because I'm curious. You have one image, [Ellen] has another image. They both are clearly describing what each of you are seeing. The question is whether you're seeing the same thing.

The interviewer has asked Lynne to repeat her explanation, but she offers more, tying her geometric description to Ellen's slingshot model:

> Lynne: One of the reasons it looks most like a slingshot is because it doesn't curve everywhere. It just stays in the same straight path.

The center of a slingshot's rubber band pulls straight back and springs straight forward. Its path is not "curvy." Geometrically speaking, point *D* moves along a line.

> Lynne: When *D* is passing through *A* and *B* like a slingshot, it doesn't go closer to *A* or closer to *B*. It just stays in the center of them both…[*drags point D so that it forms one continuous segment ADB with D at the midpoint*]…It would make sense [*dragging point A as she speaks*] because everywhere, if you can imagine, the straight line there [*gesturing with her hand to indicate the hidden perpendicular bisector line j*], they [*the lines*] would always be perpendicular.

In the above quotation, Lynne explains that regardless of the location of slingshot tips *A* and *B*, the path of point *D* will always be perpendicular to segment *AB*.

The interested reader will find an extended excerpt from this interview in Appendix D.

<p style="text-align:center;">Commentary</p>

When the interviewer asks Ellen to explain her slingshot idea, it is simple enough for her to mimic a slingshot's movement with her hands. The ultimate test of her metaphor, however, comes in what might be called the "translation phase" of the task. Recreating a slingshot-like movement with Sketchpad requires Lynne to examine how a slingshot's movement can be coded into the language of geometry. Such translation work is a new skill, one not required in traditional, static geometry.

In the second interview excerpted below, Norman must also translate his informal observations into geometric terminology. He fares less well than Lynne.

<u>Norman</u>

Figure 26 shows four snapshots of our perpendicular bisector sketch. Chapter V describes this sketch in detail, but in brief, the line between *A* and *B* is the perpendicular bisector of a hidden segment connecting the two points.

Norman explores the construction by dragging point *A* with his mouse:

> Norman: [*dragging point A to point B*] That's weird. How does it do that? How does it [point *A*] keep away from the line? It's like a magnet to the line. As I get it closer [*point A closer to point B*], the line moves farther away. And if I *really* try hard, I can get [the points] to come together in the center [*drags point A onto point B*]…But how does it do that?

The interviewer asks Norman to expand upon his magnet metaphor. Norman drags point *B* as he speaks:

> Norman: …this [point *A*] seems to be the pressure point…Only [there] can the dot [point *B*] touch the line.

> Int.: So now describe the line in relation to the two dots…

> Norman: …as you push [the line] with this dot [point *B*], it moves towards the other dot [point *A*]. Any way you do it [*he illustrates that regardless of where point B begins, the line is "pushed" towards point A as point B is dragged to point A*].
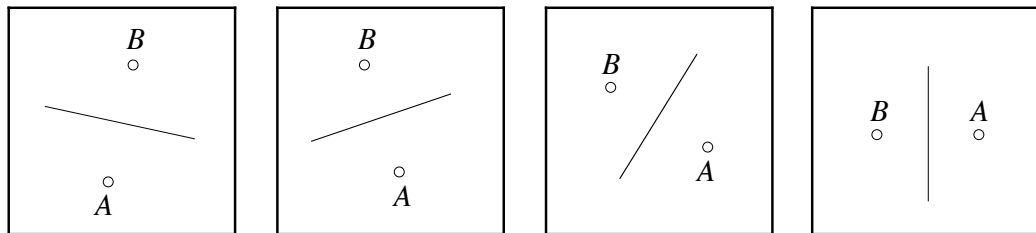

Figure 26. Four snapshots of the perpendicular bisector construction..

Later in the conversation, the question of whether Sketchpad has a command or menu item that creates magnet-like behavior arises:

> Int.: You know what? Figuring out the software is not the object of this exercise. You can ask any questions about the software.

> You tell me what it is that you want to do. You want these points to do something.

Norman: I want these points to interact with the line.

Int.: OK, in what way? Describe the relationship that you want.

Norman: Like, when the point moves nearer to the line, the line moves away and towards the other point.

Int.: There's no instruction that I know of that does exactly that.

The interested reader will find an extended excerpt of this interview in Appendix D.

<u>Commentary</u>

Norman's description of the perpendicular bisector construction is not without merits. His descriptions of the onscreen behaviors are accurate. As point *A* approaches point *B*, the line moves away from *A*. And point *A* can only touch the line when the two points, *A* and *B*, coincide.

With these observations in hand, Norman must answer an important question: what accounts for these movements? He couches his descriptions in terms of magnetism.

As one drags point *A* in the direction of point *B*, it certainly does feel as if the point "pushes" (to use Norman's word) the line away. If this black-box arrangement appeared as part of a physics simulation, then associating the pushing to a magnetic force might serve Norman well. One could imagine Norman selecting a magnetism option from the software's menu items to simulate such behavior.

But Sketchpad operates under the rules of geometry, and geometry knows nothing of pushing or magnets. There are no menu items corresponding to "pushing" and "pressure points." Norman must, as did Lynne, translate his observations from a physical model to the corresponding language of geometry. Why does it feel like point *A* pushes the line away? Because the line always passes through the midpoint of segment *AB*. See Figure 27:
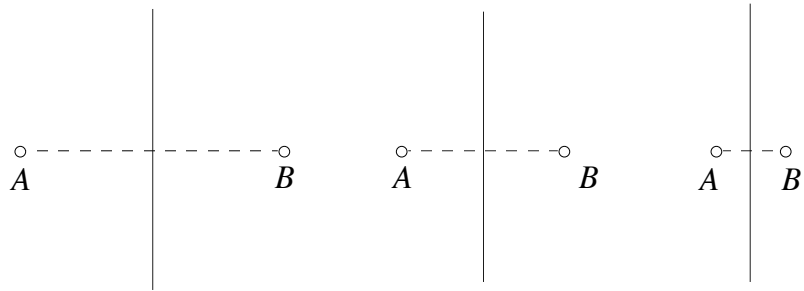
Figure 27. Three snapshots showing point *A* approaching point *B*.

Movement-inspired metaphors such as Ellen's slingshot and Norman's magnet both serve as potential entries into the process of deconstructing the geometry that lies behind black-box sketches. But users must recognize metaphors as informal, transitional tools to be reinterpreted in terms of formalized geometry. For Norman, who says that if he "*really* tries hard" he can bring points *A* and *B* together, the magnetism interpretation may be very real to him and not a metaphor at all.

CHAPTER VIII
CONSTRUCTION

With their attention to issues of language and motion, Chapters VI and VII focused on interviewees' investigative mouse dragging. Now, in this final analysis chapter, the spotlight shifts to their efforts at rebuilding black-box constructions from scratch. Specifically, it details the work of several interviewees as they assemble a Sketchpad square.

Compared to other geometric shapes, a square is a relatively simple object to construct, typically built with the aid of Sketchpad's circle tool and its Perpendicular and Parallel Line commands. The interviews excerpted below, however, offer student construction techniques that depart from standard methods. With each technique comes the same question: have the interviewees built a square? While this may sound like a simple matter to answer, it is, in fact, a thornier issue than one might suppose in the world of dynamic geometry.

What Qualifies as a Sketchpad Square?

Chapter III presented a structure for classifying constructions in a dynamic geometry setting. Recall three of Finzer and Bennett's (1995) construction categories: drawing, underconstraint, and appropriate constraints.

Figure 28 shows a square that reveals itself to be a *drawing* when points *B* and *D* are dragged. In this instance, the supposed square was created by eyeballing the location of four segments so they would appear equal in length and positioned at right angles. Without any geometric constraints built into the picture, any perturbation to the object deforms it into an arbitrary quadrilateral.
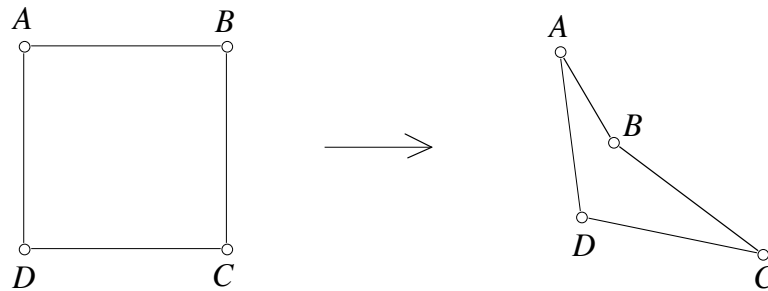

Figure 28. *ABCD* looks like a square, but isn't.

Figure 29 shows an *underconstrained* square. It has four built-in right angles, but no constraints to keep its lengths equal. Dragging vertex *B* deforms the square into a rectangle.
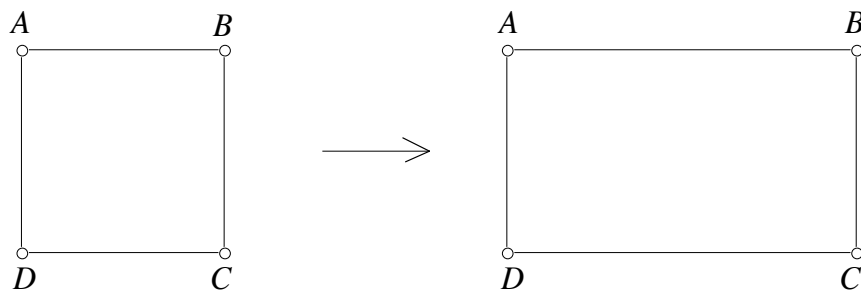

Figure 29. *ABCD* looks like a square, but is actually a rectangle.

Finally, Figure 30 shows a common method for building a bona fide square with *appropriate constraints*. Segments *DA* and *DC* are radii of a circle, ensuring the equality of *ABCD*'s lengths. Segments *AB* and *CB* are constructed perpendicular to *DA* and *DC* respectively, guaranteeing that all four of the quadrilateral's angles measure ninety degrees. With these construction features in place, *ABCD* may grow or shrink when tugged on a vertex, but will always remain square.
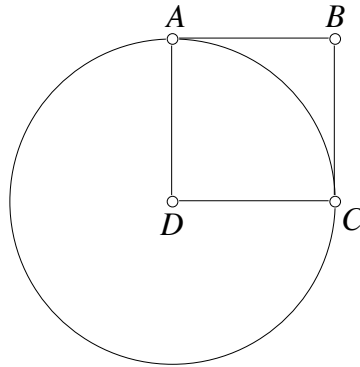
Figure 30. Constructing a Sketchpad square.

The three Finzer and Bennett categorizations outlined above function as more than pure theory. A student asked to construct a Sketchpad square must know why the options in Figure 28 and 29 do not suffice.

This chapter presents several interviewees' square-building efforts. Their methods expand upon Finzer and Bennett's construction distinctions and raise new questions concerning the definition of a DG square.

Norman's Square

When Norman begins his square construction, the interviewer tells him that he'll first need to "learn some things" about Sketchpad's menu items (such as the Perpendicular and Parallel Line commands). Undaunted, Norman assures him, "No, you can do it in a much easier way."

Norman draws a square *ABCD* by freehand, estimating the positions and lengths of the four segments. He then selects Sketchpad's circle tool and draws a circle that originates roughly in the center of *ABCD* and extends out to its four vertices (see Figure 31). Norman draws the circle so that point *A* serves as its control point.
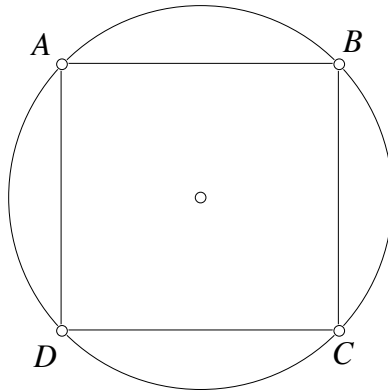
Figure 31. Norman's initial square-building method.

Norman drags point *A* and admits that his square has some problems. Since its lengths and angles were estimated rather than constructed, *ABCD* deforms into a random quadrilateral. Judged by Finzer and Bennett's classification, Norman has himself a drawing.

Yet beneath this seeming failure lies the germ of a good idea. When a circle expands, it retains its shape. If *ABCD* can be linked to the circle, it too will retain much of its shape. Because Norman drew his circle *after* forming *ABCD*, neither point *B*, *C*, nor *D* remains attached to its circumference. Norman wants all four points of *ABCD* to move in unison with the circle, so he reverses course.

Norman begins fresh by drawing a circle. Only then does he draw *ABCD*, placing each of its four vertices along the circle. Again, point *A* serves as the circle's control point.

When Norman drags point *A* of his new square randomly across the screen, the behavior of *ABCD* exhibits more regularity than his previous attempt. Now, all four points of the quadrilateral remain attached to the circle when it moves. Aside from segments *AB* and *AD*, the entire figure grows and shrinks proportionately. In other words, $BC = CD$ and $\angle BCD = 90°$ regardless of the quadrilateral's size (see Figure 32). Some careful dragging of point *A* yields more regularity still: if point *A* is dragged in a northwest direction[4], *ABCD* remains a square while expanding.

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[4] Mathematically speaking, the user must drag point *A* so that the measure of angle *DAB* remains 90 degrees.
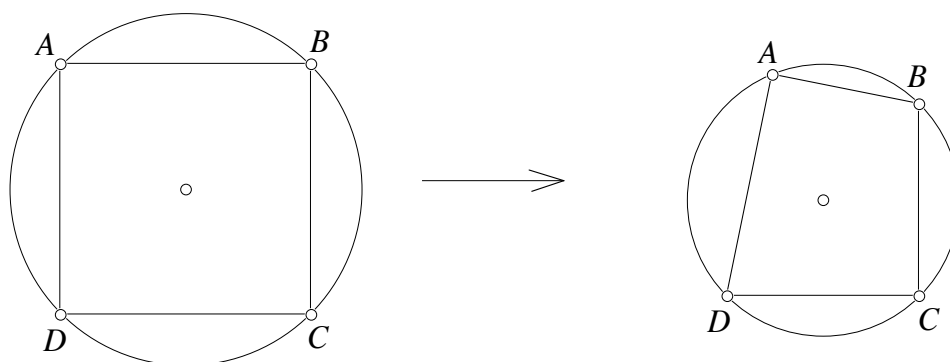
Figure 32. The result of dragging point *A*.

Commentary

Norman's square-building technique stays clear of the Perpendicular and Parallel Line construction tools. Without the use of these menu items, it is hard to imagine how anything he builds could rise above the category of drawing. Yet through the clever use of a circle, he manages to build a quadrilateral *ABCD* that grows and shrinks, all the while approximating a square, provided point *A* moves in a controlled path.

Norman's efforts fall short of an appropriately constrained object, but there is still much to admire in his work. Compared to the underconstrained square in Figure 29, his construction holds several advantages:

1. *Economy and speed*. Building Figure 29's underconstrained square demands multiple trips to Sketchpad's Perpendicular and Parallel Line commands. It also requires the user to hide construction lines and replace them by segments. By contrast, Norman's method involves no hidden lines and no menu selections. It can be completed in less than half a minute, qualifying it as a quick-and-dirty means of illustrating a multitude of squares.

2. *Mathematical integrity*. None of the construction elements built into Figure 29's underconstrained square help to keep its four sides equal in length. By contrast, Norman's method uses the symmetry of a circle to achieve four (sometimes) equal lengths.

Because Norman's construction exhibits square behavior, one might classify it as an underconstrained square. This assessment, however, leaves room for disagreement. Norman drew his square by eyeballing the correct locations for its vertices. As such, the angle measures of *ABCD* were not precisely ninety degrees, and its lengths were not perfectly equal. These inaccuracies can be corrected by measuring and adjusting the particulars of *ABCD*, but is measuring a legitimate part of the construction process? Traditionalists would likely say no.

Regardless of which side of the above debate the reader chooses to accept, the message here is more general: deciding whether an object built with Sketchpad is a square, contains some degree of "squareness," or is not a square can be surprisingly difficult and open to multiple perspectives.

David and Ben's Square

David and Ben begin their square construction by drawing a circle and placing points B and D on its circumference. They add radii $\overline{CB}$ and $\overline{CD}$ to the picture and construct perpendicular lines j and k through points B and D, respectively (see Figure 33(i))

(i)   (ii)
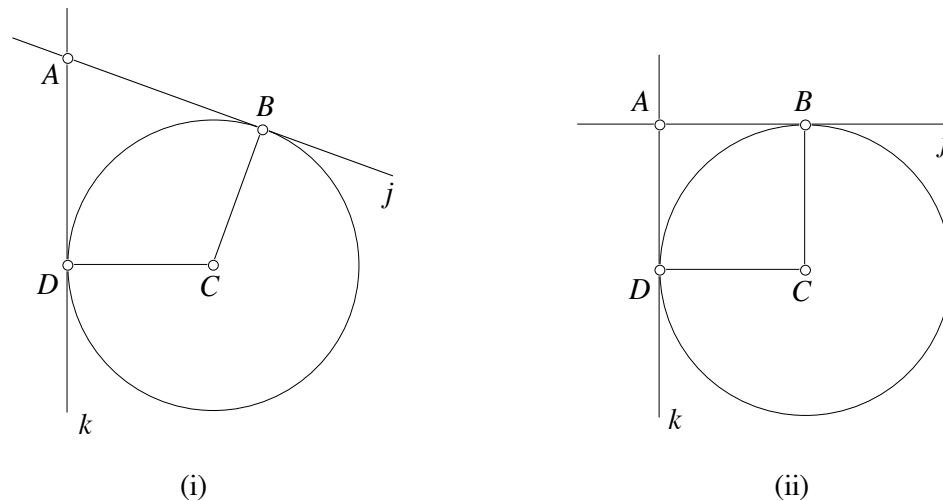
Figure 33. David and Ben's construction method.

David and Ben's construction is similar to the appropriately constrained square in Figure 30, but with one difference: whereas $\angle BAD$ in Figure 30 always measures ninety degrees, $\angle BAD$ in Figure 33 can assume a variety of measurements since points B and D move independently. Only when $\overline{CB}$ is perpendicular to $\overline{CD}$ is $ABCD$ square (Figure 33(ii)).

The interviewer questions David and Ben about $ABCD$, but soon discovers that neither interviewee intended this quadrilateral to serve as his final square:

Int.:   You're trying to build a square. What are the features you're trying to build into it?

Ben: Equal sides and ninety-degree angles, all four.

Int.:   Check this thing out by moving it in various ways. And find out whether any of the properties that you're looking for are there. For example, you do have some ninety-degree angles that stay all the time.

Ben: Well, we don't have a full square yet so we can't really tell.

David: Yeah we do. That's a full square [*points to ABCD*].

Ben: Oh yeah. I wasn't looking at that part.

David: Actually, I wasn't really thinking of that [either]. I was thinking we'd go all the way around it [the circle].

Ben: Yeah, I was too.

David and Ben plan to build a square that circumscribes (goes "all the way around") their circle. They resume their construction by placing points *E* and *F* on the circle's circumference, drawing $\overline{CE}$ and $\overline{CF}$, and constructing lines perpendicular to the two segments (see Figure 34(i)).
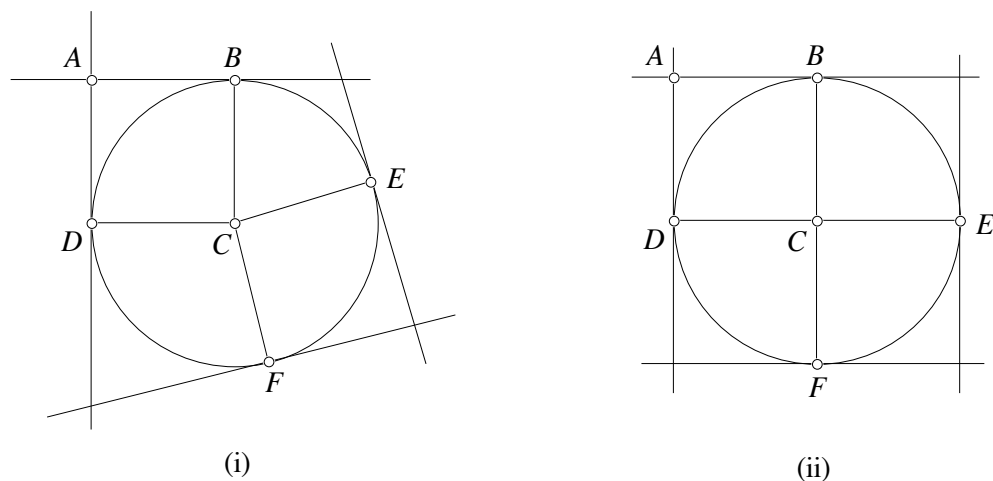

Figure 34. David and Ben continue their square construction.

By dragging points *E* and *F* to their proper locations, David and Ben are able to make their quadrilateral look like a square (see Figure 34(ii)). Of course any subsequent dragging of either *D*, *B*, *E*, or *F* will deform the square back into an arbitrary quadrilateral.

At this point, the interviewer intervenes, explaining that since segments *BF* and *DE* were drawn (as opposed to constructed) perpendicular to each other, the

illustration will never be an appropriately constrained square. Not to be deterred by this observation, David suggests a simple fix:

> David: Wait, wait, wait. I have a question. If you hide points D, A, B, E, and F, and you could only move point C, would that work?

Sure enough, when David hides all points except the circle's center (Figure 35(i)) and then drags point C, their quadrilateral grows and shrinks, always remaining a square regardless of point C's location (Figure 35(ii)). David and Ben give each other a congratulatory high-five and move on to the next black-box challenge.
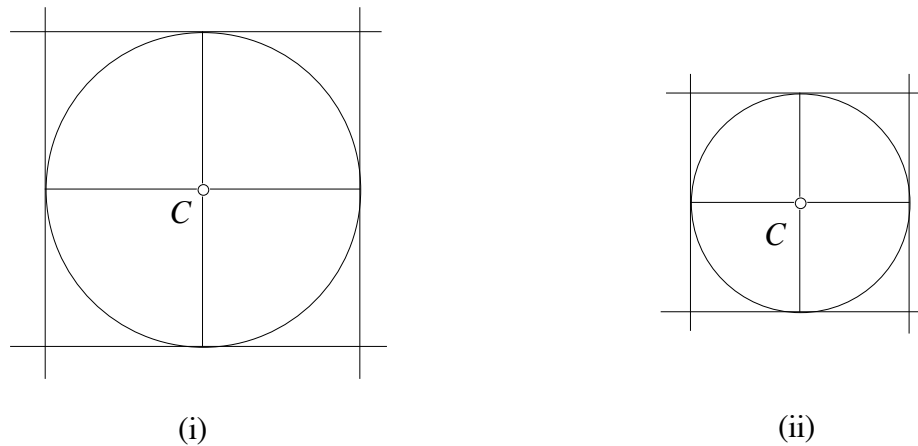

(i)   (ii)

Figure 35. Hiding every point except C yields a resizable square.

Commentary

The Finzer and Bennett (1995) categories cited throughout this chapter describe the various degrees of "squareness" that a quadrilateral can attain. From a theoretical perspective, this hierarchy seems clear enough. But in the sessions described in this chapter, these categories did not always provide a neat and tidy blueprint for the interviewer to follow.

Consider the case of David and Ben. As soon as the pair placed independent points B and D on their circle (Figure 33), I was tempted to curtail their work. My experience with appropriately constrained squares told me there was too much variability in their construction. David and Ben intended angle BCD to measure ninety degrees, but I knew a simple tug of either point B or D would alter the angle's measure.

David and Ben's solution to this problem—hiding nearly every point in their sketch (Figure 35)—struck me at first as cheating. Yes, dragging point C did grow and shrink their square perfectly well, but the hidden points B, D, E, and F could

"mess up" the shape. Was it fair to remove these points from view, thus making them inaccessible to mouse dragging?

Strictly speaking, yes. Hiding extraneous construction points is perfectly legal under the laws of Sketchpad constructions. Dragging any of the quadrilateral's vertices such as point *A does* maintain its square shape, so from this perspective, it is difficult to argue with David and Ben's technique.

But Ben made a remark prior to starting his next black-box item that caused me some concern. He asked, "Is that the only way to get it [the square]?" Throughout this dissertation, I have explained how the tools of dynamic geometry are not identical to their traditional counterparts. How one constructs a square with Sketchpad is not identical to straightedge and compass. Yet despite its differences, Sketchpad does maintain the mathematical integrity of the construction. Aside from knowing how to use such Sketchpad commands as "Perpendicular Line" and "Parallel Line," there are no special software tricks needed to build a DG square.

Sketchpad's square construction is no less mathematical than the steps required for straightedge and compass. For Ben to think that he needed to hide misbehaving points struck me as unfortunate. If David and Ben had constructed (as opposed to drawn) segments *BF* and *DE* perpendicular to each other, then there would be no need to hide points. Any point, visible or otherwise, would maintain the quadrilateral's square shape when dragged.

Sorting through the correctness of David and Ben's work is tricky business. Is their square appropriately constrained, underconstrained, or a mix of both? Or might it, like Norman's construction, qualify only as a drawing since one must measure and adjust the square's four angles to guarantee ninety-degree angle measurements at each vertex? My dissertation does not provide answers to these questions. In presenting interviewees' work, my purpose is to raise an awareness of the complexities that arise when judging the merits of a Sketchpad construction.

CHAPTER IX

CONCLUSION AND RECOMMENDATIONS

In his 1992 review of The Geometer's Sketchpad, mathematician Robert Devaney communicates an appreciation of dynamic geometry that persists among the mathematics community to this day:

> The Geometer's Sketchpad is one of the most effective pieces of software I have ever encountered. I expect that it will revolutionize the way secondary geometry is taught. Any prospective mathematics teacher, elementary, high-school, or college, should be exposed at some point in their education to this software. (p. 2)

As a long-time Sketchpad user, I, too, have experienced the satisfaction of building, animating, and investigating geometric objects on my computer screen. The interview summaries provided in this dissertation focused on students' intellectual inquiries, but they could equally well have told a story of their engagement with the software. Our interviewees expressed surprise and frequent delight with the animated images they viewed and constructed on screen. Some requested copies of their sketches so that they could share them with friends.

In the face of such enthusiastic support, it is easy to overlook questions that reveal a more nuanced appraisal of dynamic geometry. Without detracting from its obvious emotional and creative appeal, research must look past the novelty of Sketchpad's animation capabilities and ask what they contribute from an educational perspective.

Some reasonable hypotheses are easy to generate: viewing an isosceles triangle grow and shrink will help students to perceive the general features of all isosceles triangles. Building a resizable square can only be accomplished using commands like "Parallel Line."

These claims seem plausible because teachers and researchers are seasoned mathematics consumers who bring their years of static geometry experience to the world of Sketchpad. But the intuition of veteran geometers cannot compete with the information obtainable by interviewing actual students.

Below are six themes that arose from analyzing our videotape data of middle-school students exploring and recreating black-box mystery items. The remainder of this chapter interprets these findings as they relate to teachers and curriculum developers, and offers a collection of related issues for future study.

Theme One: *Students did not always draw a distinction between geometric properties specific to Sketchpad and geometric properties attributable to mathematics.*

In mathematical terms, a rectangle is a quadrilateral with four right angles. This property applies to all rectangles, whether static or "dynamic." By contrast, a Sketchpad rectangle possesses properties unique to the computer medium. Interviewees Rick and Allen (Chapter VI) noticed that each vertex of our pre-built rectangle, when dragged, behaved differently. From a mathematical perspective, such movement patterns had no bearing on the rectangle. Nonetheless, Rick and Allen viewed these properties as no less critical than the right-angle measures.

Theme Two: *When describing the properties of geometric shapes, students sometimes paid more attention to hidden construction features than to visible attributes.*

David and Ben, along with Ellen and Lynne (Chapter VI), used their mouse to explore an isosceles triangle. Such triangles possess two sides of equal length, but neither pair mentioned this visible feature. Rather, interviewees uncovered the hidden circle used to regulate the triangle's movement. This strategy, unavailable

to generations of students raised on static geometry, allowed the students to reconstruct the triangle from scratch without ever identifying it by name.

Theme Three: *When exploring pre-built objects with their mouse, students sometimes found it helpful to restrict the shapes' movement in their efforts to identify its features.*

The Geometer's Sketchpad allows students to vary the width, length, and tilt of a rectangle simultaneously. Lynne, along with Rick and Allen (Chapter VI), investigated this option, but ultimately found it more manageable to restrict the variability of the rectangle. By viewing the rectangle as its tilt and one pair of sides remained fixed, the interviewees were able to identify its geometric features.

Theme Four: *Some students described geometric shapes using metaphors that reflected how these objects moved onscreen. The success of these metaphors rested upon students' ability to shift them into geometric language.*

The sensations derived from dragging an object with the computer mouse and observing the resulting behavior led some interviewees to describe our sketches in informal, metaphorical terms. Norman classified a perpendicular bisector line as magnet-like in its movement patterns. A variation of this sketch reminded Ellen of a slingshot (Chapter VII).

The use of such metaphors recalls stage one of the van Hiele model for static geometry. Here, however, rather than classify an object by its shape (e.g., a rectangle is a door), students described objects by their behavior. How well these metaphors served our interviewees depended upon their ability to translate the metaphor into geometric terms. Lynne, for example, was able to extract from Ellen's slingshot description those geometric properties of slingshots that allowed her to reconstruct one from scratch.

Theme Five: *Students' efforts at building geometric shapes from scratch sometimes defied categorization as "right" or "wrong." The prevailing classification structure, while helpful, did not account for creative construction techniques.*

In dynamic geometry language, a square is either a drawing or a construction. More specific still, a square is either a drawing, underconstrained, overconstrained, or appropriately constrained. Theoretically, the differences among these four Finzer and Bennett (1995) categories create an unambiguous scheme for classifying students' construction attempts. In practice, these distinctions proved decidedly murky. Norman, David, and Ben (Chapter VIII) built squares without using common Sketchpad tools like "Perpendicular Line." Through clever use of a circle's symmetry, these students were able to build squares that, to varying degrees, maintained their "squareness" when dragged. Such non-traditional approaches serve as a challenge to teachers who assess the construction efforts of their students. The recommendation section below provides suggestions for addressing this issue.

<u>Theme Six:</u> *The mathematical insights that students derived from dynamic geometry software were not always those intended by the interviewers.*

This theme encompasses much of the material summarized above. In a paper-and-pencil geometry setting, students' relationship to objects is limited by the static nature of the medium. By contrast, a dynamic geometry environment allowed our interviewees to scrutinize objects with their mouse and call upon a powerful set of construction tools. With this freedom, it was perhaps natural that interviewees' responses and actions did not always yield the mathematical insights we expected. But the uncovering of a hidden circle (Chapter VI), the use of metaphors (Chapter VII), and the exploitation of a circle's symmetry (Chapter VIII) all represent unanticipated mathematical discoveries that often served our interviewees well.

<u>Recommendations for Teachers of Dynamic Geometry</u>

Based on my research findings, I have assembled recommendations that may assist teachers who use dynamic geometry software as part of their mathematics curricula.

1. As informed consumers of geometry, teachers view onscreen motion through a lens that extracts pertinent geometric information while siphoning out irrelevant, software-specific behavior. Students may have a more difficult time pinpointing salient aspects of a sketch. While an interest in point behavior may be inappropriate in one situation, it could be central to another investigation.

    Due to these concerns, teachers may wish to begin younger students in a static, rather than dynamic, realm. Displaying six different static rectangles may be preferable to one malleable rectangle. Static pictures can, of course, fall prey to the prototype effect, but typically this occurs because of space-saving concerns of textbook publishers. A well-chosen sample of six rectangles offers the variability not possible with just a single image.

2. The potential confusion arising from black-box sketches raises a question: Should teachers avoid these tasks entirely? I would suggest *any* sketch that students explore, whether self-made or teacher-constructed, contains elements of a black box. If a student draws something as simple as a segment on screen and drags it, she may know the name of the object, but only gradually come to understand how it behaves. What happens when its endpoint is dragged? How does this contrast to dragging the segment itself? A given activity may not address these questions directly, but the student will develop theories nonetheless. It is best to confront software-specific behaviors head on, rather than leave students to draw their own conclusions in silence. Black-box sketches, accompanied by class discussion, can serve this purpose.

3. Appendix E offers alternative square and isosceles triangle constructions in which the behavior of individual vertices departs from those in our interview. Teachers might assemble a sketch containing two squares or two isosceles triangles, each with different movement patterns. Students can then explore the two shapes side by side and describe those features common to both. In this manner, students can distinguish attributes of a square from features of Sketchpad.

4. The movement patterns of daily life include ceiling fans rotating at the flick of a switch and bicycle tires spinning across pavement. The exact nature of these movements may be mysterious (What mechanism drives the fan?) but their sources are identifiable: electricity fuels the fan and our feet pedal the bicycle. In the case of dynamic geometry objects, what is the source of their movement patterns? What causes two points and a line to behave (in Norman's words) like a magnet?

   For teachers, the answer is simple: geometry. Through the use of a perpendicular bisector, the points and line acquired magnet-like movements. For students, however, how odd to realize that geometry creates movement! Formal words like "perpendicular" and "bisector" acquire new significance when viewed as tools to build and control onscreen behavior. Teachers may need to emphasize this functionality to nudge students' metaphors into the domain of formal geometric language. Given a construction and its student-assigned metaphor, a teacher can ask the class to identify those geometric properties that invest the construction with its observed movement patterns. Metaphors serve as a powerful shorthand for describing onscreen activity, provided their connection to geometry receives attention.

5. Assessing students' object-building skill is a delicate balancing act. Their work may not fit strict construction standards, yet still contain nuggets of mathematical insight. If a student builds a square that sometimes, but not always, maintains its squareness (and is thus "underconstrained"), should she be commended for her work? I would argue yes. While the quadrilateral lacks "appropriate constraints," it is no small achievement to elevate a square above the category of "drawing." Teachers must look freshly at each of their students' DG constructions and not adopt a lockstep appraisal method.

   If several students in a class build squares using different techniques, a teacher can use this opportunity to ask them to compare and contrast the merits of each construction. What characteristics of each object qualify it as a square? Where does each square fall short? Can the class form a consensus on how a Sketchpad square should behave? Student-led critiques may be more effective than teacher-imposed definitions.

Recommendations for Developers of Dynamic Geometry Curricula

Curriculum developers and publishers looking to move in the direction of dynamic geometry would do well to consider the adoption road forged by graphing calculator proponents in the 1990s. In the early days of the graphing calculator, textbooks added special starred sections that provided applications and problems to be solved with these handheld devices. In this manner, those teachers who chose not to introduce the new technology could simply skip the supplemental material. Eventually, graphing calculators became so commonplace and accepted that their use became integrated throughout entire textbooks.

The gradual introduction of graphing calculators into textbooks illustrates that the adoption of a new technology need not be an all-or-nothing affair. Curriculum developers and publishers have several options open to them:

1. *Incorporating minimal DG software use into existing texts*. Traditional geometry textbooks define objects in traditional ways. A rectangle is any quadrilateral whose angles measure ninety degrees—there is no mention of dragging, underconstraint, or appropriate constraints. For those publishers who wish to test the waters with a nominal degree of dynamic geometry interspersed through their existing texts, the best advice might be to remain in traditional territory. In particular, this could mean providing students with a sketch displaying a collection of rectangles, all with different orientations and sizes. Rather than asking students to drag these shapes and resize them, a text might instead restrict students to using Sketchpad's measurement tools to answering the question, "Do the given quadrilaterals satisfy the (traditional) definition of a rectangle?" This idea ignores the power of Sketchpad's motion features, but it does allow a publisher to modify a text with a minimum of effort and without introducing new dragging-related terminology.

2. *Rewriting existing texts to reflect the definitions and capabilities of DG software*. Publishers who wish to make dynamic geometry a more central component of their texts will need to focus attention on several DG-related issues. From a terminology standpoint, texts must adapt their definitions of geometric objects to reflect the nature of dragging. Connected Geometry (Everyday Learning, 2000), for example, introduces the informal words "MessUpAble" and "UnMessUpAble" to emphasize that an object such as a rectangle must maintain its ninety-degree angle measurements when dragged. Also, given what is already a growing diversity of DG software programs, publishers must either align themselves with a particular program, or provide more generic directions that can be applied to any software package. Finally, the teacher editions that accompany these texts should document some DG-related learning issues of the type discussed in this dissertation.

3. *Addressing the issue of DG evidence and proof*. Dynamic geometry software allows publishers to create discovery-style texts in which

students create visually compelling evidence for well-known theorems. Students who connect the midpoints of a triangle's three sides to the three opposite vertices will see that these segments meet at a single point (the centroid) regardless of how they drag and reposition the triangle's vertices. Such a demonstration is compelling and allows students to develop their own conjectures rather than rely on the text or teacher to state theorems directly. Publishers must be careful, however, that students do not accept the visual evidence gathered with DG as a substitute for formal proof.

Recommendations for Future Research Study

Given its small pool of interviewees, this study does not offer theories with guaranteed universal applicability. Rather, it uncovered themes specific to the work of eight individuals. It remains for future studies to refine these themes further. With an eye towards this work, I provide ten suggestions below for the interested dynamic geometry researcher.

1. Our student interviewees were middle schoolers with a limited geometry background. Conducting identical interviews with high-school students would help to determine whether increased mathematical maturity yields different results. Do older students, for example, pay less attention to the mobile/immobile nature of Sketchpad's points?

2. Our students had never seen Sketchpad prior to their first interview session. One might choose instead to interview experienced users of the program. Are students with extended exposure to Sketchpad more adept with the tools needed to reconstruct black-box items?

3. This dissertation focused attention on The Geometer's Sketchpad. Other dynamic geometry software operates differently. The study described here could be conducted again using Cabri Geometry to gauge the stability of the outcomes.

4. The interviews conducted in this study used Sketchpad version 3.0, first released in 1995. The newest version of Sketchpad, version 4.0 (Jackiw, 2001) allows for more construction flexibility. In Sketchpad version 3.0, a point's movement can be restricted to the circumference of a circle only if the point is built directly onto the circle. With the newer Sketchpad, a point can be placed anywhere on screen and then attached to the circle later using a "merge" command. This feature, along with other software changes, might affect the interview outcomes.

5. Students in our study received no pre-test of their geometric ability. To gain a better sense of interviewees' prior knowledge, researchers might give students a list of terms to define like "rectangle", "square", "isosceles triangle", "equilateral triangle", "perpendicular bisector" prior to their work with the software. The pre-test should also cover material beyond the scope of the

black-box items so that interviewees do not make a connection between the two tasks.

6. The students we interviewed often paid undue attention to the mobile/immobile nature of points. Dragging a vertex of our black-box square might, for example, move three of the quadrilateral's four vertices, leaving the fourth vertex stationary.
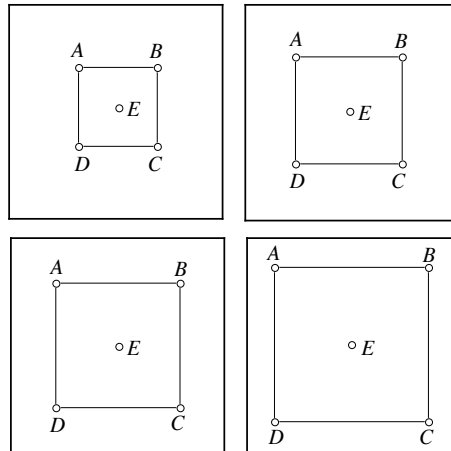

Figure 36. Square *ABCD* expands uniformly about point *E*.

Consider instead a different square with a center, *E*, that stays fixed in place as vertices *A*, *B*, *C*, and *D* move outward uniformly (see Figure 36). Might the symmetric movement of the vertices alter interviewees' perception of the shape? Appendix E contains directions for building Figure 36's square.

7. When building an isosceles triangle, our interviewees were aided by their discovery of a hidden circle. Imagine a different animation of an isosceles triangle where no point moves in a circular fashion when dragged.
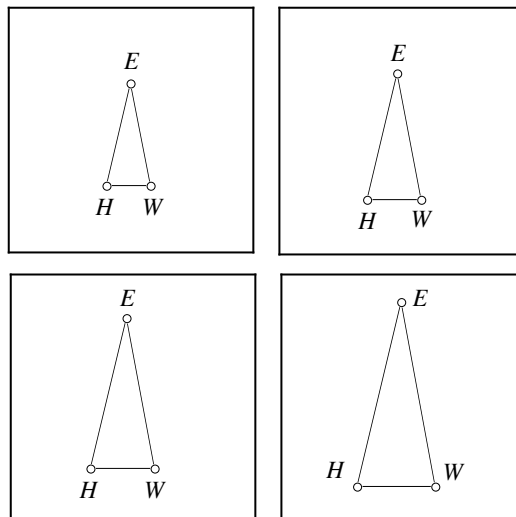

Figure 37. Four snapshots of isosceles triangle *EHW* as it expands.

Figure 37 above shows an isosceles triangle expanding uniformly, without any circular motion. Might this new movement make it more challenging for interviewees to construct the triangle? Appendix E contains directions for building Figure 37's triangle.

8. Rather than provide students with black-box animations, an interviewer might simply instruct students to build a square, rectangle, and isosceles triangle. Does the omission of the exploratory stage of the black-box task affect students' construction abilities?

9. The interviewees in our study worked solely within the computer environment of Sketchpad. Typically, however, Sketchpad is just one component of a geometry class that also includes hands-on activities. In our trial interview questions, we included a task that tested interviewees' ability to translate their work from a physical model to a Sketchpad simulation. Students received a piece of paper containing two points and were asked to fold one point onto the other. They then opened the paper and examined the resulting crease line. Moving to Sketchpad, the students drew two points on screen and constructed the same crease line geometrically. Note that this paper-folding construction is identical to our perpendicular bisector black-box item. Researchers might design more questions of this type to explore the connections between hands-on constructions and their Sketchpad counterparts.

10. As readers of this dissertation might attest, it is not always easy to follow the written descriptions of interviewees' mouse actions. A so-called "videopaper" format (Carraher, Nemirovsky, DiMattia, Lara-Meloy, & Earnest, 1999) could offer some help. Videopapers are electronic word-processing or web-based documents that contain embedded videoclips. These clips replace the role of static illustrations. A reader who wished to see the original work of interviewee Ellen, for example, could click on the appropriate videoclip and watch an excerpted portion of my videotape. Such access to the original tape not only makes it easier for readers to understand the accompanying commentary, but also allows them to detect events that might have been missed by the author.

Conclusion

   Throughout this dissertation, a single theme reappears in the contexts of motion, language, and construction. This theme can be summarized in a mathematical way: static geometry $\neq$ dynamic geometry. Static geometry is not the same as dynamic geometry. How students learn and think with a paper and pencil is not equivalent to how they function with computer screen and mouse.

Laborde (1993) reminds us of the challenges involved in introducing a new piece of software—and by extension, a new way of thinking—into the geometry classroom:

> A cognitive process does not emerge spontaneously from observation of marvelous drawings moving on the screen of a computer. It must take place in an oriented and organized environment towards learning. It means that the task of the teacher remains important as ever, the only difference being that the teacher now has a lot of powerful tools for organizing the learning environment. In one sense his/her task is more complex. (p. 60)

The complexities behind the simple statement static geometry ≠ dynamic geometry are undeniable, but DG research helps to clarify the distinctions between these decidedly different means of geometric communication.

Will the themes presented throughout these chapters resurface in future studies? It is impossible to say. But the relatively new field of dynamic geometry research needs some theories to test and ideas to debate. This dissertation serves as a springboard for further investigation and discussion.

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APPENDIX A

AN OVERVIEW OF DYNAMIC GEOMETRY

AND THE GEOMETER'S SKETCHPAD

This appendix provides a brief introduction to the terminology and features of so-called "dynamic geometry" (DG) software programs. In particular, it focuses on The Geometer's Sketchpad, the DG software used in this dissertation's interview study. There is no substitute for interacting with Sketchpad directly, but the information below conveys the essence of the program.

Dragging: A Central Aspect of Dynamic Geometry

In straightedge-and-compass geometry, the shape and size of a triangle drawn on paper remain fixed once the pencil is lifted from the page. By contrast, a triangle built with a dynamic geometry software package has more flexibility. The leftmost picture of Figure 38 shows an arbitrary triangle *ABC* as it might appear on a computer screen. With Sketchpad or any other DG program, one can click and hold point *A* with the mouse and then move the mouse across its mousepad. This action, known as "dragging," generates a continuum of onscreen triangles, all with the same base $\overline{BC}$ but with different locations of vertex *A*. The middle and rightmost diagrams of Figure 38 show the triangle as point *A* is dragged from left to right. Of course, the "snapshots" shown here cannot adequately

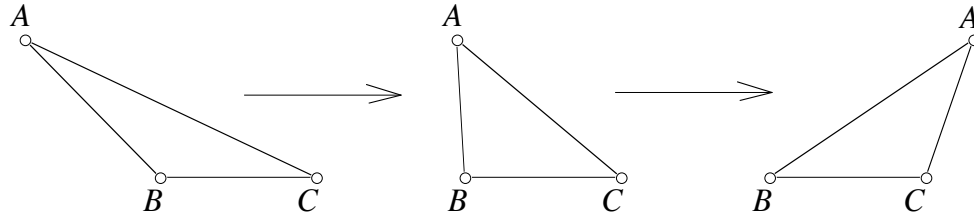capture the "dynamic" aspect of viewing the triangle reshape itself in a fluid, continuous motion.

<u>Figure 38.</u> Three snapshots of Δ*ABC* as point *A* is dragged to the right.

<u>Defining "Dynamic Geometry"</u>

Finzer and Jackiw (1998) propose three drag-related attributes as defining features of any "dynamic geometry" software program:

1. *Manipulation is direct*. When users drag point *A* of Figure 38, they do not think to themselves that they are dragging the mouse, which in turn moves point *A*. Rather, they sense that they are dragging point *A* itself.

2. *Motion is continuous*. As point *A* of Figure 38 moves, it does so without any discernible jumps or gaps in its movement. Motion flows like film animation.

3. *The environment is immersive*. The behavior of circles, squares, and other onscreen objects seems as real as their physical counterparts.

Individual dynamic geometry programs differ in terms of menu items and drawing techniques, but the features listed above are common to them all, and will most likely continue to define the software genre.

<u>Sketchpad's Tools</u>

When a Sketchpad user opens a new file, the screen appears blank, much the same as a fresh word-processing file. Whereas an author combines typing and menu commands to write and format text on a word processor, a Sketchpad user employs a mix of onscreen tools and menu commands to draw and build geometric objects. An author's completed text is a word-processing "document"; a Sketchpad user's completed work is called a "sketch."

Figure 39 shows a sketch consisting of a segment *AB*, a circle with center at point *C*, and a lone point *E*. The buttons labeled along the left side of the screen include a selection arrow along with point, circle, segment, ray, and line tools. These tools function similarly to the drawing palettes found in software illustration programs. To draw a segment, for example, the user clicks on the segment tool button and moves the cursor onto the blank screen. She then clicks the mouse button to indicate the location of one endpoint of the segment. Keeping

the mouse button depressed, she drags the mouse across the mousepad. In the process, a segment appears on screen that changes its length and direction in response to the mouse action. Releasing the mouse button sets the location of the segment's other endpoint.
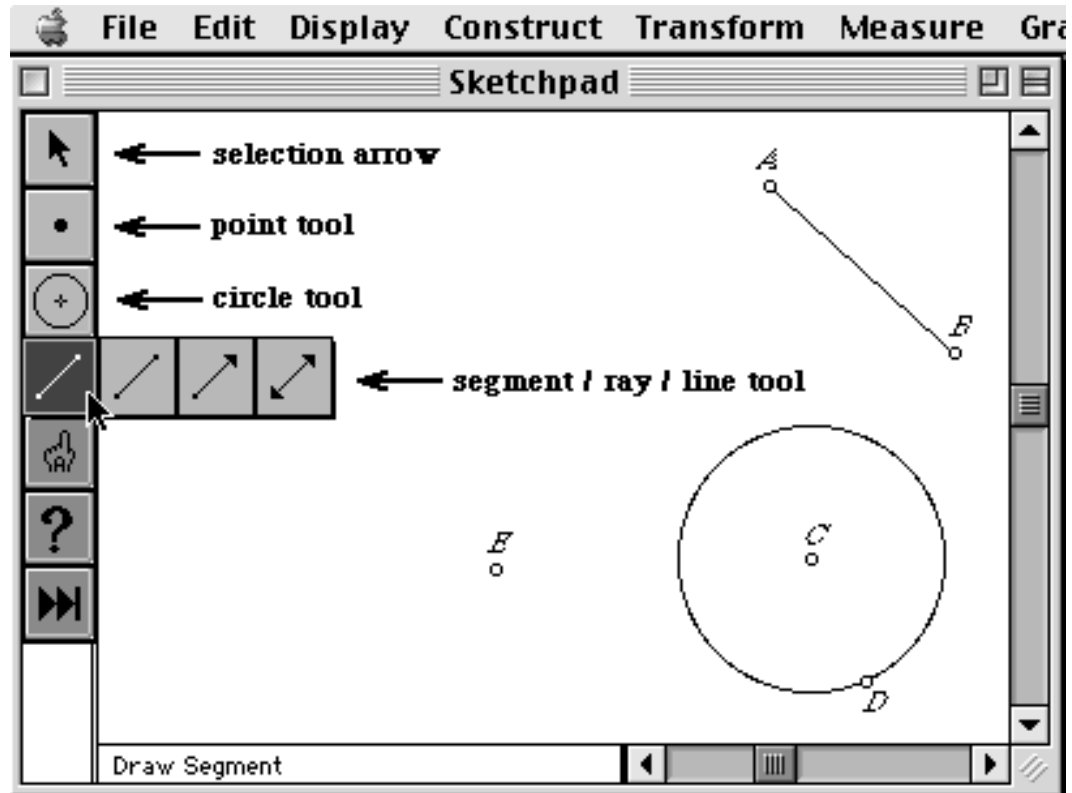

Figure 39. A typical Sketchpad window with labeled toolbar.

Altering a segment once it has been drawn can be accomplished in several ways. A user can click and drag point *A* (Figure 39), keeping point *B* fixed but changing point *A*'s location. Alternately, one can click and drag the segment itself. This action translates the segment, changing neither its length nor slope.

Just as a segment is defined by its two endpoints, the circle in Figure 39 is also defined by two points—its center, *C*, and a "control point" *D*. The same click-and-drag technique for drawing a segment applies to circles. The user clicks a location for the circle's center, drags the mouse away from the center, and releases the mouse at the desired spot for the control point. To change the size of the circle, the user can click and drag either the center or the control point, moving it across the screen while keeping the other point in place. To translate the circle across the screen without changing its size, the user clicks and drags anywhere on the circle's circumference other than its control point.

Building Geometric Relationships Between Objects

The point, segment, and circle in Figure 39 all operate independently of each other. Dragging the segment, for example, has no impact on the point or circle. It is possible, however, to use Sketchpad's tools to create more complex relationships between objects. Figure 40(i) shows a segment AB. In 40(ii), a user has selected the point tool, moved the cursor onto the segment, and clicked to create point C.

Now, when the user clicks and drags point C with the selection arrow, the point will only move along the segment (as shown in Figures 40(iii) and (iv)). It is worth noting that not all points can be made to travel strictly along a segment. If a point is placed off by itself when drawn (such as point as E in Figure 39) it cannot later be "attached" to $\overline{AB}$ by dragging it onto the segment. Such a point will remain "free" and can move anywhere on screen.
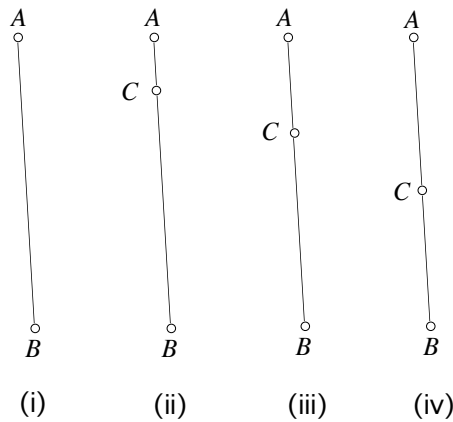

Figure 40. Point C travels only along segment AB.

Figure 41(i) shows a circle with a control point at B and three points, $C, D,$ and E, that were created on the circle itself. A user can connect points $C, D,$ and E with line segments to form a triangle CDE whose vertices lie on the circle's circumference (see Figure 41(ii)). Now, if any of the vertices are dragged, the triangle will change shape but continue to lie on the circle without changing the circle's size (see Figure 41(iii)). Note that if control point B were used as a vertex of the triangle, dragging B would not only change the shape of the triangle but also affect the size of the circle as well.

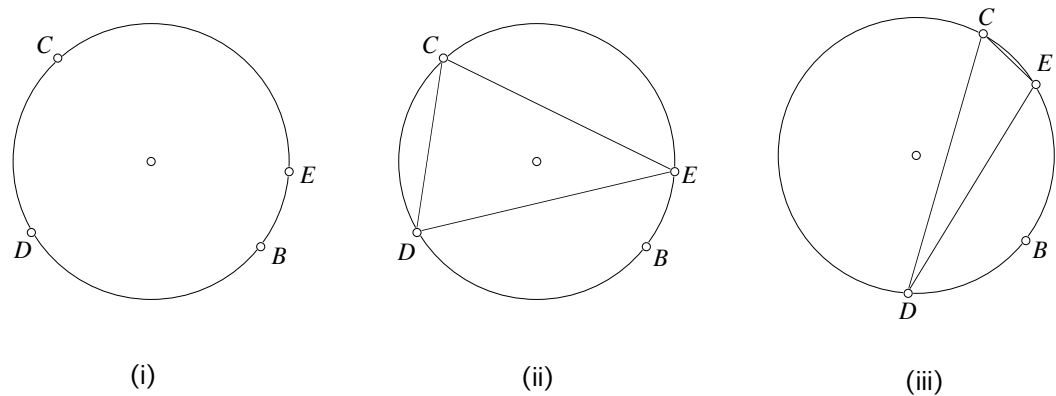(i)   (ii)   (iii)

Figure 41. Triangle *CDE's* vertices travel only along the circle's circumference.

Construction Versus Drawing

By selecting Sketchpad's segment tool, a Sketchpad user can draw easily a random triangle on screen. Choosing Sketchpad's measuring feature, the user can then measure the side lengths of her triangle and drag the vertices until the measurements indicate that two sides are equal in length. The result is an isosceles triangle. This process works through experimental refinement. If the original triangle does not have two equal sides, one tinkers with the picture until it does.

While the user's finished triangle certainly satisfies the traditional definition of isosceles, its claim to this designation is tenuous. Should one drag any of its vertices, the triangle will deform into an arbitrary triangle without two equal side lengths. Any geometric object created strictly by eyeballing, measuring, and then adjusting for the desired features is called a "drawing" (Finzer & Bennett, 1995).

By contrast, a "construction" of an isosceles triangle must satisfy a more rigorous test. Should a constructed isosceles triangle have its vertices dragged, its size or shape might change, but it will continue to remain isosceles. The circle in Figure 42(i) has its center at *C* and control point at *A*. Segments extend from *C* to point *A* and to a random point *B* created on the circle's circumference. Since segments *CA* and *CB* are both radii of the circle, their lengths are equal, making triangle *ABC* isosceles.

Dragging point *B* moves the point around the circle (Figure 42(ii)) without changing the circle's size. Since the circle's radius remains fixed in length, so do the lengths of *CA* and *CB*. Suppose, for the sake of argument, that the radius of the circle is one inch. As point *B* moves, we see all possible isosceles triangles whose equal sides are one-inch long.

By contrast, dragging control point *A* changes the size of the circle (Figure 42(iii)) and, in the process, the lengths of segments *CA* and *CB*. Since the two lengths always remain equal to each other, however, the triangle stays isosceles.
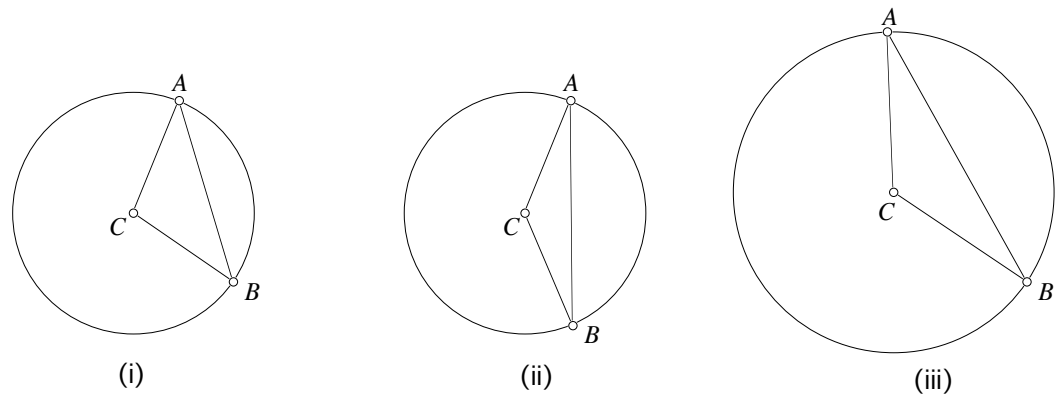
Figure 42. Three views of an isosceles triangle construction.

While the circle was necessary to build the isosceles triangle, it need not remain visible once the construction is complete. Typically, having finished a construction, a user will select Sketchpad's "hide" feature to remove from sight any of the building-block elements that unnecessarily clutter the screen. Once hidden, the circle in our isosceles triangle example continues to perform its same function—keeping segments CA and CB equal—only now out of view.

Some constructions that require multi-step procedures with straightedge and compass can be streamlined using Sketchpad's specialized menu commands. To construct the midpoint C of a segment AB for example, the user first clicks on $\overline{AB}$ with her mouse to select the segment. She then chooses "Point at Midpoint" from the "Construct" menu. Point C appears and remains at the center of $\overline{AB}$, even when the segment's length is shortened or lengthened. To construct a line perpendicular to $\overline{AB}$ through point C, the user selects $\overline{AB}$ and C with her mouse, and then chooses "Perpendicular Line" from the "Construct" menu (see Figure 43).
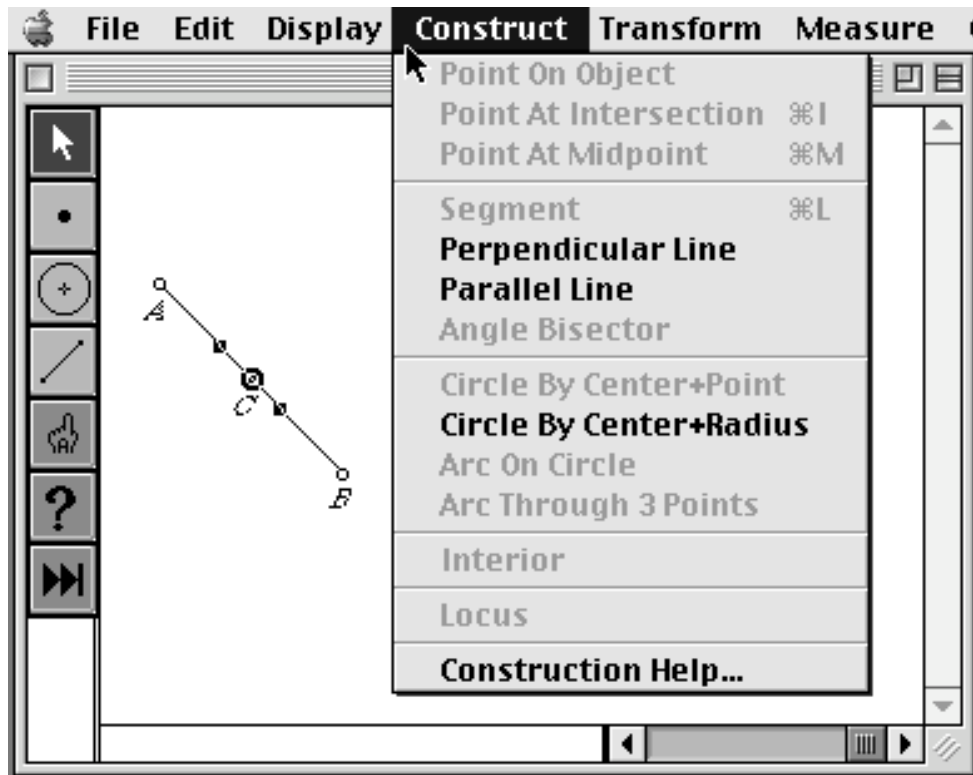
Figure 43. Constructing a perpendicular to segment *AB* through midpoint *C*.

Figure 44 shows several different orientations of the completed construction, illustrating that the perpendicular relationship is maintained when either point *A* or *B* is dragged.
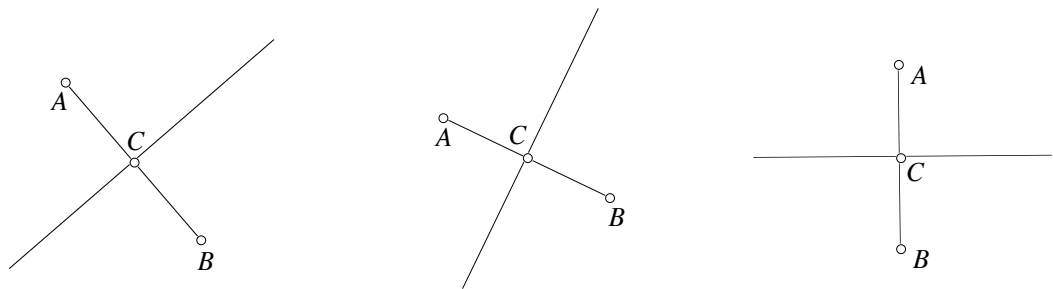

Figure 44. Three different configurations of the
perpendicular bisector construction from Figure 6.

APPENDIX B

INTERVIEWEES' INTRODUCTION TO SKETCHPAD

The items in this appendix served as interviewees' first introduction to The Geometer's Sketchpad. Interviewees received the directions and pictures reproduced below after a brief period of unstructured exploration. Throughout the session, they were free to ask the interviewer for help in interpreting the instructions and locating Sketchpad-specific features.

Part I: Getting Comfortable with Drawing Tools

1. a.   Draw some points.
    b.   Make a triangle out of line segments.
    c.   Make two circles with a line segment connecting them (place the segment's endpoints on the circles).
    d.   Make a line.
    e.   Make a ray.
    f.   Make a point that travels only along a segment.
    g.   Use the selection tool to drag the points, circles, segments, lines, and rays in the drawings above. Observe the effects.
    h.   Use Sketchpad's 'Undo' feature to remove some or all of your sketch elements.
    i.   Create a triangle with two vertices that can be moved about freely, and one vertex that can be moved only on a circle.
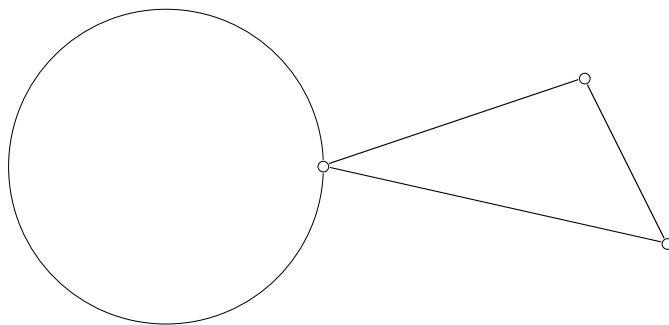
    j.   Draw two lines. Then construct a triangle with two vertices that can be moved only along one line, and one vertex that lies on the other line.
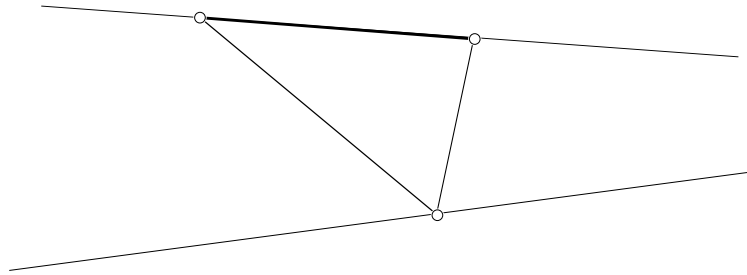
Part II: Building Geometric Constructions

2. a.   Draw a segment and construct its midpoint.
   b.   Construct two lines that are perpendicular to each other and stay
        perpendicular when either one is moved.
   c.   Construct two lines that are parallel to each other and stay parallel when
        either one is moved.

3. In this extended activity, you'll use Sketchpad to build a working model of
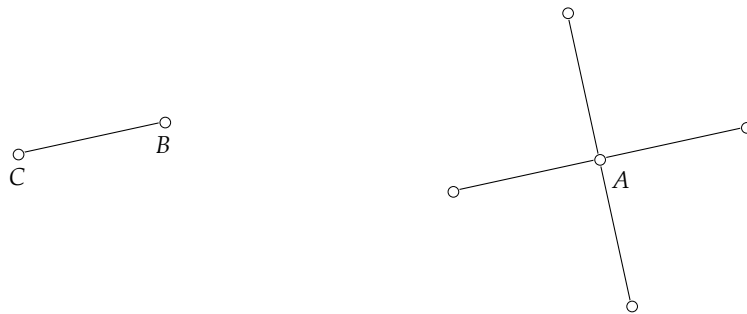a windmill.

* Open a New sketch on your computer.

* Make a point *A*, and a separate segment *BC*.

* Use the appropriate tool to *construct* a line, through point *A* and perpendicular
  to segment *BC*, that will remain perpendicular to *BC* no matter how points *A*,
  B, or *C* are moved.

* Now construct a line through point *A* and parallel to segment *BC*.

* While holding down the Shift key, click on point *A* and segment *BC* to select
  them both (click on the segment itself, not its endpoints). Now select "Circle
  By Center+Radius" from Sketchpad's Construct menu. Try dragging either
  point *B* or *C* when you're done. Stretching or shrinking the length of *BC*
  should cause the size of the circle to stretch or shrink to keep its radius equal
  to *BC*.
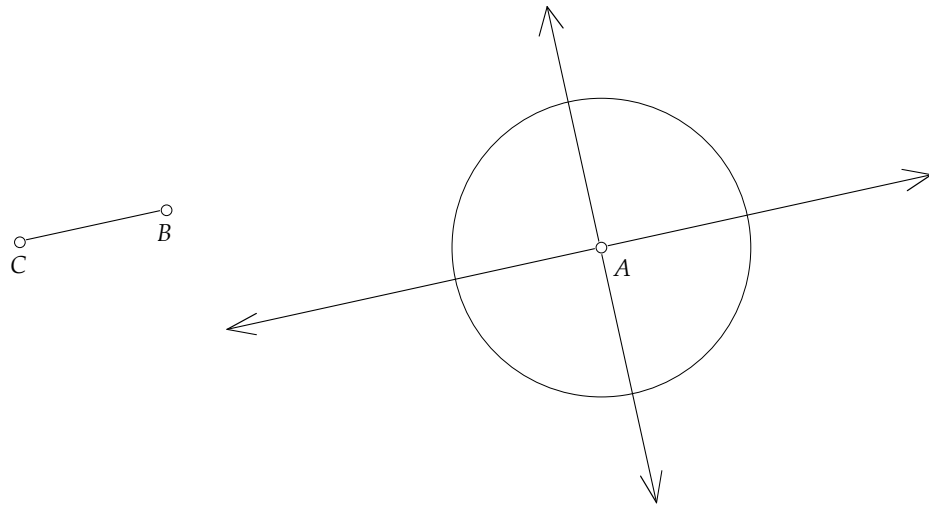
- Place points where the circle intersects the lines. Drag either point *B* or *C* to make sure these points stay at the intersections.

- Draw segments from the center of the circle to each of the four intersection points.

- Hide (don't delete) the circle and lines. Don't hide the segments. Your sketch should look like this:

- Drag point *B* and observe what happens to the segments. Now select the four points at the end of your "windmill." Go under Sketchpad's Display menu and choose "Trace Points." Drag point *B* again and observe the paths traced by the four points. Try dragging point *B* slowly and tracing your name.

APPENDIX C

THE ANALYSIS OF THE DYNAMIC
GEOMETRY INTERVIEW TAPES

The interviews conducted in our dynamic geometry study yielded 20 hours of videotape data. This appendix describes the process involved in shaping the raw footage into organized, thematic strands. Central to this task were two considerations:

1. How should the videotape data be coded into a corresponding paper format? In doing so, what level of detail should be extracted?

2. In deciding which mathematical episodes to include in this dissertation, what criteria should be used to select "noteworthy" events?

The discussion that follows addresses these areas of concern. The appendix concludes by contrasting my analysis technique to an alternative method proposed by Schoenfeld (1985).

Transferring the Interviews Into Paper Format

At the heart of any interview study are the words spoken by interviewees. Transcribing speech into written form provides the researcher with a convenient medium for subsequent analysis. If words alone, however, are the only events recorded, certain pieces of data are likely to be lost. These include pauses in speech, tone of voice, and facial expressions.

The inadequacy of word-by-word transcriptions was particularly acute in our interviews. Nearly all of the conversations between interviewer and interviewees centered on the actions occurring on a computer screen. Typical of our written transcripts was this line spoken by an interviewee: "You see, we need to move point A. You see, over here…Oh, it's X that does this…We've got this…And so far, I can move this."

Where, exactly is "here"? And what are we to make of the three references to "this"?

From the start, I knew that bringing meaning to our interview transcripts required a second level of transcription; one of a pictorial nature. As I viewed the tapes, I needed a way to capture the geometric activity that accompanied our interviewees' words.

Practically speaking, I could not draw every image that appeared on screen. Interviewees reconfigured their geometric constructions hundreds, perhaps thousands, of times during the course of their interviews, whether it was dragging

a point slightly to the left, or adding and then deleting new elements to their picture.

Film directors face a similar challenge when projecting their concept of a film beyond the words in a script. The art of storyboarding allows them to communicate visual aspects of a motion picture in a concise fashion. Each storyboard depicts a key frame of a scene, condensing the action into snapshot images.

I decided that a storyboarding approach could suit my material well. On my second viewing of the videotapes, I drew rough pictures to represent those geometric configurations that I thought best captured the actions transpiring on screen. For those pictures that required further clarification, I wrote the names of the menu and toolbar items used in the construction underneath each picture. I also added bracketed explanatory notes into the text of interviewees' words to clarify the meaning of unclear or vague expressions.

One example of my storyboarding technique appears in Chapter VII of this dissertation. In the excerpt below, Ben describes why his quadrilateral *ABCD* is not a rectangle. Appearing by themselves, the words are rather cryptic:

> Ben: Well, it's not exactly a rectangle, but if you move point *A* out, then *D* has to come out with it. If you move point *A* up, *B* has to come. If you move point *A* diagonal, then they [points *B* and *D*] have to go up and to the side.

With just three accompanying snapshots (Figure 45), however, the meaning of "out," "up," and "diagonal" becomes clearer.
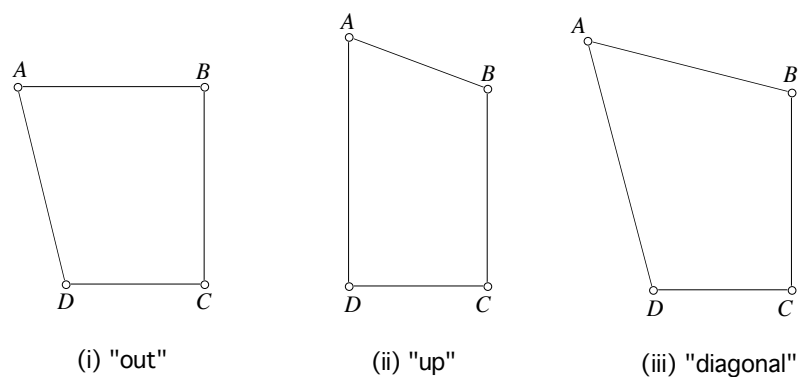

Figure 45. Three storyboard images.

Shortcomings of a Storyboard Approach

If the storyboard method was to be effective (both for myself in analyzing the data and for readers who could not view the actual tapes), I knew my transcripts and storyboards would need to stand on their own as a meaningful account of the interviews. With my hand-drawn sketches complete, I put the work aside and then returned to the data with fresh eyes.

As I read the interviewees' words and viewed my accompanying pictures, I knew that on one level, the experiment was a success—I could reconstruct the sequence of geometric actions that occurred during the course of the interview. At the same time, though, I felt that something had been lost. The exact nature of this loss remained unclear until I sat down to view the tapes again.

My storyboards presented freeze-frame views of the actions occurring onscreen. In Storyboard #1, point A might be in the top right corner of the screen. In Storyboard #2, the same point might now be in the lower left corner. How it moved from one location to another was not indicated.

In many instances, the details behind this change of location were either uninteresting or simple: point A moved from one corner to the other by being dragged in a straight line. Yet sometimes, the movement involved was more intriguing and could not be represented strictly through pictures. Just as tone of voice could be a clue to one's thoughts, so could, I realized, the speed and nature of one's mouse movements onscreen. Consider the passage below which is drawn from Chapter VI of this dissertation.

> Ben now returns to our pre-built isosceles triangle, TNJ. He begins to drag point N. His mouse movements, slow and tentative at first, cause N to rock back and forth along a small arc of the hidden circle. Point N vacillates in this manner for approximately seven times, never spanning more than half the entire circle's arc.

> David and the interviewer continue to talk while Ben stares at point N's movement. Soon Ben interjects, "OK, that's a circle!" Once Ben announces his discovery, he begins to move point N rapidly, now twirling it around the entire circle. Ben spins the point again and again, as if to confirm his statement.

Neither a picture of ΔTNJ nor Ben's quotation ("OK, that's a circle!) are enough to appreciate Ben's gradual uncovering of the hidden circle.

I decided on a case-by-case basis whether my storyboards and written transcriptions required additional descriptive commentary. If the descriptions became too long, I simply flagged the corresponding videotape excerpt as something that I would need to watch rather than annotate.

The reader can find two of my annotated interview excerpts in Appendix D.

What Counts as a Noteworthy Event?

With my descriptive coding complete, I began to consider a broad question: How did interviewees perform on our black-box tasks? The answer to this question depended on the lens I applied to the data. If I chose to focus on whether interviewees knew the technicalities of each Sketchpad command and menu item, there were plenty of minor mistakes to report. These included:

- Interviewees did not hold down the "Shift" key when trying to select more than one object on screen.

- In trying to create points that traveled along lines, interviewees mistakenly constructed the points before the lines.

From a software design perspective, such observations could be valuable (indeed the newest version of Sketchpad eliminates the need for the "Shift" key and for creating lines before points). But these findings seemed bland in light of our study's goal—to uncover those areas of students' geometric thinking that were shaped by the dynamic geometry software.

While much of our interviewees' work was routine in nature, there were excerpts of videotape—some spanning no more than a minute, others longer— where interviewees performed a certain action or gave a verbal description that clearly surprised the interviewer. These were places where interviewees' ideas did not fit either the "normal" approach or a predictable misstep (such as the bulleted items above). Confrey (1991) describes the value of finding such occurrences:

> Seldom are students' responses careless or capricious. We must seek out their systematic qualities which are typically grounded in the conceptions of the student…Frequently when students' responses deviate from our expectations, they possess the seeds of alternative approaches which can be compelling, historically supported and legitimate if we are willing to challenge our assumptions…It is at points of contact, at moments of discrepancy, that we have the highest probability of gaining insight into another person's perspective. (p. 122)

Readers of this dissertation will find that nearly all of the data excerpted throughout this work fits the general criteria below:

- The interviewees' work is unorthodox. While neglecting to hold down the "Shift" key would not be a great surprise to anyone familiar

with the software, the methods employed by our interviewees were not predictable prior to conducting the study.

- The interviewees' work is clever. This is not to say their methods were always productive (see in particular Chapter VII for Jonathan's problems with a magnet metaphor). But in all cases, interviewees approached their constructions in entirely reasonable ways.

- The geometric ways of thinking involved are specific to dynamic geometry software. Uncovering hidden paths (Chapter VI), describing objects in terms of movement metaphors (Chapter VII), and finding alternative ways to build a square (Chapter VIII) all depend on the tools made available by the software.

In choosing examples that conformed to these criteria, my challenge, working from Confrey's perspective described above, was to chronicle the unforeseen and sensible use of Sketchpad, as compared to its misuse.

This goal was aided by the style of our interviews. As interviewers with years of experience using the software, we were comfortable enough with Sketchpad so that interviewees' tiny missteps did not bother us. We gave aid where appropriate and did not dismiss ideas that others with less software knowledge might have deemed unworkable.

Developing My Interpretations

At this stage in the analysis process, I knew which interview excerpts I would develop into narratives. I envisioned each narrative as consisting of three parts. First, I would describe for the reader the "normal" manner in which a particular construction could be accomplished with Sketchpad. Then, I would offer an interviewee's method, highlighting the ways in which it differed from the norm. Finally, I would analyze the interviewee's work, attempting to uncover the merits of her reasoning.

This entire interpretive process was shaped by an admonition of Confrey (1993) similar to the one I quoted previously:

> …when interpreting data, the interviewer must demonstrate his/her own willingness to examine his/her own assumptions and challenge the validity of those assumptions…one can come to see how frequently what is labeled as student's inadequacy is really the result of our own inflexibility in considering alternative perspectives. (p. 6)

Chapter VIII of this dissertation provides an example of this approach. It would have been easy to dismiss the work of Jonathan, Marcus, and Sam as incorrect

ways of building a square. Instead, I found aspects of their work that while problematic, still displayed ingenuity.

Sharing My Interpretations

Having developed my own theories of interviewees' motivations, I needed a way to validate my findings. To gain feedback and alternative perspectives, I requested the assistance of a fellow Mathematics Education Ph.D. student in New York University's Department of Teaching and Learning.

For our first few meetings together, I did not bring videotapes. Rather, I sat with my colleague at a computer and introduced her to the features of Sketchpad. When she developed some familiarity with the software, I presented her with the various black-box items from our interviews and allowed her to explore and then recreate them on her own.

At this stage, I was ready to share the videotapes. For each excerpt, I first allowed my colleague to view the tapes without any interruption. However, in places where she needed a brief clarifying comment, I paused or rewound the tape to offer an explanation. I then assumed the role of interviewer as I asked my colleague to analyze what she saw and heard on the tapes. The questions of most interest to me in nearly all cases were, "What do you think interviewees are trying to accomplish here? What ideas are guiding their exploration/construction? Do their ideas seem reasonable?"

This interviewing process gave way to discussions in which I shared, defended, and reworked my own interpretations of the video data. Through this collaboration, I was able to provide support to my theories as well as consider new possibilities and themes suggested by my colleague.

Soliciting Feedback from Authorities in the Field

In January of 2001, I returned to my former workplace, Education Development Center, Inc. in Newton, Massachusetts, to share and discuss my dissertation draft with my former research partner E. Paul Goldenberg. I also forwarded my draft to William Finzer, another leading authority in the field of dynamic geometry curriculum and research. The feedback that I received from both individuals helped to confirm the reasonableness of my interpretations. In the case of William Finzer, for example, he noted that although my descriptions of square constructions provided a challenge to his categories of drawing, underconstraint, and appropriate constraints (Chapter VIII), he felt my analysis added some worthwhile nuances to the categories.

An Alternative Approach

One of the more detailed descriptions of interview analysis techniques appears in Schoenfeld's *Mathematical Problem Solving* (1985). It is instructive to consider his approach as a means of highlighting where (and why) I pursued a different course.

Similar to our dynamic geometry study, Schoenfeld's interviews focus on geometry, though of the paper-and-pencil variety. He, too, provides transcripts of the sessions along with illustrative pictures. But there the similarity ends.

Schoenfeld's interest lies in charting the cognitive strategies of students. For these purposes, he describes six problem-solving categories: Read, Analyze, Explore, Plan, Implement, and Verify. Each interview is "parsed" into a timeline showing which category best describes an inerviewee's behavior at any given moment. Particular attention is paid to junctures where students switch from one strategy to another.

With this coding, the larger scope and progression of a student's work assume priority over specific incidents:

> At the risk of flogging a dead horse, I wish to stress that…matters of detail…are virtually irrelevant. A coding scheme should highlight major decisions. (p. 289)

Because the coding stays general in nature, Schoenfeld maintains that the process of analyzing an interview can be remarkably standardized, even for those without graduate training. He says, "A team of undergraduate coders can be trained to parse the protocols with accuracy and reliability" (p. 315).

Schoenfeld's broad coding allows him to address issues of metacognition—specifically, "the overall quality of the students' monitoring, assessing, and executive decision making" (p. 310). Sometimes the quality can be low, leading students on "wild goose chases" or to choosing "ill-chosen approaches" (p. 282).

There is a definite appeal to Schoenfeld's analysis technique. It introduces some of the reliability found in quantitative methods into qualitative work. It does not force us to reconsider our own methods of solving mathematical problems.
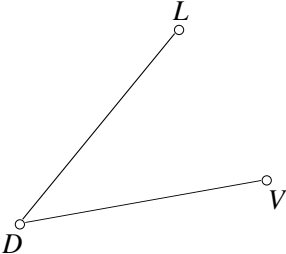
I accept that Schoenfeld's method may yield more uniformity than my approach. But I also maintain that this technique overlooks as much as it finds. I believe that small incidents *do* matter: perhaps Jonathan's magnet description of perpendicularity in Chapter VII is a "wild goose chase," but it's a delightful one because his use of a movement metaphor tells us something about the software we did not know before.
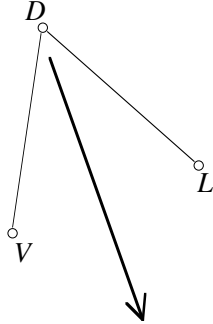
APPENDIX D

TWO ANNOTATED INTERVIEW EXCERPTS

The two extended interview excerpts below are both drawn from Chapter VII of this dissertation. Each focuses on the theme of metaphor. In the first excerpt, Lynne connects her mathematical description of a construction to her partner Ellen's slingshot metaphor. In the second excerpt, Norman analyzes a perpendicular bisector construction that he finds magnet-like in its behavior.
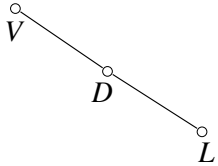

Extended Excerpt One

	122.	INT	So here is another one [*opens a black box sketch*].
"D gets smaller" = D moves closer to L and V [*interesting…Lynne says the point itself gets smaller—it doesn't*]. "it gets big" = the lengths of LD and VD get longer while D moves away from L and V.	123.	LYNNE	Should we experiment? If you move V closer into the center [*pt. L*], it gets smaller, when you move it outwards, the D stretches more out, but L stays put. And when you move L into V, D gets smaller. When you move it outwards again, it gets big. When you move D, it just—
Ellen motions with her hand showing the motion of a rubber band as it's pulled back and then released. *METAPHOR!*	124.	ELLEN	It's one of those like…what do you call them? Slingshots. You throw a rubber band and then it—
	125.	LYNNE	That's cool [*drags L*]. I think that…Here, would you [*Ellen*] like to play with it? I have a hypothesis. I like that

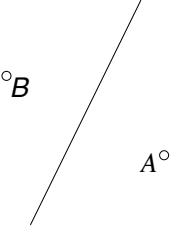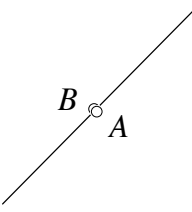|			
			word now. Ever since yesterday.
	126.	ELLEN	This looks cool.
"everything goes around L" = L stays in place while D and V move.	127.	LYNNE	Well, when you move V, everything goes around L. Ah, where is it?…and when you move L, everything goes around V. And when you move D, it does a little slingshot thing.
"D and V might be on a line segment" = when D and V are far apart, it doesn't appear that the distance between them is changing (it is—it's just hard to see). But when L and V get closer together, Lynne realizes the distance between D and V isn't constant. [*Again, another example of informal language. It's also not always easy to judge whether certain segments are constant or variable in length just by eyeing them.*]	128.	ELLEN	Yes. I think that maybe, actually …well, I think…I think that maybe…that's strange. Well, I was about to say that D and V might be on a line segment. When they go like this, they don't move apart. But when you move them in and they go closer. So it is not true.
	129.	INT	Wait, say more about— you said you tried something. You said, "Ooh, strange." But that meant it wasn't like what you were thinking. But I didn't know what you were thinking.
Lynne demonstrates that the distance between D and V isn't always the same.	130.	LYNNE	I was about to say—I don't believe this is correct—but I was about to say that I just —when I was going like that, I didn't really see the width between D and L changing. I was about to say they were on a line segment. But they do change when you do move it in and out. So I don't think that

			is right. But…I think…
	131.	INT	What makes it look like a slingshot to you?
	132.	ELLEN	To me?
	133.	INT	Yeah. That is way you described it. What about it is—
When Ellen says, "..and then when you throw it," she yanks point *D* through *L* and *Z*, showing the path of a projectile as it's released by her "slingshot." She then pulls *D* back to illustrate her statement "…and then comes back to you."	134.	ELLEN	A slingshot? Well, OK, I was thinking of looking at a slingshot from the top. From the upper point. And these are the two. You know how it's like a 'Y'? [*uses her hands to trace a big 'Y' in the air*]. That's the two points of the 'Y' from the top [*points to L and V onscreen*] and this is the rubber band [*points to D*]. So when you like throw it and you pull it back—and this would be it pulled back and then when you throw it—it goes forward and then comes back to you. Like a yo-yo or something.
	135.	LYNNE	I think I might have an idea. I think *L* and—
	136.	INT	Lynne is in the middle of something. I don't want to interrupt that. But I want to follow up that idea later on. So we'll come back to that regardless of the outcome of this experiment.
"And that they are perpendicular to a different line that *D* is on" = Lynne moves the selection arrow back and forth between *L* and *V* to indicate a	137.	LYNNE	OK, well I think that *L* and *V* are on the same…are on the same line. And that they are

hidden segment. She then uses the arrow to indicate a hidden line passing through point *D*, perpendicular to *LV*. "So that when you move…when you move *V* up…" = Lynne shows that when *V* is in a different position, the same perpendicular relationship still holds.			perpendicular to a different line that *D* is on. So that when you move…when you move *V* up, the line would be going across through *L* and *V*. And then if you moved *D* like that, another line would be going through point *D* and they would be perpendicular.
	138.	INT	Would you want to try making a construction like that somewhere to see whether that works that way? And first of all, by the way, does that make sense to you [*Ellen*]?
	139.	ELLEN	Well I got some of it. There is going to be a perpendicular line like that and like that and then there will be one and then another against each other sort of?
	140.	INT	I was wondering whether that… Yes, I think that is what she said. But I was actually wondering whether that fit—whether that matched with your picture of the slingshot. As she was describing that, can you fit that to the picture that you already had?
	141.	ELLEN	Um…I think so. I mean I don't really know how it's like that.
	142.	INT	Describe your thing again [*to Lynne*]. You see if you can, while she

			is describing what she is describing, you see if you can picture your slingshot. And see if the things that she describes fits that picture. Because I'm curious. You have one image, you have another image. They both are clearly describing what each of you are seeing. The question is whether you are seeing the same thing.
When Lynne says, "I mean, when *D* is passing through *L* and *V* like a slingshot, it doesn't go closer to *L* or closer to *V*. It just stays in the center of them both" she positions *D* like this: 	143.	LYNNE	Okay, well I think…It looks…one of the reasons it looks most like a slingshot is because it doesn't curve everywhere. It just stays in the same straight path, *D* does, when it is going through *L* and *V* [*drags D back and forth*]. I mean, when *D* is passing through *L* and *V* like a slingshot, it doesn't go closer to *L* or closer to *V*. It just stays in the center of them both.
Lynne shows that regardless of the location of *L* and *V*, the path of point *D* is always perpendicular to the hidden segment *LV*.		LYNNE	So I think that it would make sense if *D* was on a straight line going through *L* and *V*. And *L* and *V*…and if a straight line was going through *D*, and a straight line was going through *L* and *V*, it would make sense because everywhere, if you could imagine the straight lines there,

			they'd always be perpendicular. And so could I try constructing it?
	144.	INT	Sure. Do you see the slingshot in her description? [*Ellen nods*] Yeah. So near as you can tell anyway, until she constructs it, it fits your picture…Sure, give it a construction.

Extended Excerpt Two

METAPHOR! "How does it keep away from the line?" = when Norman drags *A* away from *B*, the line follows *A*, but does not ever reach it.	6.	NRMN	That's weird. How does it do that? How does it [*pt. A*] keep away from the line? It's like a magnet to the line.
"As I get it closer…" = when Norman drags *A* towards *B*, the line moves away from *A*, closer to *B*. Notice how Norman emphasizes the word "really"—he seems invested in		NRMN	As I get it [*pt. A*] closer, the line moves farther away. And if I *really* try hard, I can get them to come together in the center [*drags pt. A onto pt. B*]…But how does it do that?

the magnet metaphor.			
	7.	INT	Well, describe the 'that' again. What's what? You're getting the—
"why won't the dot just go right to the line.." = point *A* causes the line to move; the line doesn't stay still allowing the "dot" to reach it. The two are connected and move in unison. informal language: point = "dot"	8.	NRMN	Well, how does it—why won't the dot just go right to the line rather than take a while to— why does the line move back? [*when pt. A is dragged away from pt. B*]
	9.	INT	Aha, what is the line doing as you're moving this back?
	10.	NRMN	Yeah.
	11.	INT	What is the line doing as you're moving it back?
	12.	NRMN	It's moving away from the dot.
	13.	INT	Is that true for both dots?
Norman now drags point *B* and shows that it, too, causes the same behavior as point *A*. Note that point *A* is now stationary.	14.	NRMN	Yeah. See, look. It moves away.
	15.	INT	What if you move the dot away from the line?
Norman shows that if he drags point *B*, then point *A* (the "other dot") is a "pressure point"—i.e., only when point *B* rests directly on top of point *A* do the points and the line all coincide.	16.	NRMN	The line follows it. And, like, this [pt. *A*] seems to be the pressure point where the other— where the other dot is, it seems to be the place where the dot— you can't touch the line, but only there can the dot touch the line; only where the other dot is can the dot touch the line.
	17.	INT	So now describe the line in relation to the two

			dots. You've been describing it in relation to one. How does it fit in relation to two of them?
Norman shows that regardless of where point *B* is located in relation to point *A*, the effects of dragging it towards or away from point *A* are the same. "Pushing" = as point *B* approaches point *A*, the line is pushed towards point *A*. "Pulling" = as point *B* moves away from point *A*, the line is pulled away from point *A*, too. Norman sees the symmetry of the construction. Interesting use of the words "push" and "pull" here. The movements of the line make the mouse actions feel like pushing and pulling as opposed to just "dragging." The magnet metaphor remains strong.	18.	NRMN	It—oh, wait, wait. Let me see something. The line—as you push it with this dot, it moves towards the other dot. Any way you do it. If you push this way…Oh no, if you push the line with the dot it moves towards the other dot. See, look: I'm over here, and I'm pushing—moving the mouse that way, and it meets right there. But now I'm pulling…I'm moving it the other way and it comes apart.
	19.	INT	And what affects the line's direction? So you're now talking about the line's distance, right, from the dot?
	20.	NRMN	And this dot affects the line's direction…But I want to know how you do it.
	21.	INT	Okay, hang on just a second. Which dot affects the line's direction? Explain once more.
Whichever point is being dragged determines the direction of the line.	22.	NRMN	The dot that I'm not holding onto. No, the

		dot that I am holding onto.

APPENDIX E

CONSTRUCTION VARIATIONS

Two recommendations in Chapter IX suggest testing alternative constructions of the square and isosceles triangle. Below are step-by-step instructions for building these objects.

<u>The Square</u> (see Figure 46)

1. Draw a circle with center, A, and control point B.
2. Construct a diameter BC through points A and B.
3. Construct a diameter DE perpendicular to $\overline{BC}$ through point A.
4. Connect C, E, B, and D to form a square.
5. Hide point A, the two diameters, and the circle.
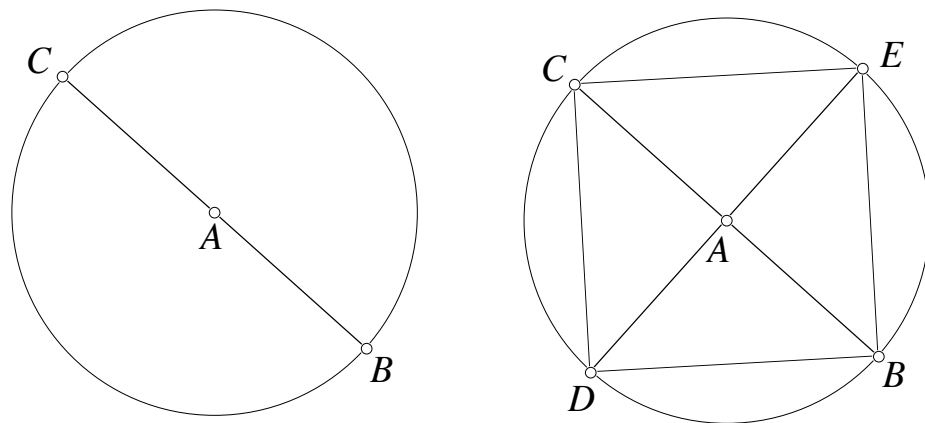6. Drag point B to expand and shrink the square.

<u>Figure 46.</u> Constructing square *CEBD*.

<u>The Isosceles Triangle</u> (see Figure 47)

1. Draw a circle with center, *A*, and control point *B*.
2. Place a random point *C* on the circle's circumference. Connect *A*, *B*, and *C* to form isosceles triangle *ABC*.
3. Hide the circle.
4. Place a random point *D* on $\overline{AB}$. Construct segment *DE* parallel to $\overline{BC}$.
5. Hide $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Connect *A* to *D* and *A* to *E* to form isosceles triangle *ADE*.
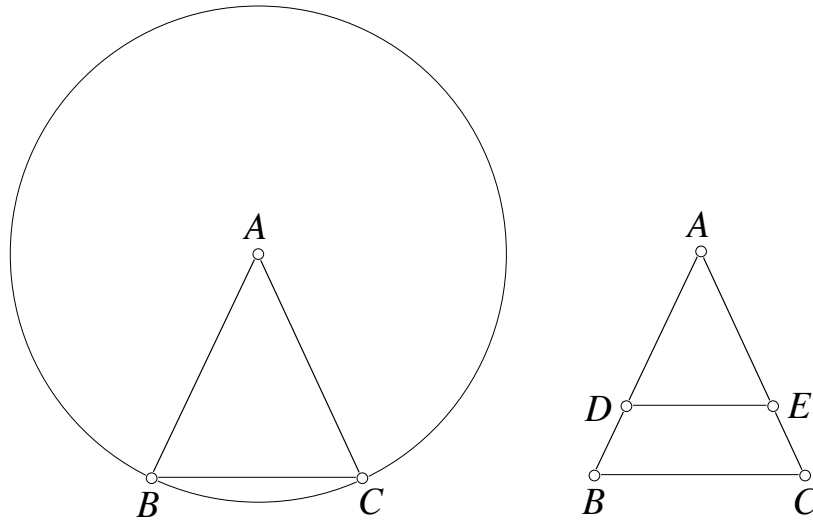6. Drag point *A* or *D* to expand and shrink the triangle.

Figure 47. Constructing isosceles triangle *ADE*.