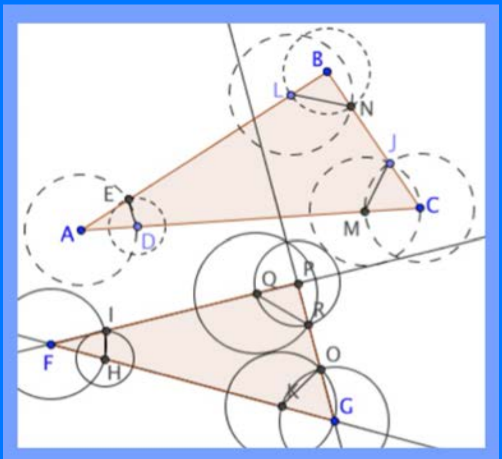
Gerry Stahl's assembled texts volume #14

Adventures in Dynamic Geometry



Gerry Stahl

Gerry Stahl's Assembled Texts

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Adventures In Dynamic Geometry

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2015

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Introduction

The Virtual Math Teams (VMT) Project has developed a collaboration environment and integrated a powerful dynamic mathematics application into it, namely the open-source GeoGebra, which integrates geometry, algebra and other forms of math in a dynamic computational environment. The project made the incorporated GeoGebra multi-user, so that small groups of students can share their mathematical explorations and co-construct geometric figures online. In support of teacher and student use of this collaboration environment, we have developed several versions of a set of activities to systematically introduce people to dynamic geometry, including core concepts from Euclid, standard geometry textbooks and the Common Core Standards for Geometry.

The topics for VMT with GeoGebra are available free for download in several versions. They all include GeoGebra tasks to work on collaboratively and tutorials on the use of VMT and GeoGebra software. The best version is the active GeoGebraBook version at http://ggbtu.be/b140867. Here, you can try out all the activities yourself. (VMT-mobile allows you to do those same activities collaboratively with chat in persistent online rooms.)

The dynamic geometry activities were developed through a series of versions, which were tested and then revised based on the testing. Here are the major historic versions:

1. The original version was developed for use with teachers in Fall 2012. Their students worked on the first 5 activities in Spring 2013. It is a workbook with 21 topics for collaborative dynamic-geometry sessions. It includes Tours (tutorials) interspersed with the Activities. The Tours refer to the VMT software environment including GeoGebra (but not the VMT-mobile version developed in 2015). This was the version used in WinterFest 2013 and analyzed in *Translating Euclid* and in *Constructing Dynamic Triangles Together*.

Stahl, G. (2012). *Dynamic-geometry activities with GeoGebra for virtual math teams*. Web: http://GerryStahl.net/elibrary/topics/activities.pdf.

2. The next version was developed for use with teachers in Fall 2013. It includes 18 topics, including a number of advanced, open-ended investigations, as well as 10 tutorials or appendixes. The 18 topics involve 79 GeoGebra tabs, which are shared figures or tasks for groups to work on using GeoGebra. Teams of math teachers worked on many of these.

Stahl, G. (2013). *Topics in dynamic geometry for virtual math teams*. Web: http://GerryStahl.net/elibrary/topics/geometry.pdf.

3. The third version was developed for use by teachers with their students in Spring 2014. It is more tightly focused on the notion of designing dependencies into geometric constructions. It is envisioned for a series of 10-12 hour-long online sessions of groups of 3-5 students who may not have yet studied geometry. It is especially appropriate for an after-school math club. It includes 12 topics with a total of 34 GeoGebra tabs, plus 6 tutorials or appendixes. The introduction from this version is included below.

Stahl, G. (2014). *Explore dynamic geometry together*. Web: http://GerryStahl.net/elibrary/topics/explore.pdf.

4. The fourth version was developed for use by teachers in Fall 2014. It is focused on identifying and constructing dependencies in geometric figures. It includes a total of 50 GeoGebra activities, divided into 13 topics. The first and last topic are for individual work. Six topics form a core introductory sequence and five topics are optional for selection by the teacher.

Stahl, G. (2014). *Construct dynamic geometry together*. Web: http://GerryStahl.net/elibrary/topics/construct.pdf.
This pdf version corresponds to the GeoGebraBook at http://ggbtu.be/b140867

5. The fifth version was developed for use by teams of students in Spring 2015. It takes a game approach. It includes a similar sequence of 50 GeoGebra challenges, divided into 13 topics. It is still focused on identifying and constructing dependencies in geometric figures.

Stahl, G. (2015). *The construction crew game*. Web: http://GerryStahl.net/elibrary/topics/game.pdf.
This pdf version corresponds to the GeoGebraBook at http://ggbtu.be/b154045.

In this volume, four of the preceding versions are reproduced, with the most recent first:

1. *The construction crew game*. This is available as an active GeoGebraBook on GeoGebraTube. It presents the topics as challenges in a video-game-based style. http://ggbtu.be/b154045
2. *Explore dynamic geometry together*. This version was based on the analysis and recommendations in *Translating Euclid* and in *Constructing Dynamic Triangles*

Together. It included the tutorials as an appendix. (The tutorials have been eliminated from this volume to save space.)

3. *Topics in dynamic geometry for virtual math teams*. This version had several advanced topics. It included the tutorials as an appendix. (The tutorials have been eliminated from this volume to save space.)

4. *Dynamic-geometry activities with GeoGebra for virtual math teams*. This was the first major curriculum version. It was used in WinterFest 2013. It included tutorials in VMT and GeoGebra, interspersed with the activities.

Contents

1. The Construction Crew Game

Welcome to the Construction Zone

The Construction Crew Game is a series of challenges for your team to construct interesting and fun geometric figures. Many of the figures will have hidden features and your team will learn how to design them. So put together your Construction Crew with three, four or five people from anywhere in the world who want to play the game together online.[1]

The Construction Crew Game consists of several *levels* of play, each with a set of *challenges* to do together in your special online construction zone. The challenges in the beginning levels do not require any previous knowledge about geometry or skill in working together. Playing the challenges in the order they are given will prepare you with everything you need to know for the more advanced levels. *Be creative and have fun*. See if you can invent new ways to do the challenges.

Try each challenge at your level until everyone in your crew understands how to meet the challenges. Then move on to the next level. Take your time until everyone has mastered the level. Then agree as a team to go to the next level. Most levels assume that everyone has mastered the previous level. The levels become harder and harder – see how far your team can go.

Geometry has always been about *constructing dependencies* into geometric figures and discovering *relationships* that are therefore *necessarily* true and provable. Dynamic geometry (like GeoGebra) makes the construction of dependencies clear. The game challenges at each level will help you to think about geometry this way and to design constructions with the necessary dependencies. The sequence of levels is designed to give you the knowledge and skills you need to think about dynamic-geometric dependencies and to construct figures with them.

[1] The challenges in the Construction Crew Game are designed for a crew of people playing together. You can read the challenges in this document (http://GerryStahl.net/elibrary/topics/game.pdf) or GeoGebraBook (http://ggbtu.be/b154045), and explore the example figures yourself in GeoGebra (http://geogebra.org). However, to work on a challenge, you should meet your construction crew in a Virtual Math Teams (VMT) chat room: http://vmt.mathforum.org. This requires a computer or laptop. A version of VMT-mobile for iPads, tablets and other mobile devices is also available.

Your "construction crew" can accomplish more than any one of you could on your own. You can chat about what you are doing, and why. You can discuss what you notice and wonder about the dynamic figures. Playing as part of a team will prevent you from becoming stuck. If you do not understand a geometry word or a challenge description, someone else in the team may have a suggestion. If you cannot figure out the next step in a problem or a construction, discuss it with your team. Decide how to proceed as a team.

Enjoy playing, exploring, discussing and constructing!

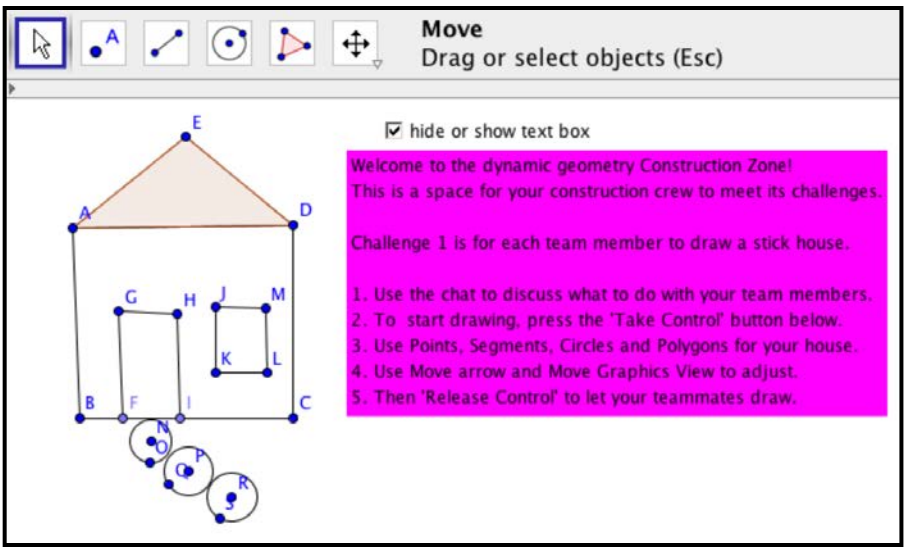
1. Beginner Level

The first three challenges are to construct and play around with points, lines, circles and triangles that are connected to each other in different ways.

Play House

Each person on your construction crew should build a house with the tools available in the tool bar. Chat about how to use the tools. Press the "Take Control" button at the bottom, then select the "Move Tool" arrow at the left end of the Tool Bar near the top. Touch or click on a point in the geometry figure to drag it.

Next, see who can build the coolest house. Build your own house using the buttons in the Tool Bar. Create points, lines, circles and triangles. Drag them around to make a drawing of a house. To create a triangle, use the polygon tool and click on three points to define the triangle and then click again on the first point to complete the triangle. Play around with the parts of your house to see how connected lines behave.

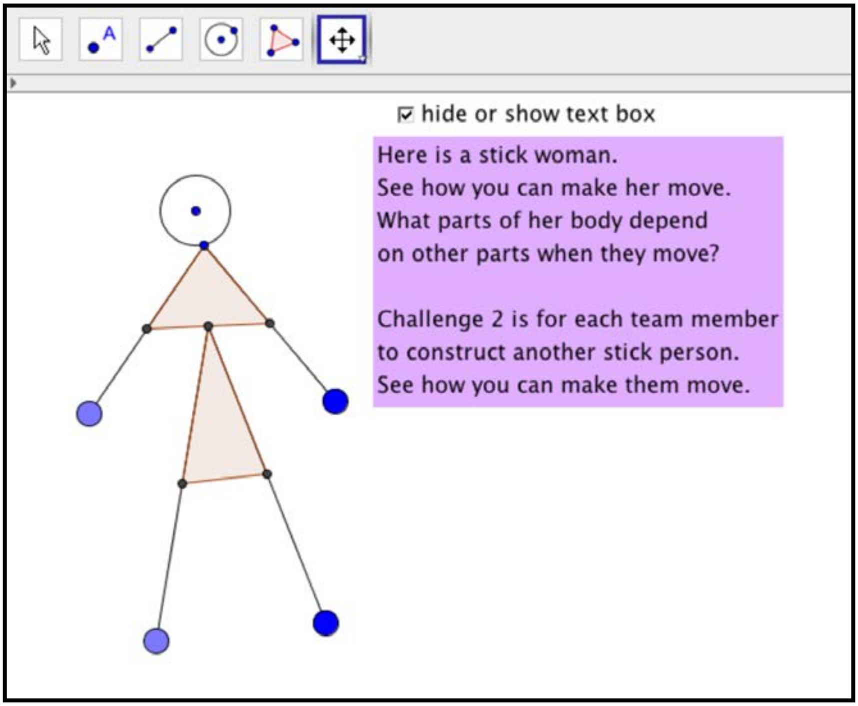


Play with Stick People

You can be creative in GeoGebra. This challenge shows a Stick Woman constructed in GeoGebra. Can you make other stick people and move them around?

Notice that some points or lines are **dependent on** other points or lines. For instance, the position on one hand or foot **depends** upon the position of the other. Dependencies like this are very important in dynamic geometry. Your construction crew will explore how to analyze, construct and discuss dependencies in the following challenges. So, do not expect to be able to construct dependencies like this yet.

Do not forget that you have to press the "Take Control" button to take actions in GeoGebra.



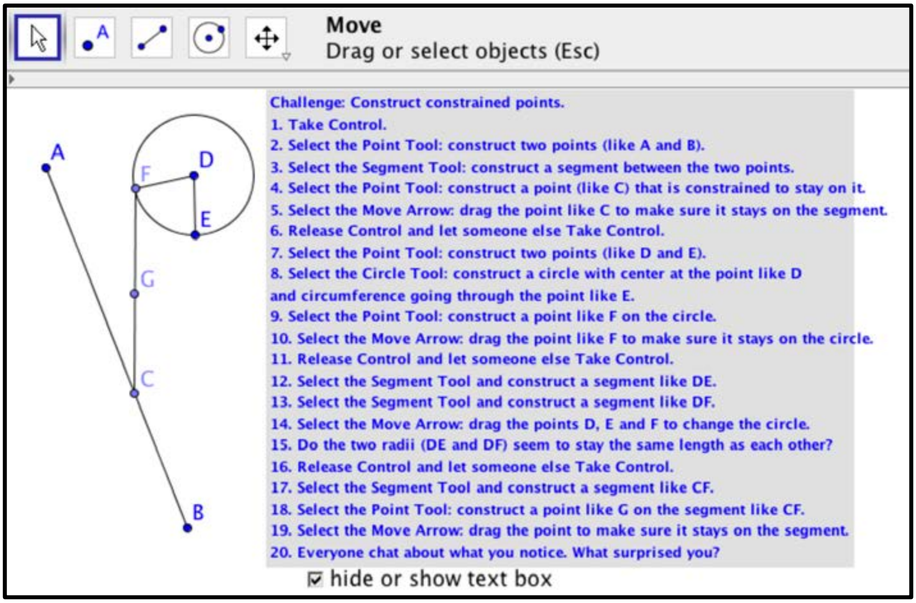
Play around with Points, Lines and Circles

Everything in geometry is built up from simple **points**. In *dynamic geometry*, a point can be dragged from its current position to any other location. A line **segment** is made up of all the points along a straight path between two points (the **endpoints** of the segment).

A **circle** is all the points ("**circumference**") that are a certain distance ("**radius**") from one point ("**center**"). Therefore, any line segment from the center point of a circle to its circumference is a radius of the circle and *is necessarily the same length as every other radius of that circle*. Even if you drag the circle and change its size and the length of its radius, every radius of that circle will again be the same length as every other radius.

For this Challenge, create some points that are **constrained** to stay on a segment or on a circle.

Decide as a team when you have completed the challenge. Make sure everyone agrees on how to do it.



2.Construction Level

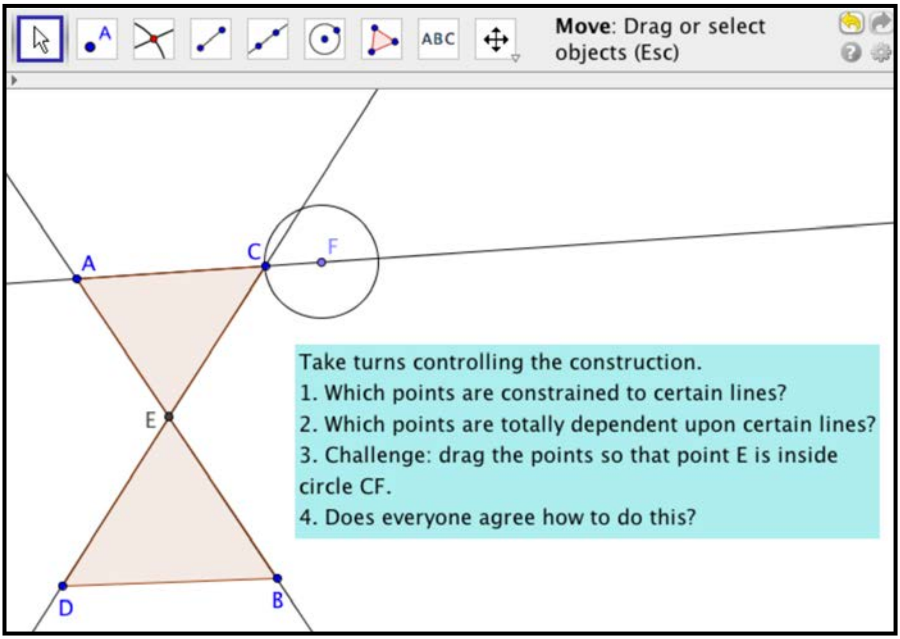
Play totally online. Discuss what you are doing in the chat. This way you have a record of your ideas. Even if you are sitting near your teammates, do not talk out loud or point. Do everything through the computer system. In general, try to say in the chat what you plan to do before you do it in GeoGebra. Then, chat about what you did in GeoGebra and how you did it. Let other people try to do it too. Finally, chat together about what you all did. Take turns doing steps. Play together as a team, rather than just trying to figure things out by yourself.

Play by Dragging Connections

When you construct a point to be on a line (or on a segment, or ray, or circle) in dynamic geometry, it is "*constrained*" to stay on that line; its location is "*dependent*" upon the location of that line, which can be dragged to a new location.

Use the "**drag test**" to check if a point really is constrained to the line: select the Move tool (the first tool in the Tool Bar, with the arrow icon), click on your point and try to drag it; see if it stays on the line. Drag an endpoint of the line. Drag the whole line. What happens to the point?

Drag points and segments to play with them. The drag test is one of the most important things to do in dynamic geometry. Make sure that everyone in your team can drag objects around.



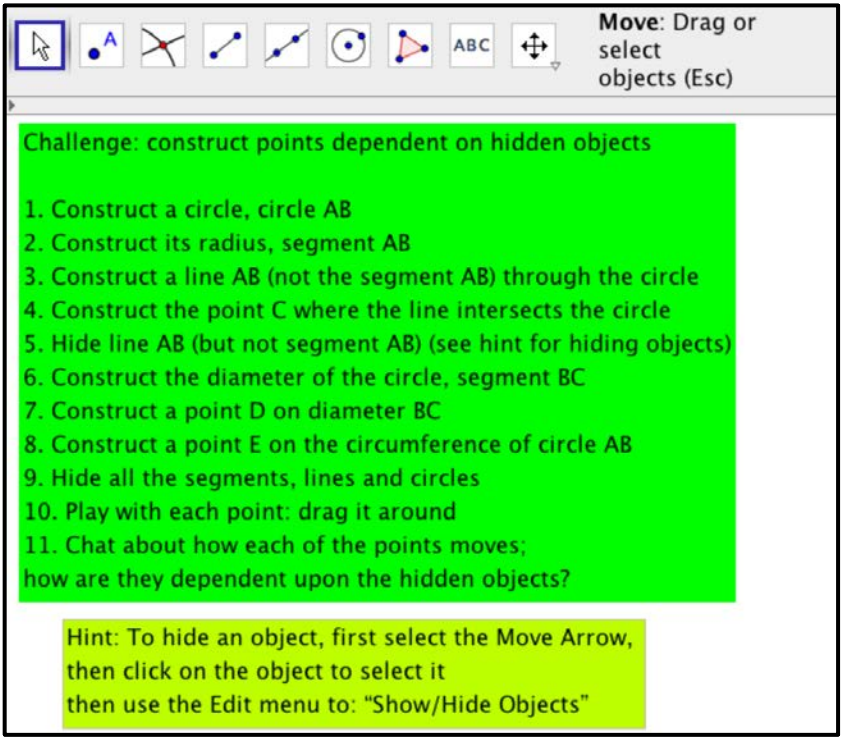
Play with Hidden Objects

Take control and construct some lines and segments with some points on them. Note that "segments" are different from "lines." Lines continue indefinitely beyond the points that define them.
The segment tool, line tool and ray tool are together on the tool bar. Click on the little arrow to select the tool you need.

Notice how:

- Some points can be dragged **freely**,
- Some can only be dragged in certain ways (we say they are partially "**constrained**") and
- Others cannot be dragged directly at all (we say they are fully "**dependent**").

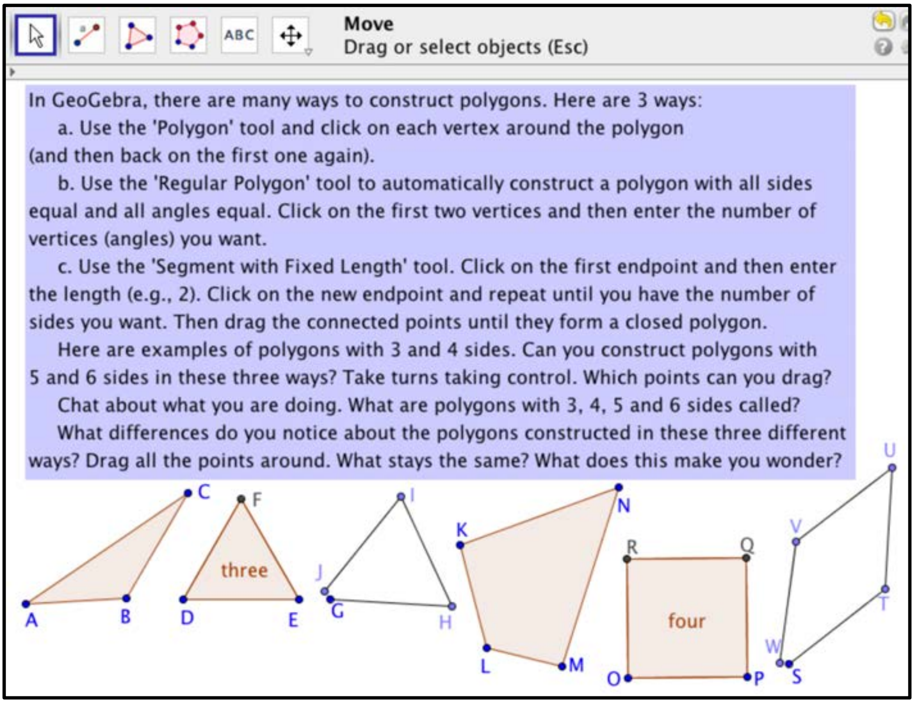
The dependencies are still there if the objects are hidden, even when the hidden objects are dragged.



Construct Polygons in Different Ways

Try to do things in different ways in GeoGebra. This challenge shows different ways to construct "**polygons**" (figures with several straight sides). Use the "**drag test**" to check how the different methods make a difference in the dependencies of the lengths of the sides. Then use the three methods to make your own figures with five or six sides.

Note that these polygons can *look* different as you drag their points. However, they always remain the same kind of polygons, with certain *constraints*—like equal sides, depending on how they were constructed.



3. Triangle Level

The construction of an **equilateral triangle** illustrates some of the most important ideas in dynamic geometry. With this challenge, you will play with that construction. Before working on this challenge with your team, watch a brief **YouTube clip** that shows clearly how to construct an equilateral triangle: http://www.youtube.com/watch?v=ORIaWNQSM_E

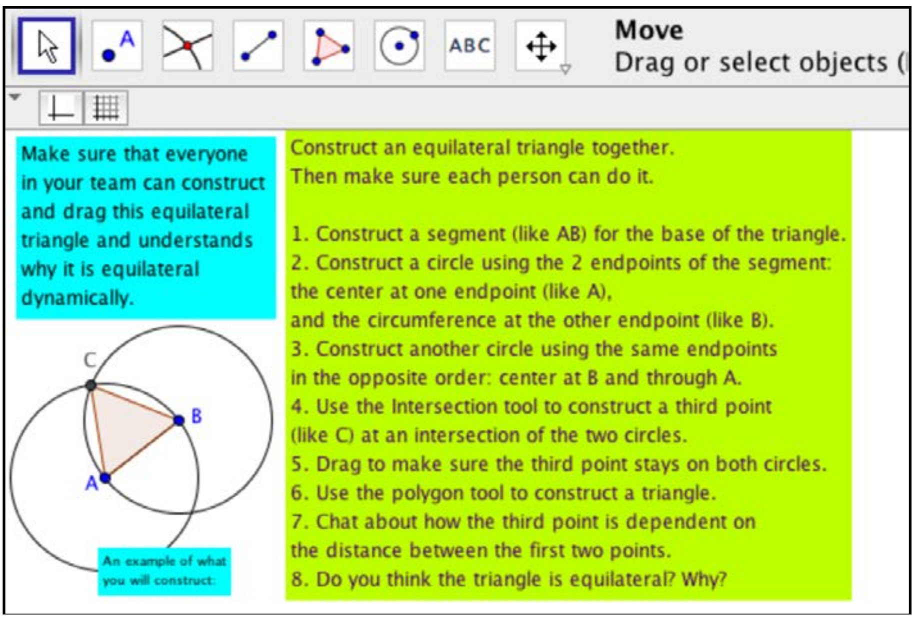
Remember, a **circle** is all the points ("circumference") that are a certain distance ("**radius**") from one point ("**center**"). Therefore, any line segment from the center point of a circle to its circumference is a radius of the circle and *is necessarily the same length as every other radius of that circle*. Even if you drag the circle and change its size and the length of its radius, every radius of that circle will again be the same length as every other radius.

Construct an Equilateral Triangle

This simple but beautiful example shows the most important features of dynamic geometry. Using just a few points, segments and circles (strategically related), it constructs a triangle whose sides are always equal no matter how the points, segments or circles are dragged. Using this construction, you will know that the triangle **must** be equilateral (without you having to measure the sides or the angles).

Everyone on the team should construct an equilateral triangle. Play with (drag) the one that is there first to see how it works. Take turns controlling the GeoGebra tools. In GeoGebra, you construct a circle with center on point A and passing through point B by selecting the circle tool, then clicking on point A and then clicking on point B. Then if you drag either point, the size and location of the circle may change, but it always is centered on point A and always passes through point B because those points define the circle.

Euclid began his book on geometry with the construction of an equilateral triangle 2,300 years ago.



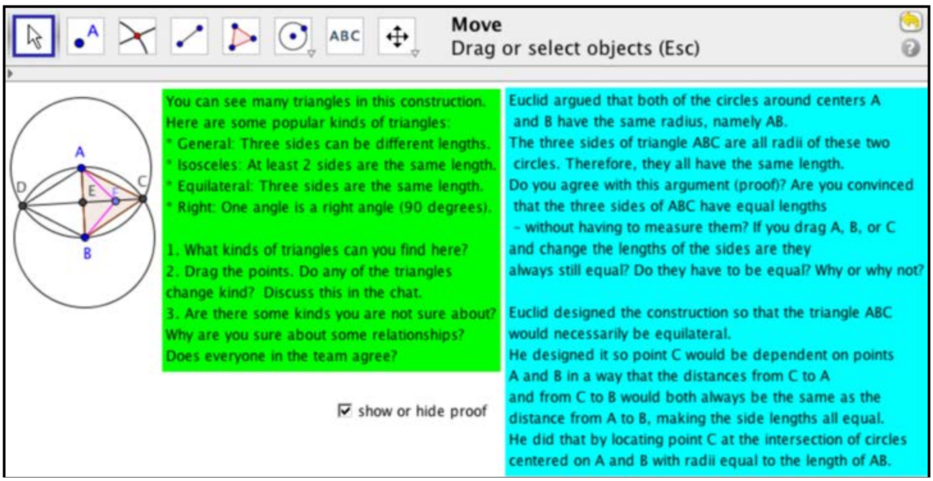
Find Dynamic Triangles

What relationships are created in the construction of the equilateral triangle? Explore some of the relationships that are created among line segments in this more complicated figure. What line segments do you think are equal length – without having to measure them? What angles do you think are equal? Try dragging different points; do these equalities and relationships stay dynamically? Can you see how the construction of the figure made these segments or angles equal?

When you drag point F, what happens to triangle ABF or triangle AEF? In some positions, it can *look* like a different kind of figure, but it always has certain relationships.

What kinds of angles can you find? Are there right angles? Are there lines perpendicular to other lines? Are they always that way? Do they have to be? Can you explain why they are?

Can you prove why triangle ABC is always equilateral (see hint on proof)?



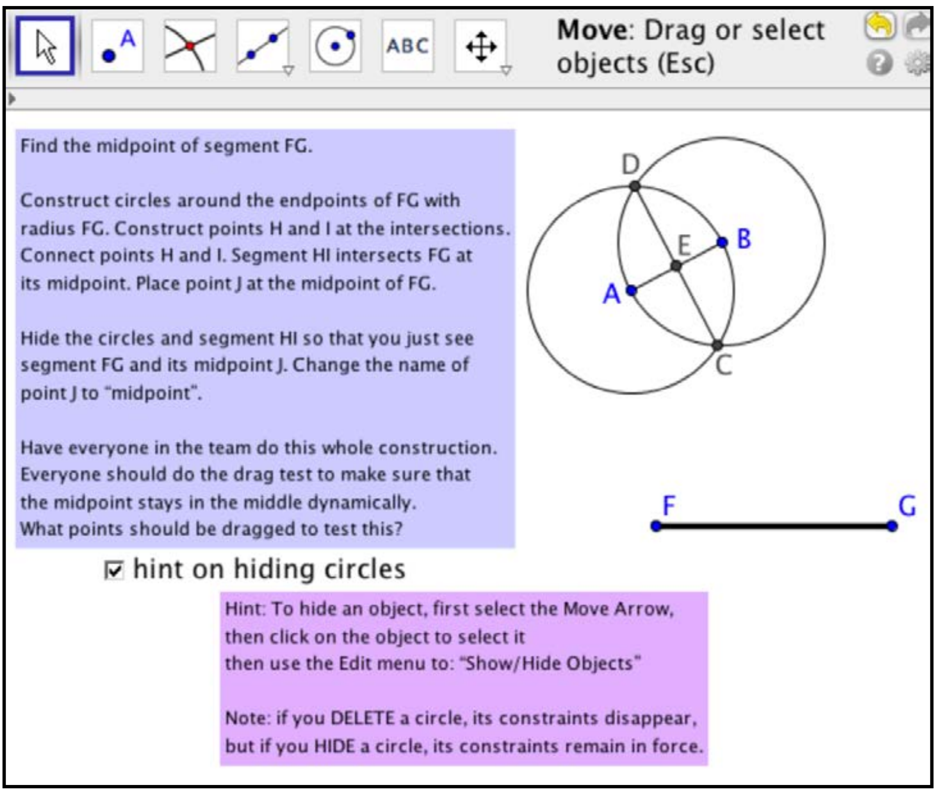
4. Circle Level

Circles are very useful for constructing figures with dependencies. Because all radii of a circle are the same length, you can make the length of one segment be dependent on the length of another segment by constructing both segments to be radii of the same circle. This is what Euclid did to construct an equilateral triangle. Circles can also be used for constructing the midpoint of a segment or perpendicular lines or parallel lines, for instance.

Construct the Midpoint

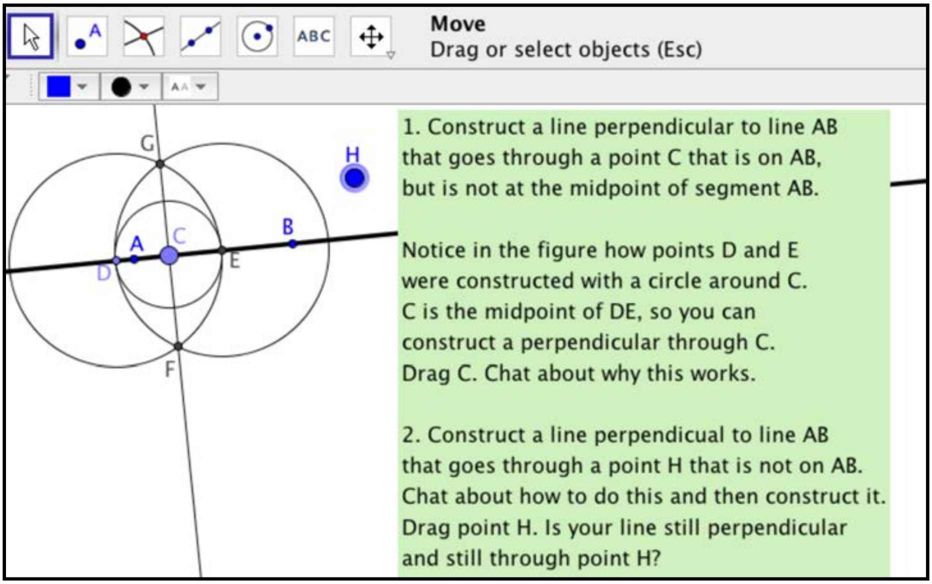
As you already saw, the construction process for an equilateral triangle creates a number of interesting relationships among different points and segments. In this challenge, points A, B and C form an equilateral triangle. Segment AB crosses segment CD at the exact **midpoint** of CD and the angles between these two segments are all right angles (90 degrees). We say that AB is the "**perpendicular bisector**" of CD—meaning that AB cuts CD at its midpoint, evenly into two equal-length segments, and that AB is perpendicular (meaning, at a right angle) to CD.

To find the midpoint of a segment AB, construct circles of radius AB centered on A and on B. Construct points C and D at the intersections of the circles. Segment CD intersects segment AB at the midpoint of AB.



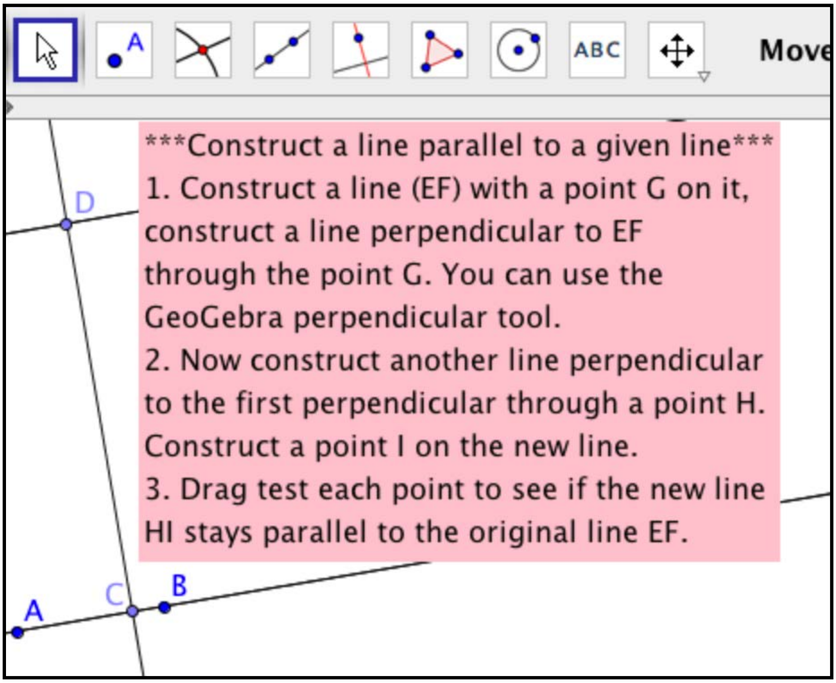
Construct a Perpendicular Line

For many geometry constructions (like constructing a right angle, a right triangle or an altitude), it is necessary to construct a new line perpendicular to an existing line (like line AB). In particular, you may need to have the perpendicular go through the line at a certain point (like C or H), which is not the midpoint of segment FG. Can your team figure out how to do that?



Construct a Parallel Line

In geometry, once you learn how to do one construction, you can use that as one step in a larger construction. For instance, two parallel lines are both perpendicular to the same line. So, if you have a line AB, you can construct a new line DE that is parallel to AB as follows. First, construct a line CD that is perpendicular to the original line AB. Then construct a line DE that is perpendicular to line CD. Now line DE is parallel to line AB. No matter how long you draw lines AB and DE, they will never cross.

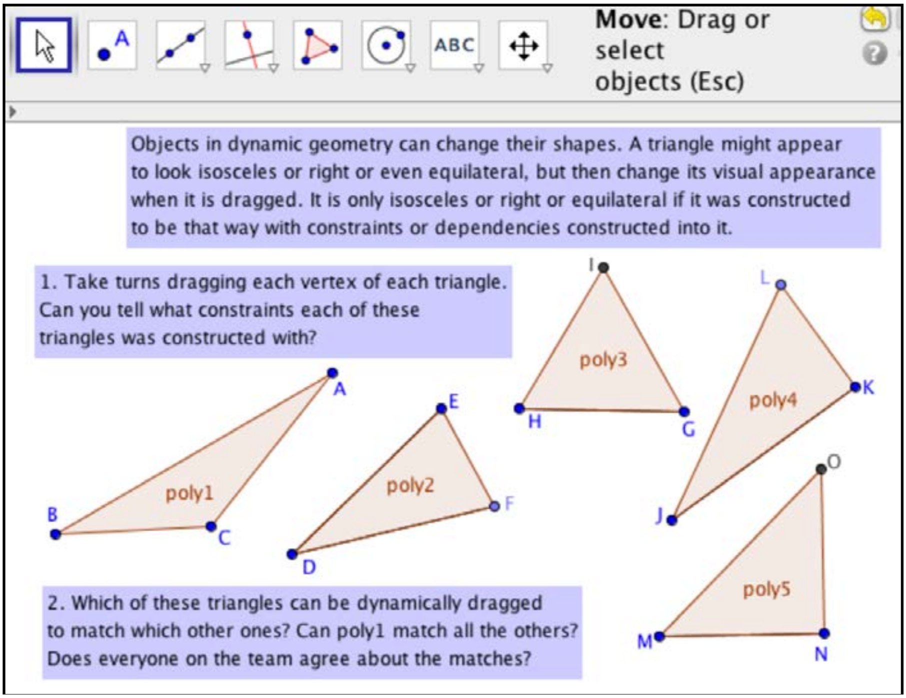


5. Dependency Level

A triangle seems to be a relatively simple geometric construction: simply join three segments at their endpoints. Yet, there are many surprising and complex relationships possible in triangles with different dependencies designed into them.

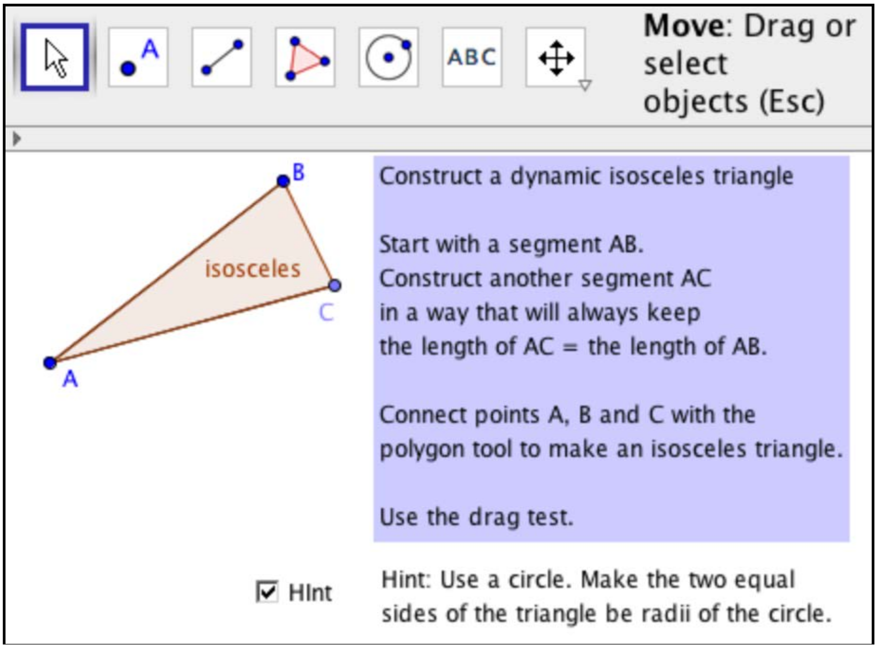
Triangles with Dependencies

Here are some triangles that were constructed with hidden dependencies. Can your team figure out what the dependencies are and how they might have been constructed?



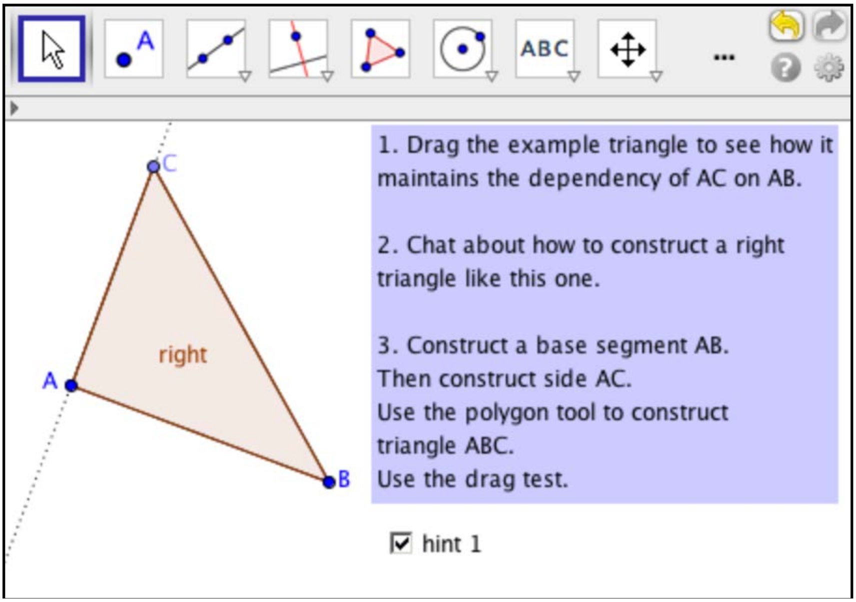
An Isosceles Triangle

An isosceles triangle has two sides that are always the same length. Can you simplify the method of constructing an equilateral triangle so that the length of one side is dependent on the length of another side, but the third side can be any length? Can you drag your figure into every possible isosceles triangle?



A Right Triangle

A right triangle has a right angle (90 degrees) as one of its vertices. Can you construct a right triangle so that one side is always perpendicular to another, but the sides can be any lengths? Can you drag your figure into every possible right triangle?

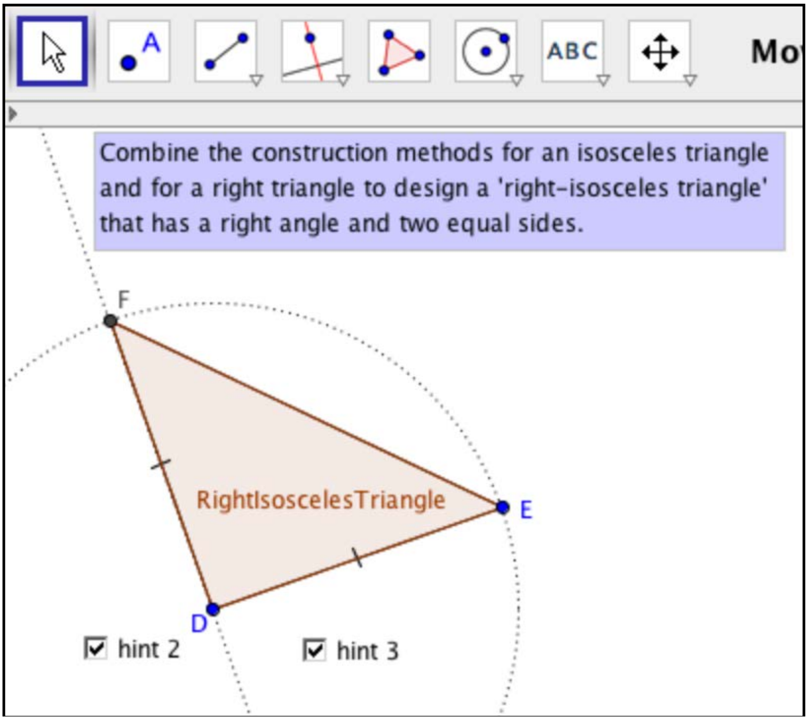


An Isosceles-Right Triangle

Combine the construction methods for an isosceles triangle and for a right triangle to design a triangle that has a right angle and two equal sides.

Be sure to take turns controlling the GeoGebra tools and chat about what you are doing. Work together—do not try to solve something yourself and then explain it to your teammates. Talk it out in the chat:

- Make sure everyone in the team understands what was done and can do it.
- Discuss what you noticed and what you wondered about the construction.
- Discuss why it worked: why is the triangle you constructed always isosceles and always right-angled no matter how you drag it?

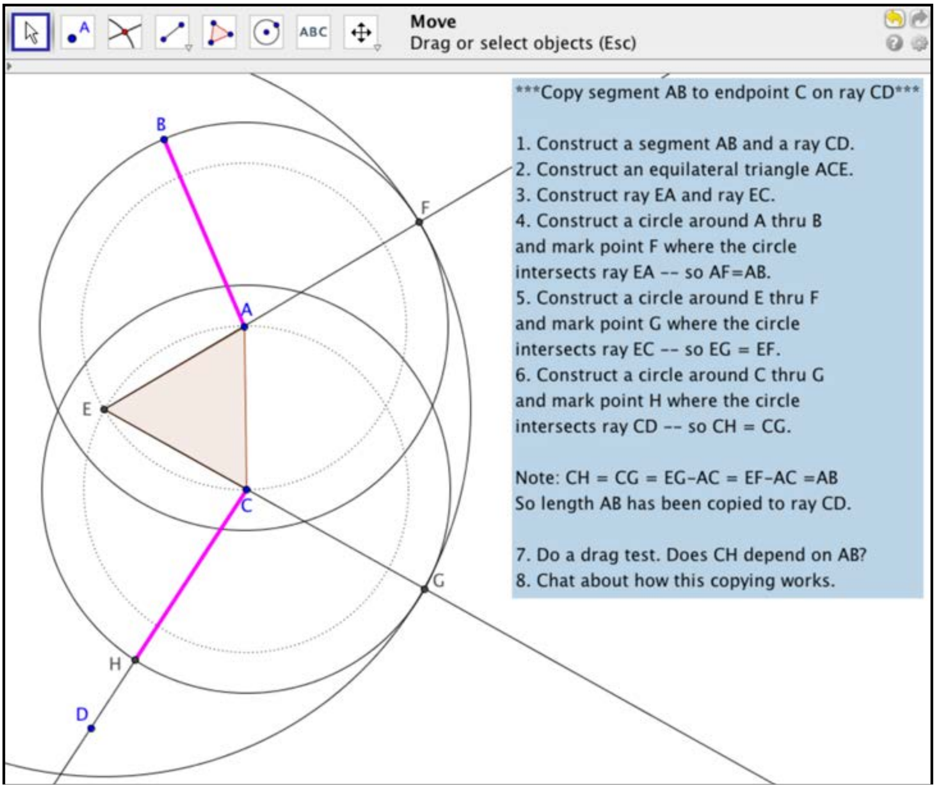


6. Compass Level

Constructing one segment to be the same length as another segment is different from just **copying** the segment. Euclid used circles to construct a segment that was the same length as another segment, but located at a different endpoint. That was a complicated procedure. In GeoGebra, we can use the **compass tool** to make that construction easier.

Copy a Length

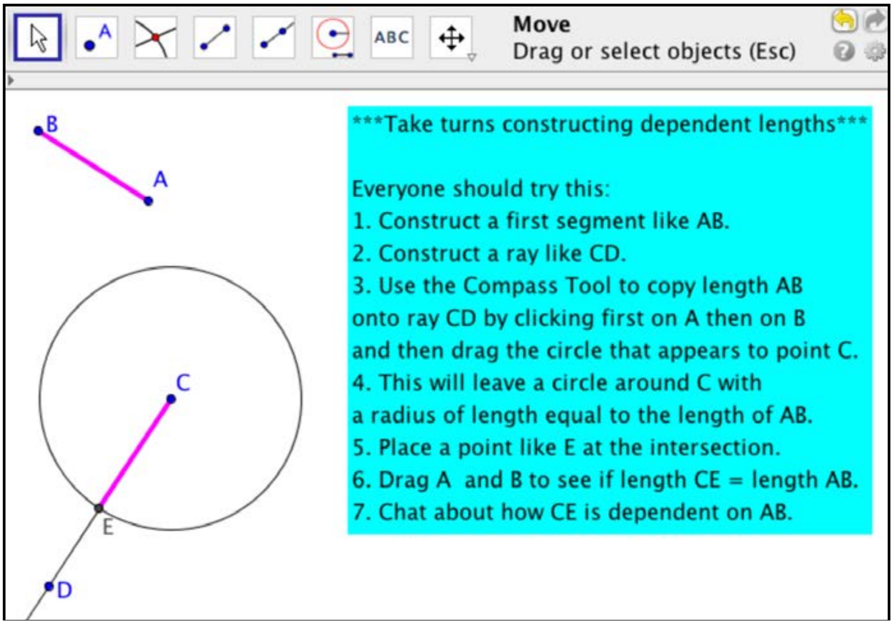
After Euclid did his construction of an equilateral triangle, he used that to show how to copy a segment length to another location. In dynamic geometry, this means making the length of a second segment (CH) dependent on the length of the first segment (AB).



Use the Compass Tool

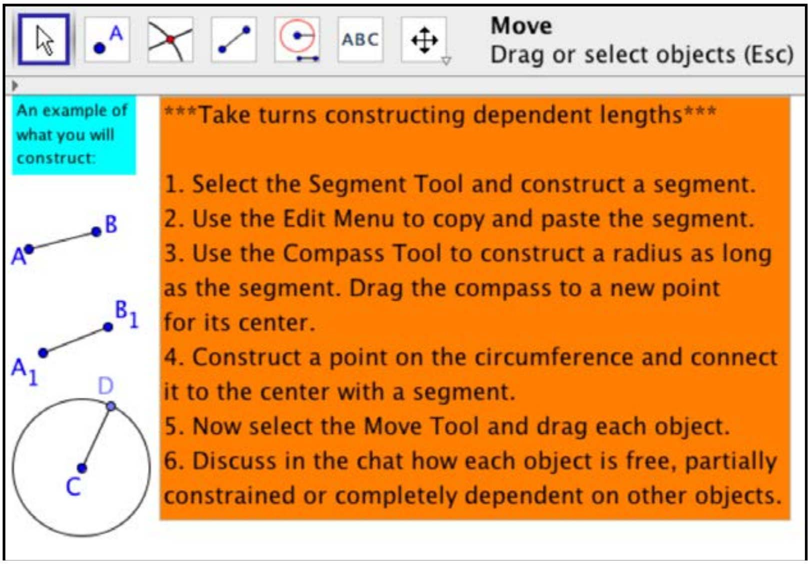
First, get a good introduction to using the GeoGebra **compass tool** from this YouTube clip: http://www.youtube.com/watch?v=AdBNfEOEVco.

Everyone on the team should play with using the compass tool. It is tricky to learn, but very useful.



Make Dependent Segments

Compare copying a segment length with *copy-and-paste* to constructing a segment with the same length using the *compass* tool. First, construct a segment like AB. Then select the segment and use the Edit menu to copy it and paste it to create a copy like A_1B_1. Next, use the compass tool, first define its radius (by clicking on points A and B) and then drag it to a new center for it, like C. Drag point A to change the length of segment AB. Does the copy A_1B_1 made with copy-and-paste change its length automatically? Does the radius CD change its length automatically when AB is changed? Why do you think this happens?

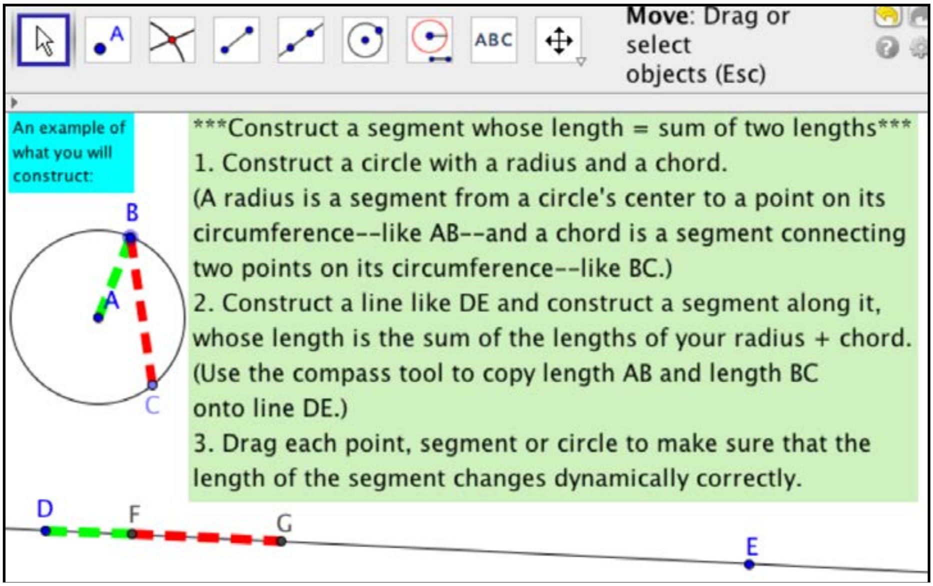


Add Segment Lengths

In dynamic geometry, you can construct figures that have complicated dependencies of some objects on other objects. Here you will construct one segment whose length is dependent on the lengths of two other segments.

Use the compass tool to copy the lengths of the line segments. Creating line segment length DG = AB + BC provides a good visual image when you drag point B.

Discuss with your team how to do each step, especially step 2. Do you see how you can use the compass tool to lay out segments with given lengths (like the lengths of AB and BC) along a given line (like DE)? Chat with your teammates about the difference between the circle tool and the compass tool.



Copy vs. Construct a Congruent Triangle

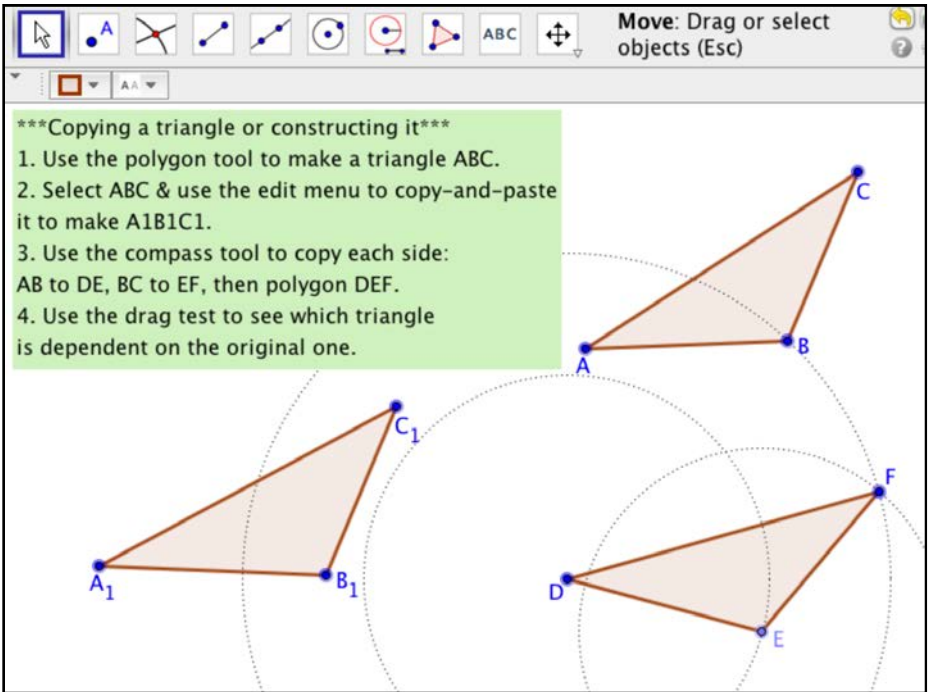
Two segments are called "**congruent**" if they are equal length.

Two triangles are called "**congruent**" if all their corresponding sides are congruent.

In this challenge, a triangle ABC is copied two ways: with copy-and-paste to create $A_1B_1C_1$ and by copying each side length with the compass tool to construct triangle DEF.

To copy and paste ABC, first select the triangle or the whole area around the triangle with the Move arrow tool and then use the Edit menu—first Copy and then Paste—and move the copy where you want it.

Which triangles do you think will stay congruent with the drag test?



Construct a Congruent Angle

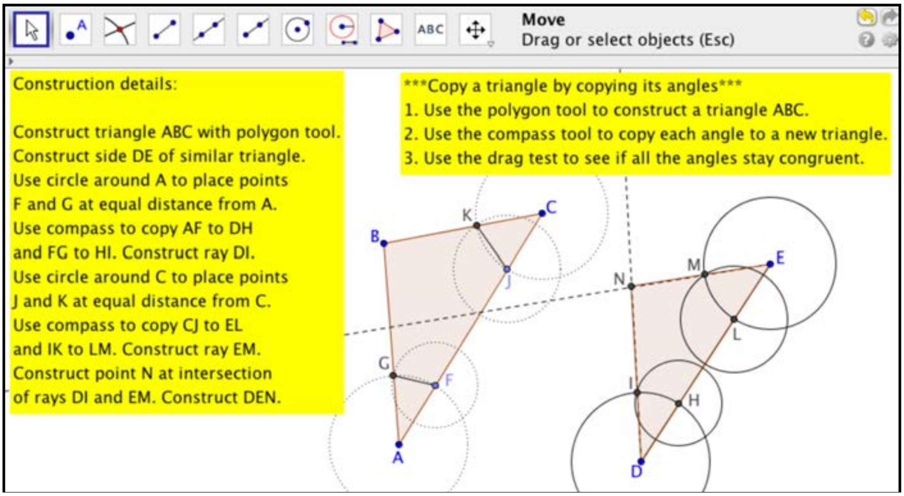
Two *angles* are called "**congruent**" if they are equal. Two *triangles* are called "**similar**" if all their corresponding *angles* are congruent. Two *triangles* are called "**congruent**" if all their corresponding *sides* (and all their corresponding angles) are congruent.

Do you think that if two triangles are *similar* then they are *congruent*? Do you think that if two triangles are *congruent* then they are *similar*?

There is no tool in GeoGebra for copying an angle, but you can use the compass tool to do it in a couple of steps. Chat about why this works. Watch this **YouTube clip**:

http://www.youtube.com/watch?v=qngWUFgyyHc. It gives a clear view of one way to copy an angle.

When you drag point A, do the sides of triangle DEN stay congruent to the corresponding sides of ABC? Do the vertex angles of triangle DEN stay congruent to the corresponding angles of ABC?



7. Congruence Level

Two triangles are called "congruent" if all their corresponding angles and sides are equal size. However, you can constrain two triangles to be congruent by just constraining 3 of their corresponding parts to be equal – for certain combinations of 3 parts. Dynamic geometry helps you to visualize, to understand and to remember these different combinations.

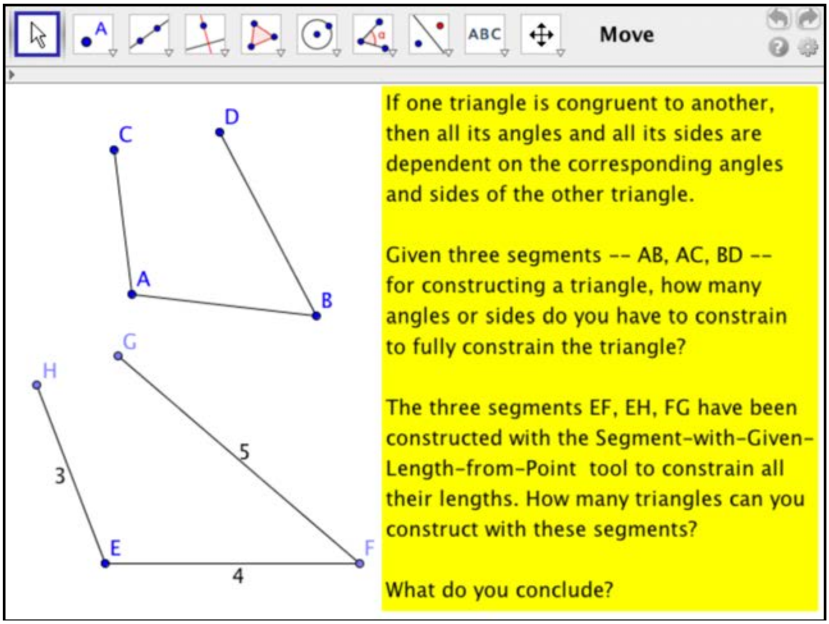
Combinations of Sides and Angles of Triangles

What constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle? Two congruent triangles have 6 corresponding parts equal (3 sides and 3 angles), but you do not have to constrain all 6 parts to be equal in order to make sure the triangles are congruent.

For instance, two triangles with their corresponding 3 angles equal are called "similar" but they may not be "congruent." You can drag one of them to be larger that the other one. They will still have the same shape, but the corresponding side lengths of one will all be larger than the side lengths of the other triangle.

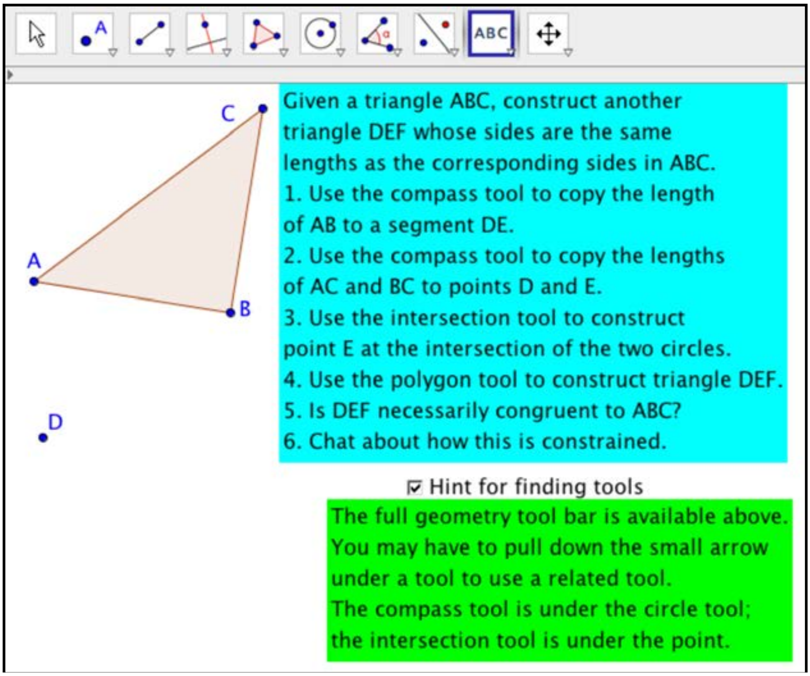
If two triangles have their corresponding angles constrained to be equal and then you constrain two corresponding sides to be equal length, will the triangles necessarily be congruent? Suppose you only constrained one of the corresponding sides to be equal?

Explore different combinations of 4 or 5 of the 6 corresponding triangle parts being constrained to be equal. Which combinations guarantee that the triangles are congruent?



Side-Side-Side (SSS)

If all three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent. This is called the "Side-Side-Side" (or "SSS") rule. In dynamic geometry, you can make the 3 side lengths of one triangle be dependent on the corresponding 3 side lengths of another triangle, ensuring that the two triangles will always be congruent.

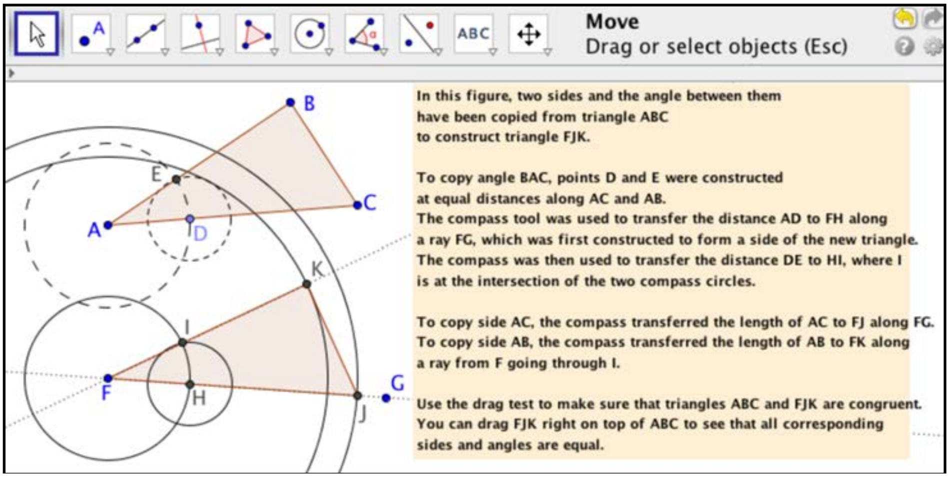


Side-Angle-Side (SAS)

If **two sides** and the angle **between them** of one triangle are equal to the corresponding sides and angle **between them** of another triangle, then the two triangles are congruent.

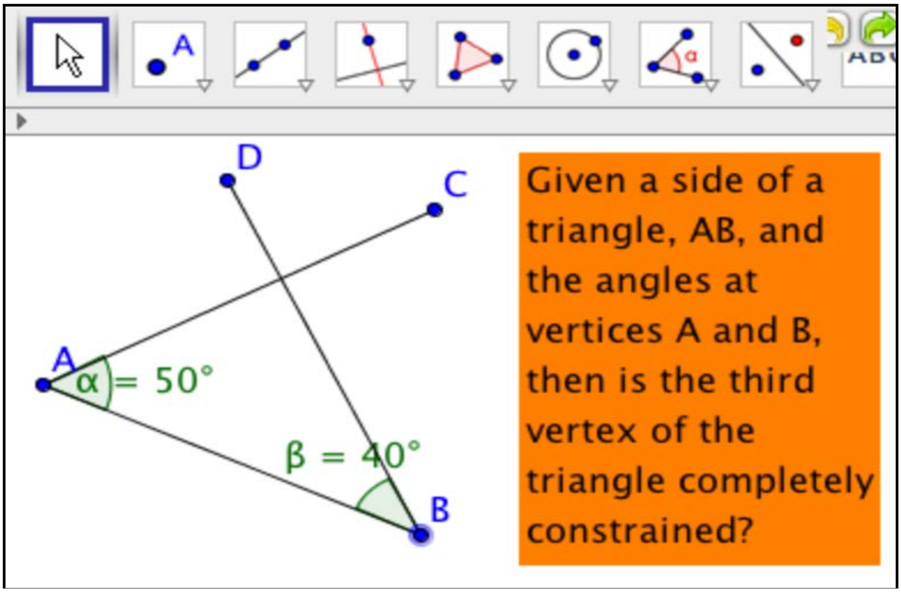
In this challenge, you will have to copy an angle. You can copy an angle in dynamic geometry using the compass tool, but it is a bit more complicated than copying a segment.

Watch this **YouTube clip**: http://www.youtube.com/watch?v=qngWUFgyyHc if you did not already. It gives a clear view of one way to use GeoGebra to copy an angle using tools that are equivalent to traditional geometry construction with a physical straightedge (for making line segments) and compass (for making circles).



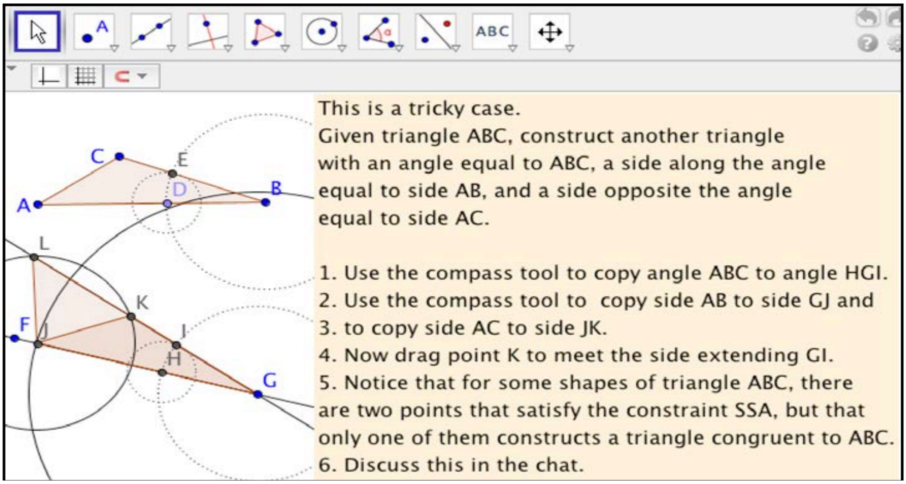
Angle-Side-Angle (ASA)

If two angles and the side included **between them** of one triangle are equal to the corresponding two angles and side **between them** of another triangle, then the two triangles are congruent. This is called the Angle-Side-Angle or ASA rule.



Side-Side-Angle (SSA)

What if two corresponding sides and an angle are equal, but it is not the angle included between the two sides?



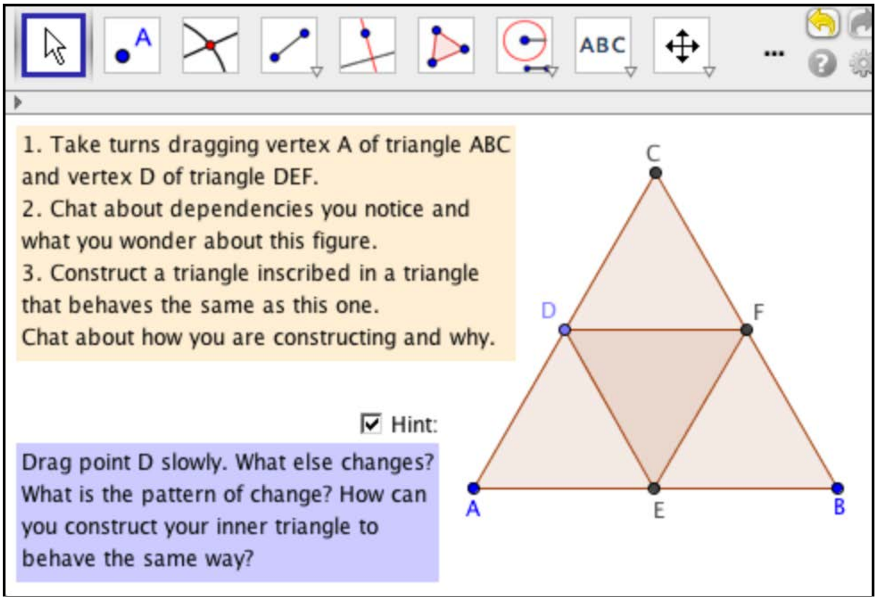
8. Inscribed Polygon Level

This section presents challenging problems for your team to construct.

The Inscribed Triangles Challenge Problem

Construct a pair of inscribed triangles. First, explore the given figure. Note the dependencies in the figure. Then construct your own pair of inscribed triangles that behaves the same way. Triangle DEF is called "**inscribed**" in triangle ABC if its vertices D, E and F are on the sides of triangle ABC.

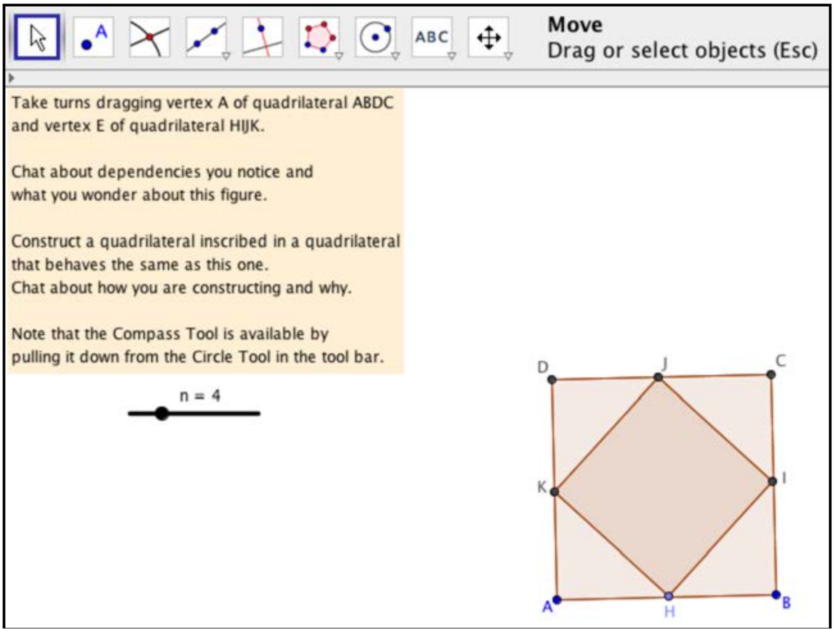
If you solved this, did you construct the lengths of the sides of the outer triangle to be directly dependent upon each other? Did you construct the lengths of the sides of the inner triangle to be directly dependent upon each other? Do you think that the triangles are equilateral? How do you know?



The Inscribed Quadrilaterals Problem

Try the same problem with inscribed quadrilaterals.

Note: The slider lets you try the problem for other regular polygons. A "regular" polygon is a polygon with congruent sides (all the same length) and congruent angles.

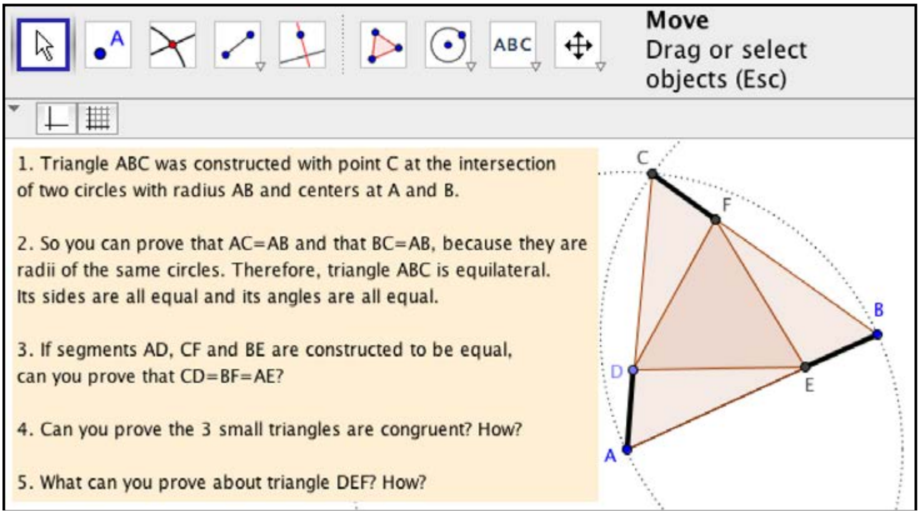


Prove Inscribed Triangles

Constructing figures in dynamic geometry—like the inscribed triangles—requires thinking about dependencies among points, segments and circles. You can talk about these dependencies in the form of proofs, which explain why the relationships among the points, segments and circles are always, necessarily true, even when any of the points are dragged around.

Many proofs in geometry involve showing that some triangles are congruent to others. You can prove that the inscribed triangles are equilateral by proving that certain triangles are congruent to each other. Chat about what you can prove and how you know that certain relationships are necessarily true in your figure. Explain your proof to your team. Does everyone agree?

If you have not studied congruent triangles yet, then you may not be able to complete the proof. Come back to this after you study the activities on congruent triangles.



9. Transformation Level

There are several "**rigid transformations**" in dynamic geometry, which move an object around without changing its size or shape. See the tool menu for GeoGebra transformation tools.

* **Translate by Vector** -- creates a copy of the object at a distance and in a direction determined by a "**vector**" (a segment pointing in a direction).

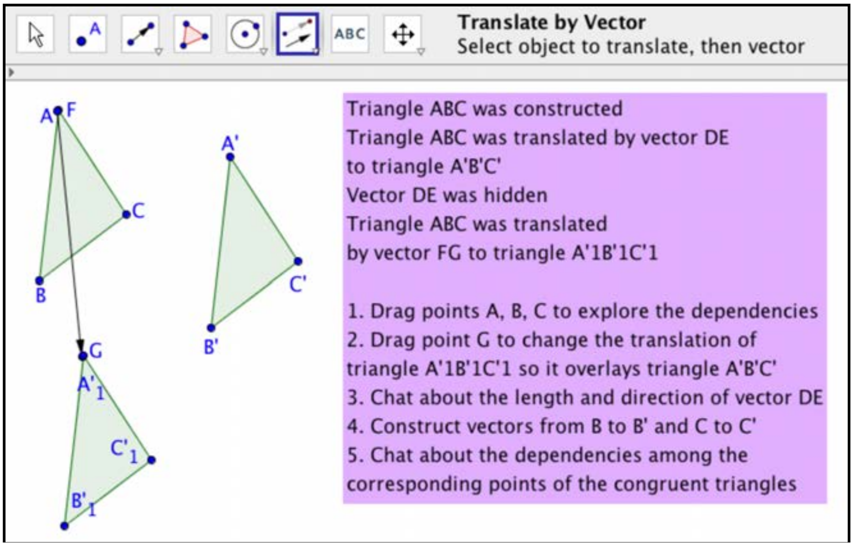
* **Reflect about Line** -- creates a copy of the object flipped across a line.

* **Rotate around Point** -- creates a copy of the object rotated around a point.

Translate by a Vector

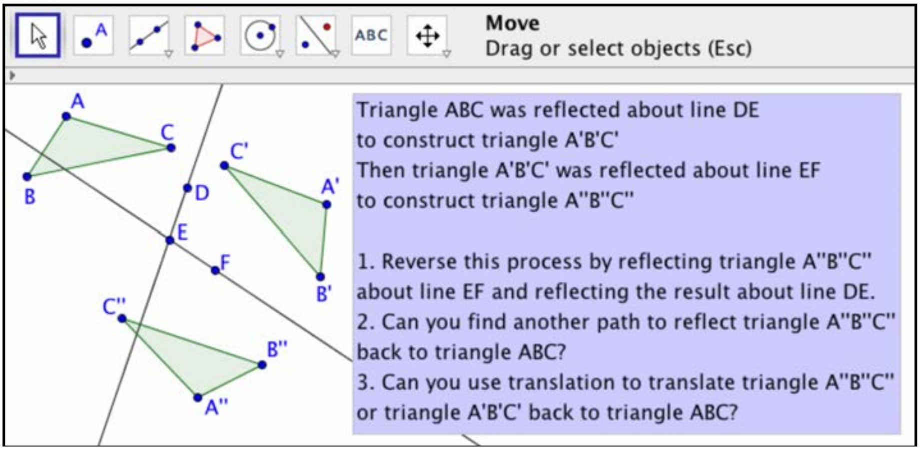
In GeoGebra, a transformation of a figure like triangle ABC creates a congruent figure that is dependent on the original figure. In a translation, the new figure is moved or displaced in the distance and direction indicated by a vector. A vector is like a segment except that it has a particular direction as well as a length.

In this challenge, triangle ABC is translated by vectors DE and FG. Vector DE is hidden. Drag point G until vector FG is the same as vector DE.



Reflect About a Line

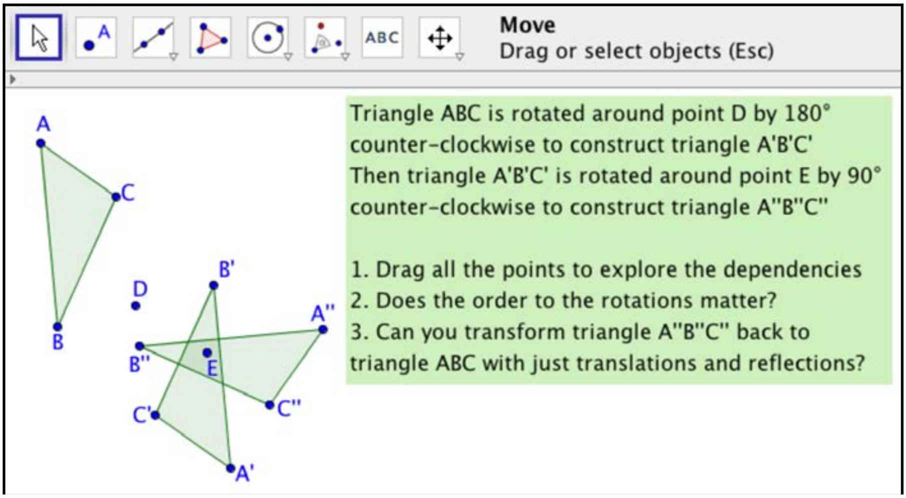
This challenge involves reflections. Take turns and chat about what you are doing and what you notice.



Rotate Around a Point

In GeoGebra, you can rotate a figure around a point. The point can be either on or off the figure. You can rotate any number of degrees—either clockwise or counterclockwise. 360 degrees rotates a figure completely around back to where it started. 180 degrees rotates it halfway around.

Chat about this challenge.

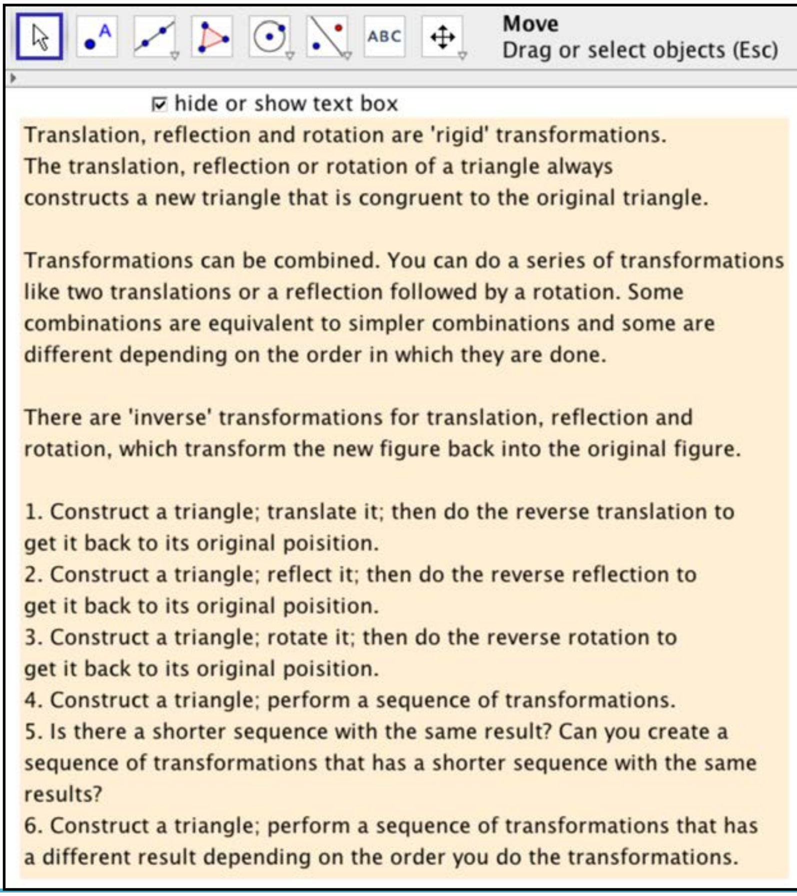


Combine Transformations

You can combine transformations. For instance, first translate and then rotate an image. Do you get the same result if you rotate first and then translate?

When you combine two or more different kinds of transformations, which transformations have to be done in a certain order and which combinations are the same in any order?

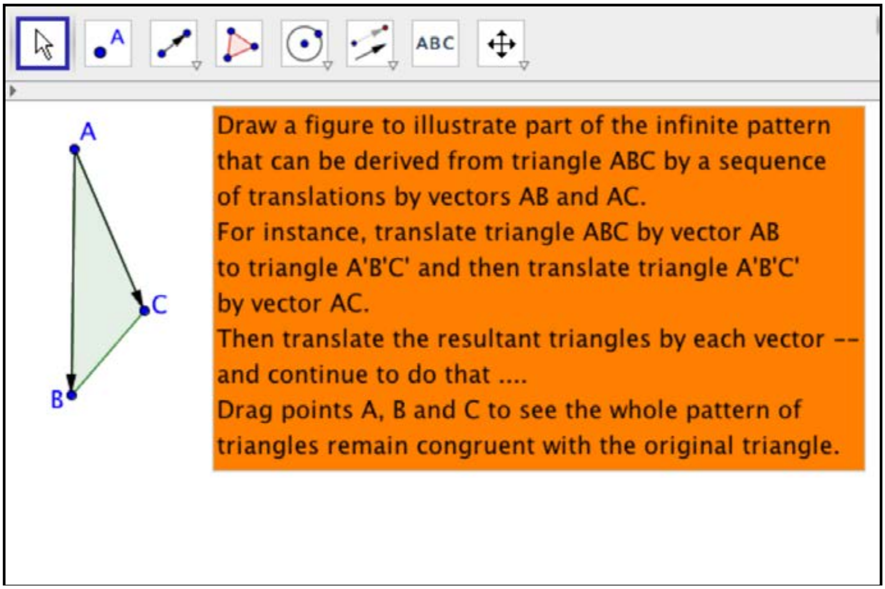
Can you get the same result with a rotation about a point as with a translation followed by two reflections?



Create Dynamic Patterns

Using transformations, you can create interesting dynamic patterns. They are dynamic because if you drag a vertex of the original figure, then all the congruent figures also change.

Try using different figures to start with. Hide some of the intermediate steps so just the pattern and some control points are visible. Add different colors. Be creative!



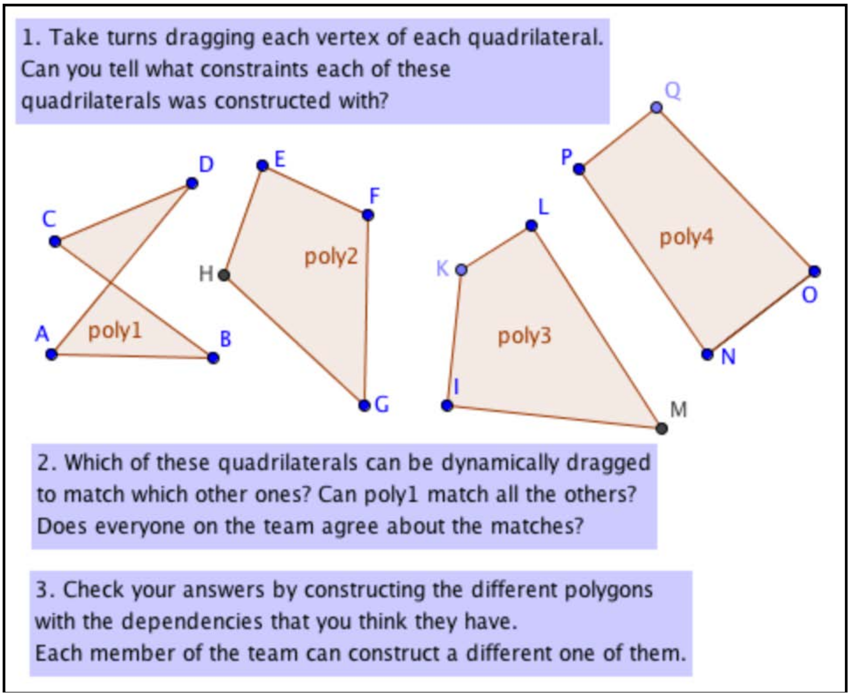
10. Quadrilateral Level

A quadrilateral is a polygon with four sides. It is much more complicated than a triangle. However, you can explore dynamic quadrilaterals the same ways you explored dynamic triangles. You can construct dynamic quadrilaterals the same ways you constructed dynamic triangles.

Construct Quadrilaterals with Constraints

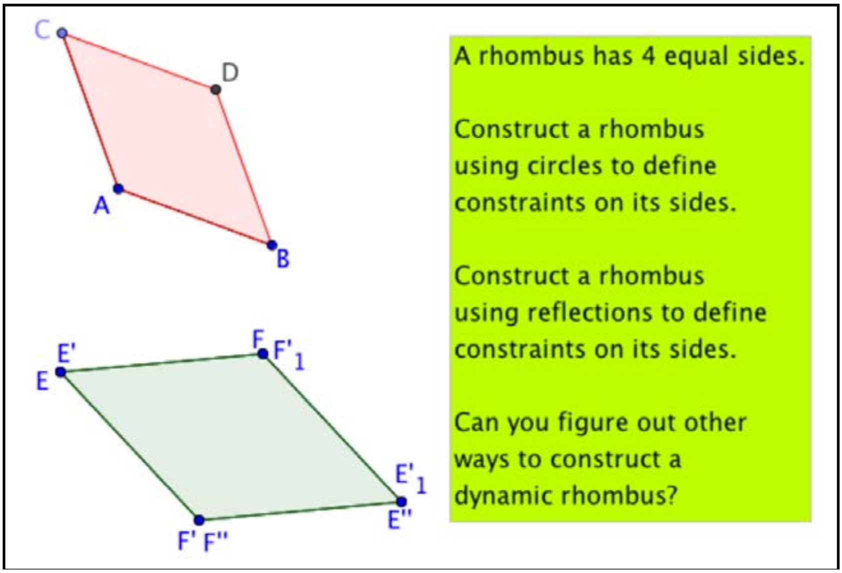
Chat about the constraints that have been constructed into these quadrilaterals.

Can you figure out how they were constructed? Try to construct them and test them by dragging.



Construct a Rhombus

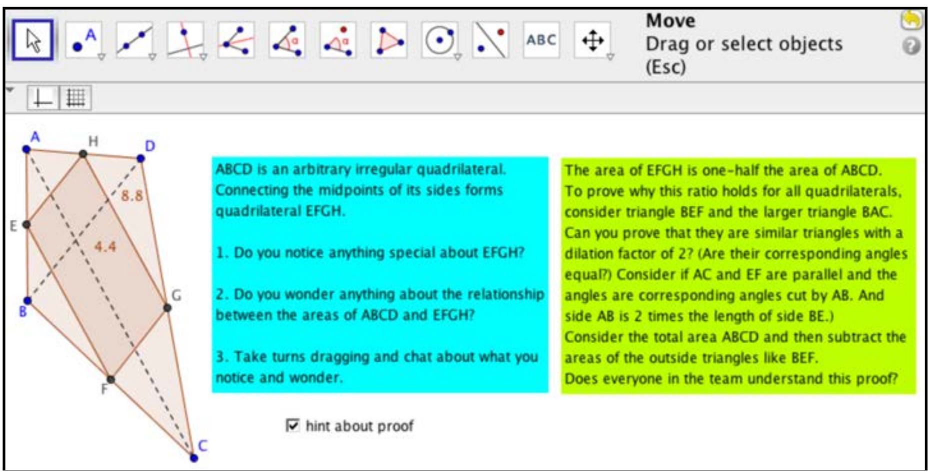
Chat about different ways to construct a rhombus. Try them out. Check by dragging.



Quadrilateral Areas

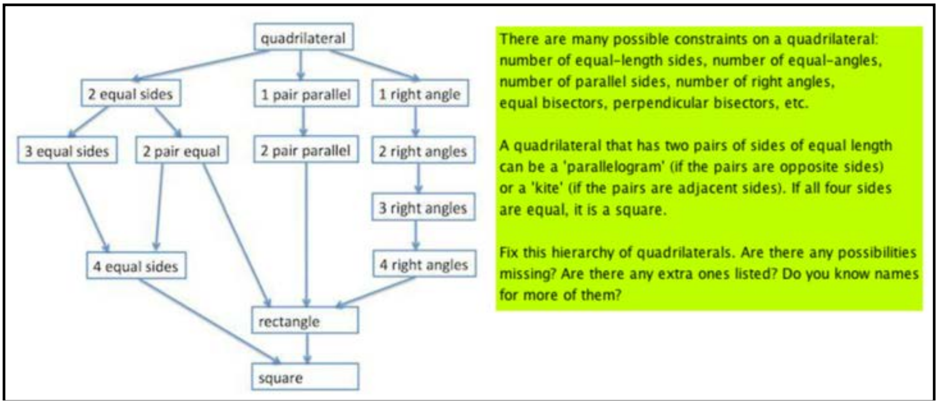
There are many interesting relationships in quadrilaterals. It is often useful to construct their bisectors, connecting opposite angles. In certain types of quadrilaterals, the bisectors have special characteristics.

This challenge involves a surprising characteristic of all quadrilaterals. Can you see what it is? Can you understand why this should always be true?



Build a Hierarchy of Quadrilaterals

This challenge contains the start of a hierarchy diagram of different kinds of quadrilaterals. Chat about the different kinds you can think of. Revise the diagram to be more complete and accurate.



11. Advanced Geometer Level

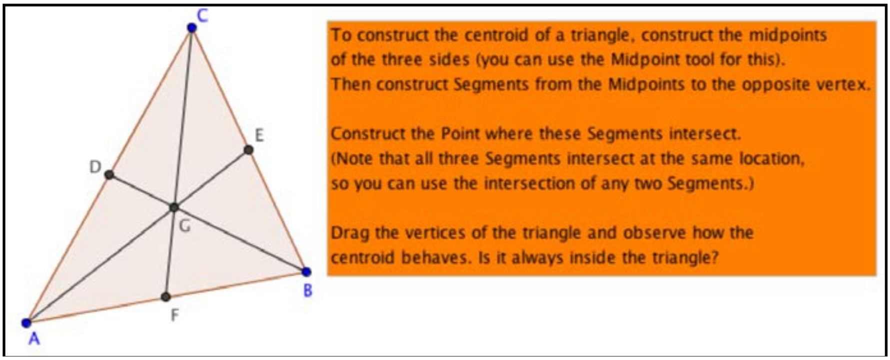
There are a number of interesting ways to define the "center" of a triangle, each with its own special properties.

The Centroid of a Triangle

The "centroid" of a triangle is the meeting point of the three lines from the midpoints of the triangle's sides to the opposite vertex.

The centroid is the triangle's "center of gravity." (If you made a cardboard triangle, it would balance on the centroid.)

If you are using a version of GeoGebra that allows you to create custom tools, then create your own tool for constructing a centroid of a triangle. Also, create your own tools for the circumcenter, orthocenter and incenter of a triangle and use these custom tools to construct the Euler segment and the nine-point circle.

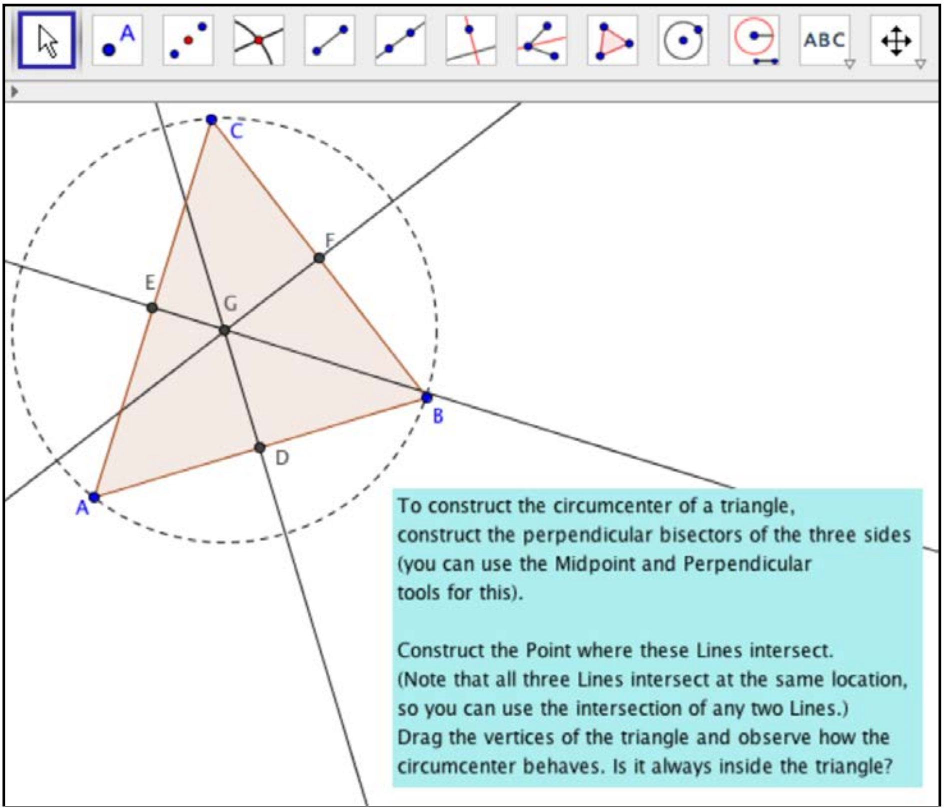


The Circumcenter of a Triangle

The "circumcenter" of a triangle is the meeting point of the three perpendicular bisectors of the sides of the triangle.

The circumcenter is an equal distance from the three vertices of the triangle.

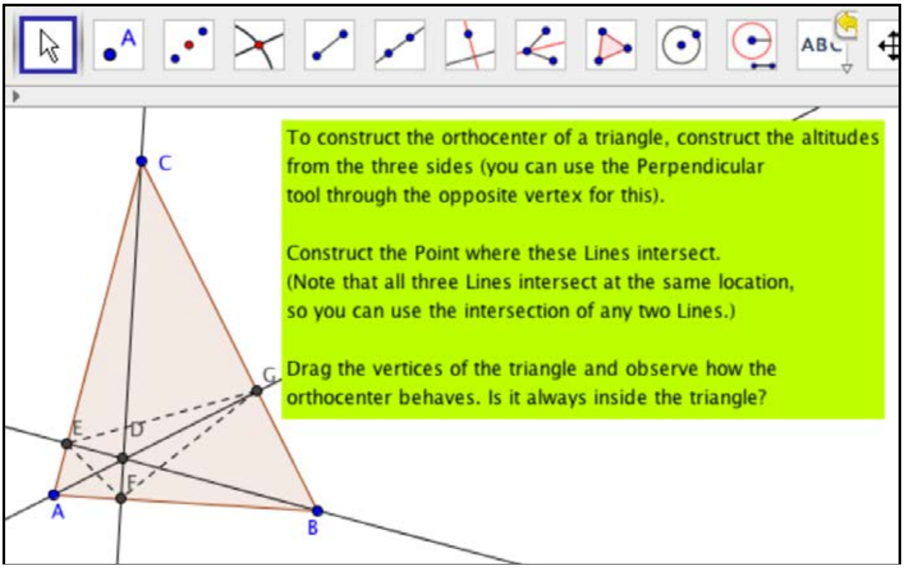
Note: The circumcenter of a triangle is the center of a circle circumscribed about the triangle (drawn around it). You can create a circle around the triangle, passing through all three vertices by constructing a circle centered on the circumcenter and passing through one of the vertices.



The Orthocenter of a Triangle

The "orthocenter" of a triangle is the meeting point of the three altitudes of the triangle. An "altitude" of a triangle is the segment that is perpendicular to a side and goes to the opposite vertex.

If you consider the four points consisting of the triangle's three vertices and the orthocenter, then the orthocenter of any triangle formed by three of those points will be the fourth point. Can you see why this is true?

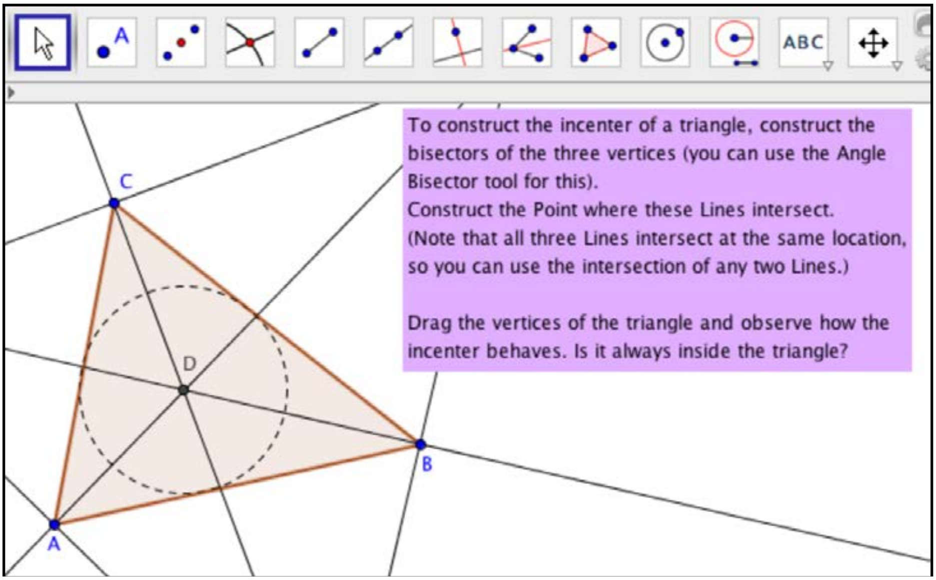


The Incenter of a Triangle

The "incenter" of a triangle is the meeting point of the three angle bisectors of the angles at the triangle's vertices.

The incenter is an equal distance from the three sides of the triangle.

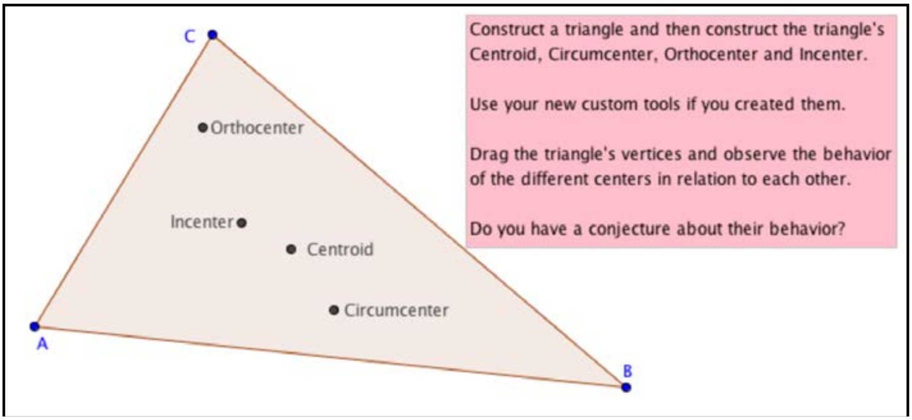
Note: The incenter of a triangle is the center of a circle inscribed in the triangle (the largest circle that fits inside the triangle. A radius of the inscribed circle is tangent to each side of the triangle, so you can construct a perpendicular from the incenter to a side to find the inscribed circle's point of tangency – and then use this point to construct the inscribed circle.



The Euler Segment of a Triangle

The relationships presented in this challenge seem quite surprising. However, they are results of the dependencies imposed in the triangle by the constructions of the different centers. Sometimes combinations of complex dependencies have surprising results.

A Swiss mathematician named Euler (his last name is pronounced "oiler") discovered a relationship among three of the centers that you created in the previous activities. Can you discover what he did? He did this in the 1700s—without dynamic-geometry tools. Euler's work renewed interest in geometry and led to many discoveries beyond Euclid's.



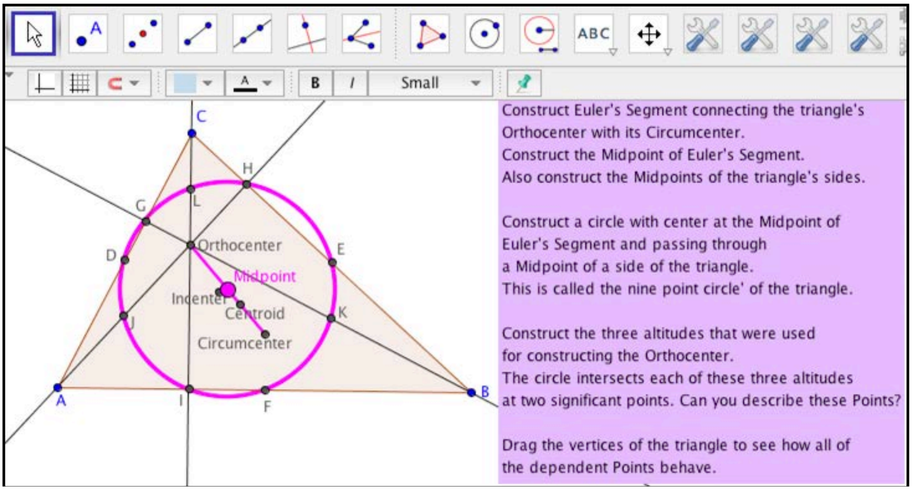
The Nine-Point Circle of a Triangle

You can construct a circle that passes through a number of special points in a triangle. Connect the orthocenter to the circumcenter: this is "Euler's Segment." The Centroid lies on this segment (at a third of the distance from the circumcenter to the orthocenter). A number of centers and related points of a triangle are all closely related by Euler's Segment and its Nine-Point Circle for any triangle. Create an Euler Segment and its related Nine-Point Circle, whose center is the midpoint of the Euler Segment.

You can watch a six-minute video of this segment and circle at: www.khanacademy.org/math/geometry/triangle-properties/triangle_property_review/v/euler-line.

The video shows a hand-drawn figure, but you can drag your dynamic figure to explore the relationships more accurately and dynamically.

Are you amazed at the complex relationships that this figure has? How can a simple generic triangle have all these special points with such complex relationships? Could this result from the dependencies that are constructed when you define the different centers in your custom tools?



12. Problem Solver Level

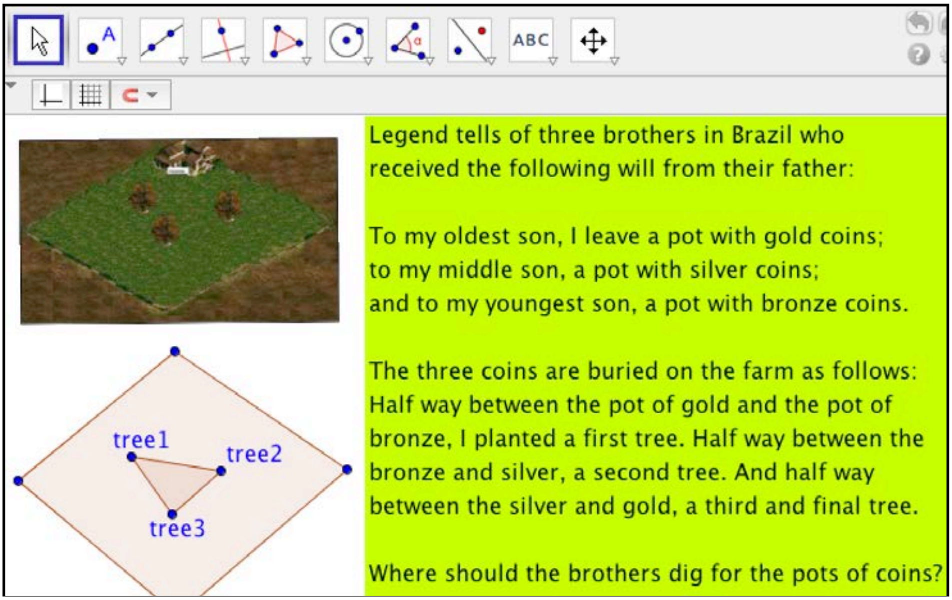
Here is a set of challenge problems for your team. Have fun!

By mastering the previous levels, your construction crew has learned many techniques useful for solving typical geometry problems.

If the team does not solve some of these during its session, try to solve them on your own and report your findings in the next team session.

Treasure Hunt

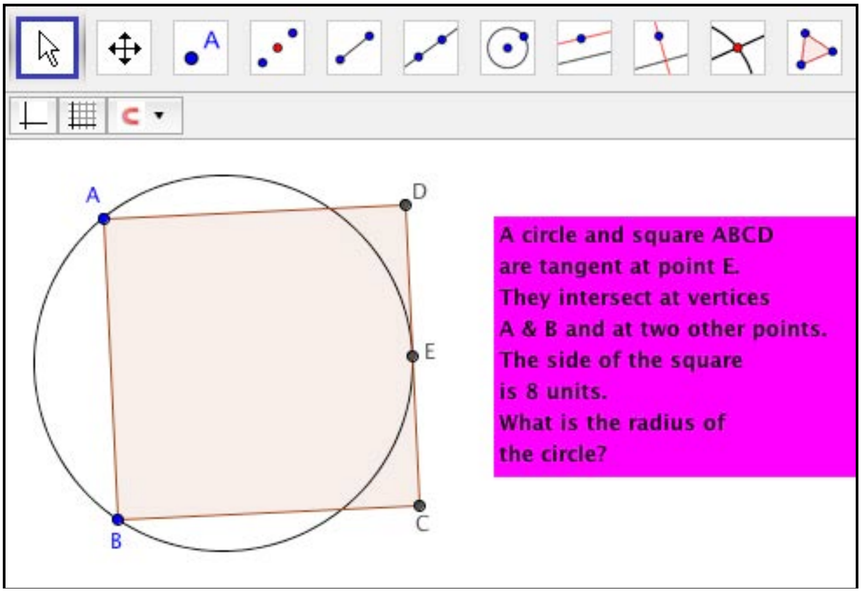
Can you discover the pot of gold in this tale told by Thales de Lelis Martins Pereira, a high school teacher in Brazil? You might want to construct some extra lines in the challenge.



Square and Circle

Can you determine the radius of the circle? You should not have to measure. What if the side of the square is "s" rather than "8"?

Hint: To solve this kind of problem, it is usually useful to construct some extra lines and explore triangles and relationships that are created. If you know basic algebra, you might set up some equations based on the relationships in the figure.

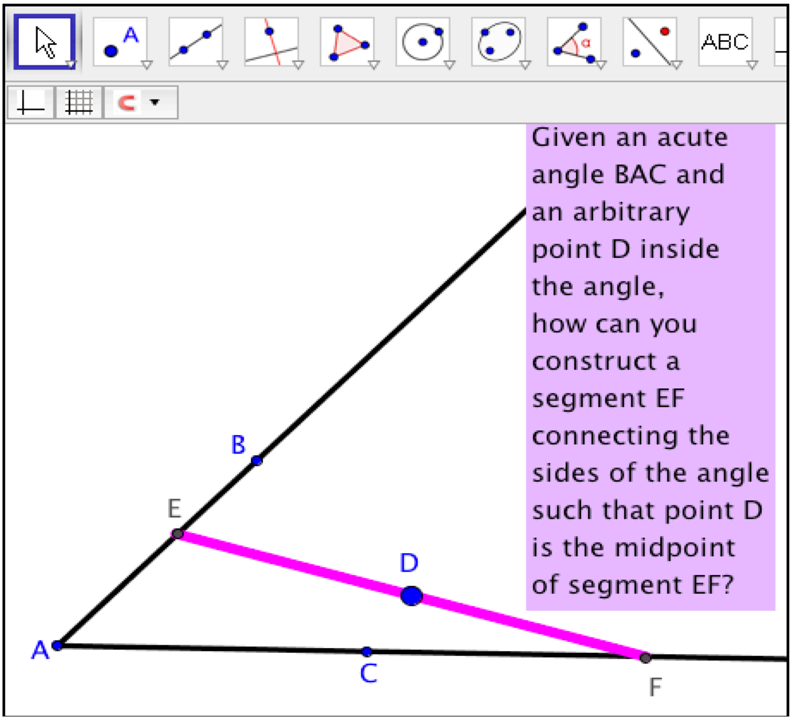


Cross an Angle

Can you construct the segment crossing the angle with the given midpoint?

Note that the angle and the location of point D are fixed. Given them, you must construct segment EF to have D as its midpoint.

Hint: This is a challenging problem. Try to add some strategic lines and drag the figure in the challenge to see what would help to construct the segment EF at the right place.



13. Expert Level

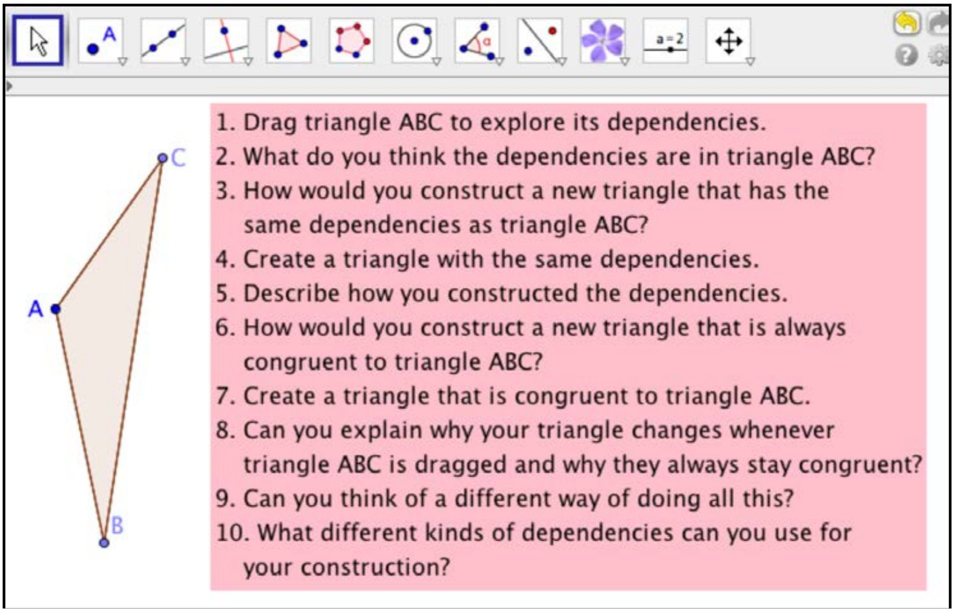
Here is a set of challenge problems for you or your construction crew to work on. Have fun!

They bring together the idea of designing dynamic-geometry constructions with dependencies.

Try to solve them and report your findings.

How Many Ways Can You Invent?

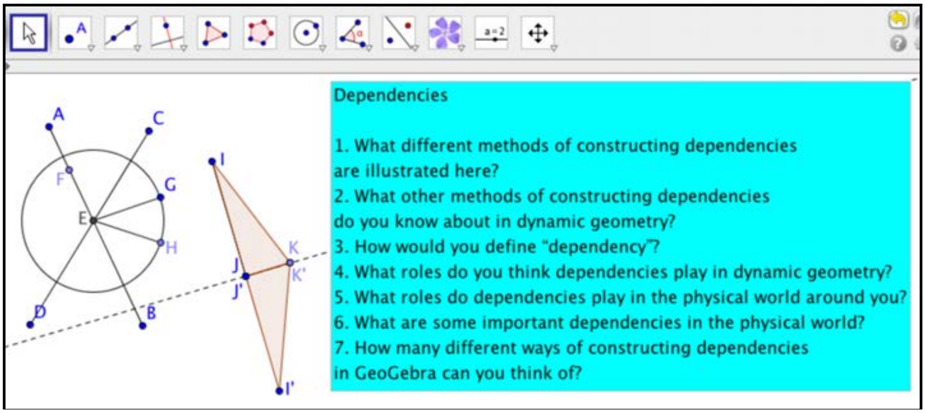
In this challenge, triangle ABC was constructed with one or more dependencies.



Dependencies in the World

The previous challenge statements have used the term "dependency" often. That is because dynamic geometry can best be understood in terms of the constraints or dependencies that are constructed into geometric figures.

Understanding dependencies in the world of physical objects and ideas is also important—especially in science, technology, engineering and mathematics. Designers of all kinds think about dependencies.

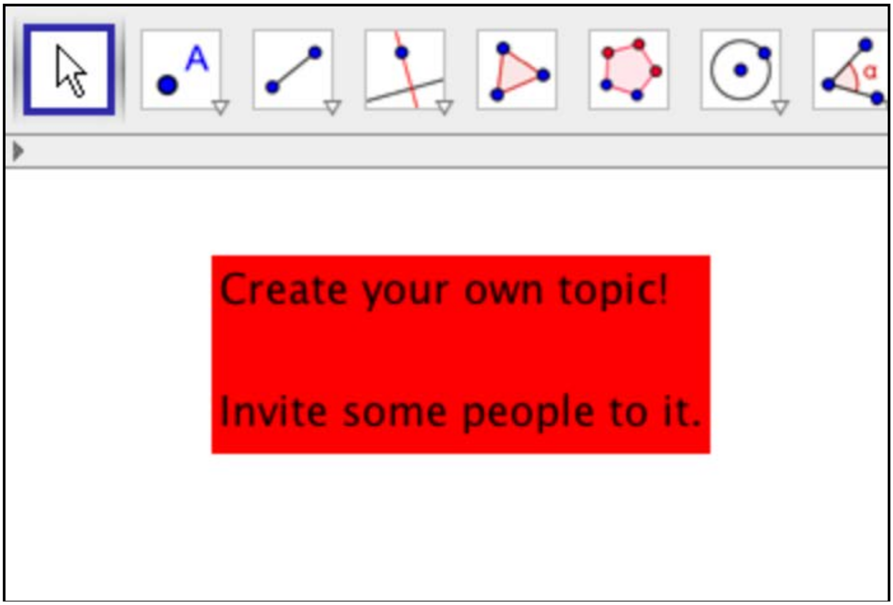


Into the Future

Congratulations on completing "The Construction Crew Game"!

If you and your construction crew have successfully completed these challenges, then you are ready to go off on your own—either as a team or individually—to explore geometry using GeoGebra. Actually, GeoGebra also supports algebra, 3-D geometry, trigonometry, pre-calculus, calculus and other forms of mathematics. So, you will be able to use it for the rest of your life as a tool for better understanding mathematical concepts and relationships.

You are welcome to continue to use VMT for collaborative events. You can create your own rooms and invite people to them to explore activities that you define. You can also download GeoGebra from www.GeoGebra.org to your computer, tablet or phone to use on your own.



Try it. Create a new VMT room. Make up a challenge for a group to do a construction together. Invite some people to join you in the room. Construct dynamic geometry together.

Enjoy!

2. Explore Dynamic Geometry Together

Introduction

Dynamic geometry is a new form of mathematics—and you can be a pioneer in it, exploring, discovering and creating new insights and tools. Dynamic geometry realizes some of the potential that has been hidden in geometry for thousands of years: by constructing dynamic-geometry figures that incorporate carefully designed mathematical relationships and dependencies, you can drag geometric objects around to investigate their general properties.

Geometry has always been about **constructing dependencies** into geometric figures and discovering **relationships** that are therefore **necessarily** true and provable. Dynamic geometry makes the construction of dependencies clear. The topics here will teach you to think about geometry this way and to design constructions with the necessary dependencies. The sequence of topics is designed to give you the basic knowledge to think about dynamic-geometric dependencies and to construct figures with them.

The activities in this workbook raise questions about geometric relationships in figures. You can answer them in terms of how a figure was **designed** and **constructed** with **dependencies** among its parts:

- Does that figure **have** to have that relationship?
- Does it **remain** true when dragged?
- Is the relationship **necessarily** true?
- How do you **know** if it is true?
- Can you demonstrate it or **construct** it?
- Can you understand, explain or **prove** why it has to be true?

In addition, the approach of these activities allows you to take advantage of the power of **collaboration**. Your “virtual math team” (VMT) can accomplish more than any one of you could on your own. You can chat about what you are doing, and why. You can discus what you notice and wonder about the dynamic figures. Working in a team will prevent you from becoming stuck. If you do not understand a geometry term or a task description, someone else in the team may have a suggestion. If you cannot figure out the next step in a problem or a construction, discuss it with your team. Decide how to proceed as a team.

To harness the truly awesome power of **collaborative dynamic geometry** requires patience, playfulness and persistence. It will pay off by providing you with skills, tools and understanding that will be useful for a lifetime. You will need to learn how to construct complicated figures; this will be tricky and require practice. You will need to think about the hidden dependencies among dynamic points that make geometry work; this may keep you up at night. You will even create your own custom construction tools to extend the GeoGebra software for dynamic geometry; this will put you in control of mathematics.

The following activities present a special approach to dynamic geometry, which may be quite different from other approaches to learning geometry. They focus specifically on the core idea of dynamic geometry: **how to design the construction of figures with dependencies**. They can help you understand geometry better, whether you already studied geometry, are studying it now or will study it in school later. They will give you an understanding of necessary relationships in geometric figures. Dragging points around in pre-constructed GeoGebra apps can help you to discover and visualize relationships within a figure. However, such apps may also hide the dependencies that maintain the relationships. You should know how to construct those dependencies yourself. Chatting about the topics in a small group will help you to think about geometry and problem solving on your own in the future.

Each of the topics here is designed for an online team to work on together for about one hour. The sequence of topics introduces you and your team to GeoGebra and guides you in the exploration of dynamic geometry. Several **tours** are included at the end of this document, which provide tutorials in important features of the VMT and GeoGebra software. The pages of this document can be used as **worksheets** to keep notes on and if the instructions in the chat rooms become erased.

Individual Warm-up Activity

You can start on this Warm-up activity by yourself at any time. Do this topic on the computer you will be using with your team—before your first collaborative group session. This will check that your computer is properly set up and that you are able to enter VMT chat rooms. It also introduces you to GeoGebra, our dynamic-geometry software system.

Here is a *general procedure* you might want to follow for each of the following topics: Before the time assigned for your group session, read the topic description in this document. It might suggest watching a brief video or taking a tour at the end of this document as important preparation. Think about the topic on your own. Then, in the group session, chat about the topic and work together on the various tasks. Discuss what you are doing in the chat and respond to the questions posed in the topic. You can mark down in your copy of this document what you noticed that surprised you and what you wondered about that you want to think more about later. Do not just rush though the topic; discuss what is important in it. The point is to learn about collaborative dynamic geometry, not just to get through the topic steps. When the session is over, try to work on your own on any parts that your team did not get to. Report to your team what you discover.

Each topic is designed for a team to spend about one hour discussing together. Some teams may want to spend more or less time on certain topics. Decide in your team if you want to go back to the chat room for a previous topic before continuing to a new topic. The rooms will remain available for you to go to either alone or with your team.

First, you can get a good introduction to using the GeoGebra software from this 5 minute ***YouTube clip****:* *http://www.youtube.com/watch?v=2NqblDIPl38. It gives a clear view of how to use GeoGebra. It also provides tips on GeoGebra tools that are equivalent to traditional geometry construction with a physical straightedge (for making line segments) and compass (for making circles).*

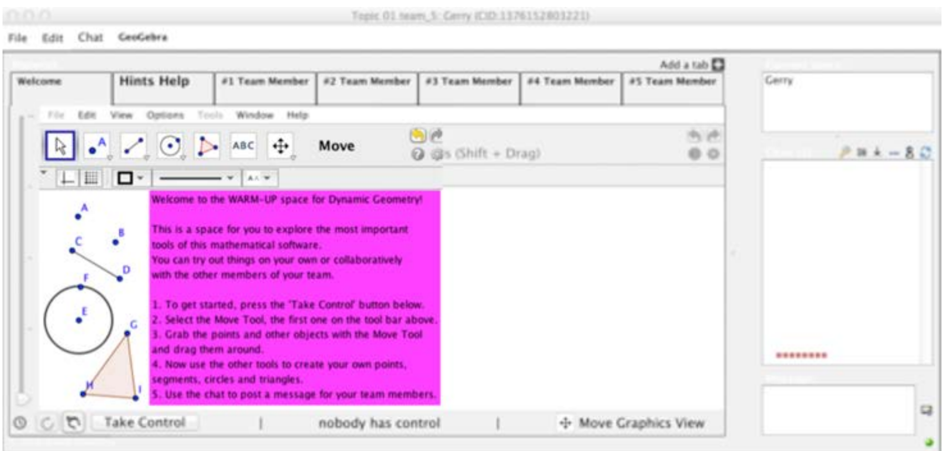
Then, read ***"Tour 1: Joining a Virtual Math Team."*** *This will introduce you to using the VMT software to register, login, find a chat room, etc. The Tours are all found toward the end of this booklet, after all the Topics.*

Welcome!

Start in the Welcome tab (shown below) of the Warm-Up “Topic 0” chat room by reading the instructions and then dragging the objects shown. Press the “Take Control” button at the bottom, then select the “Move Tool” arrow in the Tool Bar near the top and click on a point in the geometry figure to drag it.

Next, create your own similar geometry objects using the buttons in the Tool Bar. Create points, lines, circles and triangles. Drag them around and notice how they change. To create a triangle, use the polygon tool and click on three points to define the triangle and then click again on the first point to complete the triangle.

You can change to one of the other tabs, like “#1 Team Member” and press the “Take Control” button there to create your own points and lines in your own empty GeoGebra workspace. Try out each of the tools in the Tool Bar.



If there is not an unused tab available to create your points and lines in, you can create a new tab with the “Add a tab +” button in the upper right corner of the VMT window. You can use the “ABC” text tool to construct a text box with your name and a title for your figure: Select the text tool and then click in the tab about where you want to have the text box. A form will open for you to enter your text.

Helpful Hints

In the “Helpful Hints” tab, there are some hints for taking advantage of the limited space on your computer screen. Read these hints and adjust the size of your VMT window and its GeoGebra tab.

There are a number of Zoom Tools, which can be pulled down from the Move Graphic Tool (the crossed arrows on the right end of the Tool Bar).

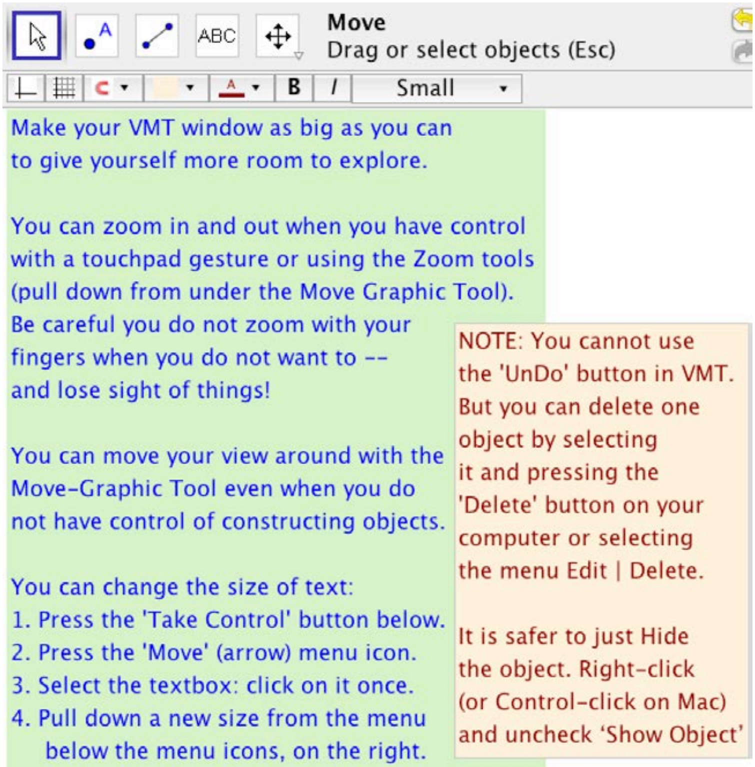
If you are using a touchpad on your computer, you can zoom with a two-finger touchpad gesture. **Caution**: It is easy to move things around without wanting to as your fingers move on the touchpad. Things can quickly zoom out of sight. This can happen even when you do not have control of construction. Then you will not see the same instructions and figures that other people on your team see.

Note: Most Zoom Tools will only effect what **you** see on your computer screen, **not** what your teammates see on theirs. It is possible for you to create points on an area of your screen that your teammates do not see. If this happens, ask everyone if they want you to adjust everyone's screen to the same zoom level and then select the menu item "GeoGebra" | "Share Your View."

When you work in a team, it is important to have everyone agree before changes are made that affect everyone. If you want to delete a point or other object – especially one that you did not create – be sure that everyone agrees it should be deleted.

Warning: Be very careful when **deleting** points or lines. If any other objects are dependent on them, those objects will also be deleted. It is easy to unintentionally delete a lot of your group's work.

Instead of deleting objects, you should usually **hide** them. Then the objects that are dependent on them will still be there and the dependencies will still be in effect. Use control-click (on a Mac) or right-click (in Windows) with the cursor on a point or line to bring up the context-menu. Select "Show Object" to unclick that option and hide the object. You can also use this context-menu to hide the object's label, rename it, etc.



You may want to **save** the current state of your GeoGebra tab to a .ggb file on your desktop sometimes so you can load it back if things are deleted. Use the menu "File" | "Save" to save your work periodically. Use the menu "File" | "Open" to load the latest saved version back into the current tab. Check with everyone in your team because this will change the content of the tab for everyone.

If the instructions in a tab are somehow erased, you can look them up in this document. You can also scroll back in the history of the tab to see what it looked like in the past. Finally, you can have your whole team go to a different room if one is available in the VMT Lobby that is not assigned to another team.

If you have **technical problems** with the chat or the figures in the tabs not showing properly, you should probably close your VMT window and go back to the VMT Lobby to open the room again.

You can use this booklet to **keep notes** on what you learn or wonder, either on paper or on your computer. Here is a space for notes on Topic 0:

Notes:

Creating Dynamic Points, Lines, Circles

Dynamic geometry is an innovative form of mathematics that is only possible using computers. It is based on traditional Euclidean geometry, but has interesting objects, tools, techniques, characteristics and behaviors of its own. Understanding dynamic geometry will help you think about other forms of geometry and mathematics.

In this topic, you will practice some basic skills in dynamic geometry. There are several tabs to work through in most topics; try to do them all with your team. Pace yourselves. Make sure that everyone in your team understands the important ideas in a tab and then have everyone move to the next tab.

*First, if you did not already do this, watch this **YouTube clip**:*

http://www.youtube.com/watch?v=2NqblDIPl38.

It gives a clear view of how to use GeoGebra. It provides important tips on GeoGebra tools that are equivalent to traditional geometry construction with a physical straightedge (for making line segments) and compass (for making circles).

*Also look through **"Tour 2: GeoGebra for Dynamic Math."** This will provide more details about the GeoGebra system for dynamic geometry. For instance, it will describe many of the buttons shown in the Tool Bar. You can always go back to these tours if you become stuck using VMT or GeoGebra.*

As your teammates and you work on the Topics, discuss in the chat explicitly what you *notice*, what it *means* to you and what you *wonder* about. Chat about what geometric *relationships* among the objects you notice. Afterward, list what you wonder about these relations. Right answers are not the main goal of these activities, so use only the ideas that your teammates and you jointly developed, not ideas from the Internet or elsewhere.

Working Collaboratively. Since the goal of this course is for you to engage your teammates in thinking together about mathematical objects and relations and to collaborate productively, you need to communicate effectively. As you interact in VMT, keep in mind the following general guidelines for collaborative work:

Read other people's chat postings to:	Write your own chat postings to:
• Be prepared to refer to and connect to someone else's ideas. • Get thoughts on open questions. • Get new perspectives.	• Make your thinking available for the group. • Develop your thinking. • Get feedback on your ideas. • Give feedback to others.

In general, try to include in you chat: (a) what you think should be done, (b) what you are doing and (c) the significance of what you did. Take turns doing steps. Work together as a group, rather than just trying to do the whole thing by yourself.

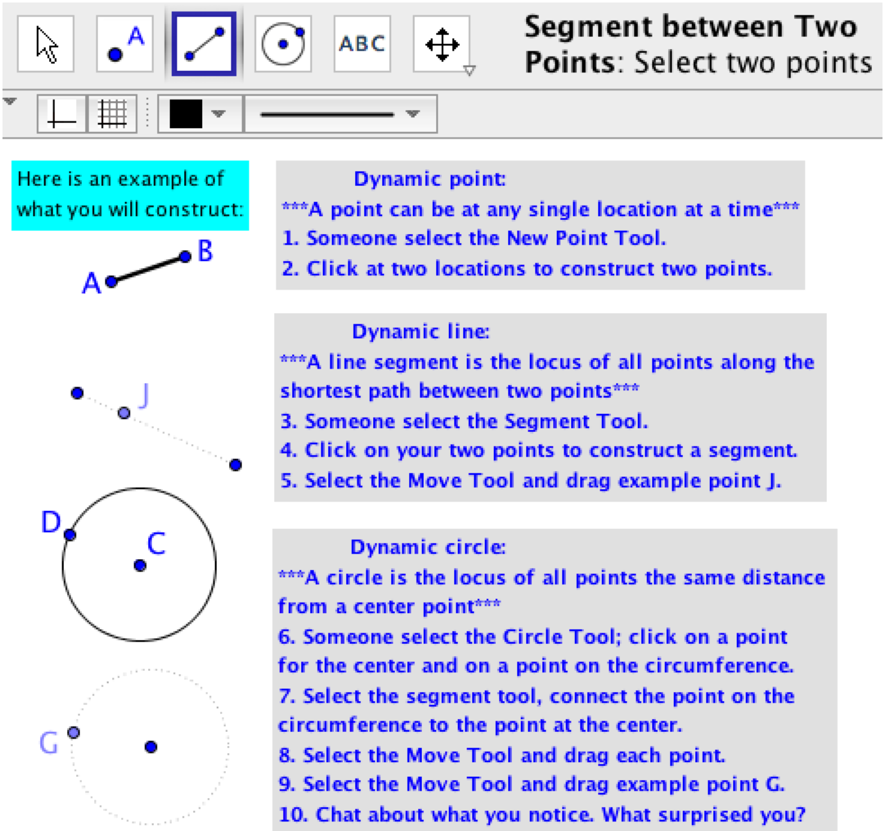
Dynamic Points, Lines & Circles

Geometry begins with a simple point. A point is just the designation of a particular location. In dynamic geometry, a point can be dragged from its current location to any other location.

Everything in geometry is built up from simple points. For instance, a line segment is made up of all the points (the "locus") along the shortest (direct, straight) path between two points (the endpoints of the segment).

A circle is all the points ("circumference," "locus") that are a certain distance ("radius") from one point ("center"). Therefore, any line segment from the center point of a circle to its circumference is a radius of the circle and is necessarily the same length as every other radius of that circle. Even if you drag the circle and change its size and the length of its radius, every radius will again be the same length as every other radius of that circle.

In this tab, create some basic dynamic-geometry objects and drag them to observe their behavior. Take turns taking control and creating objects like the ones you see.



Don't forget that you have to press the "Take Control" button to do actions in GeoGebra. Chat about who should take control for each step. Be sure you "Release Control" when you are done so someone else can take control.

When you drag point J between the two points, the locus of a line segment will be colored in. (This locus will only appear on your computer screen, so everyone in the team has to try it themselves).

The same for dragging point G around the locus of the circle.

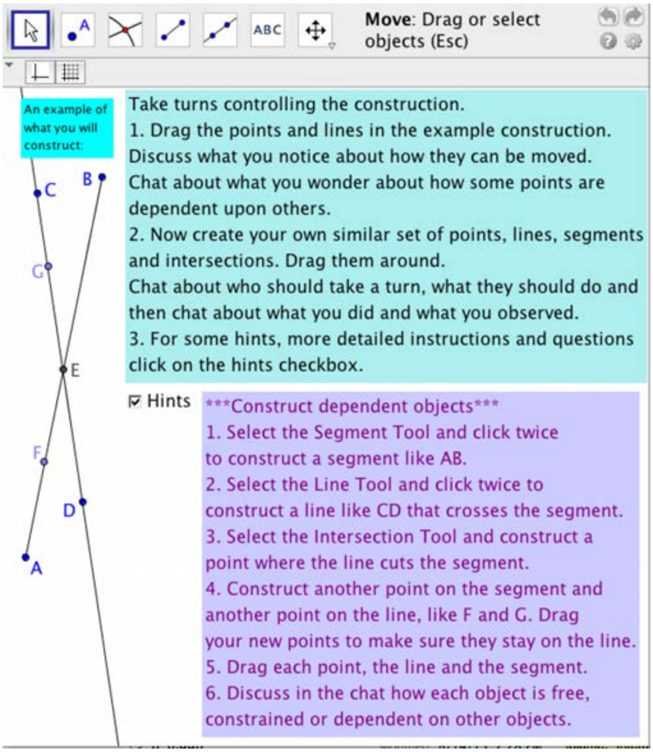
With these simple constructions, you are starting to build up, explore and understand the system of dynamic geometry. You started with simple points and now you have line segments and circles too.

Note: You can change the "properties" of a dynamic-geometry object by first Taking Control and then control-clicking (on a Mac computer: hold down the "control" key and click) or right-clicking (on a Windows computer) on the object. You will get a pop-up menu. You can turn the Trace (locus) on/off, show/hide the object (but its constraints still remain), show/hide its label information, change its name or alter its other properties (like color and line style). Try these different options.

Dynamic Dragging

When you construct a point to be on a line (or on a segment, or ray, or circle) in dynamic geometry, it is constrained to stay on that line; its location is dependent upon the location of that line, which can be dragged to a new location. Use the "**drag test**" to check if a point really is constrained to the line: select the Move tool (the first tool in the Tool Bar, with the arrow icon), click on your point and try to drag it; see if it stays on the line. Drag an endpoint of the line. Drag the whole line. What happens to the point?

Take control and construct some lines and segments with some points on them, like in the example shown in the tab. Notice how some points can be dragged **freely**, some can only be dragged in certain ways (we say they are partially "**constrained**") and others cannot be dragged directly at all (we say they are fully "**dependent**").



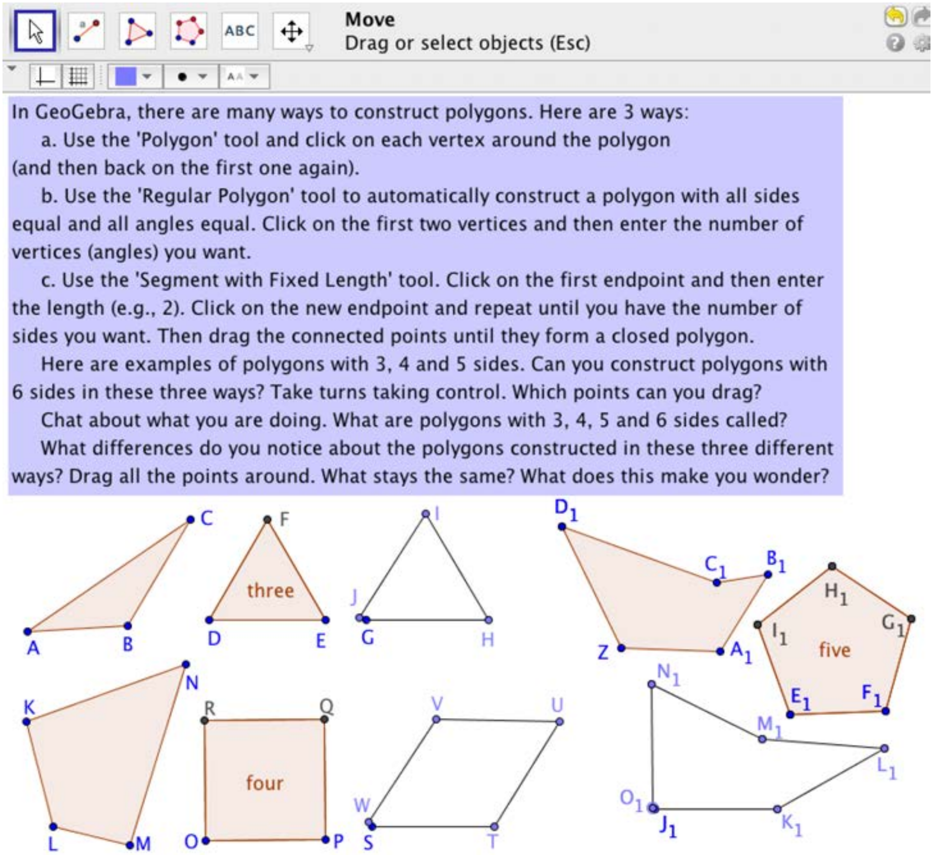
In GeoGebra, there are different ways to do things:

1. Construct a segment like AB by first constructing the two points with the point tool and then connecting them with the segment tool.
2. Then construct a segment by clicking at two locations with the segment tool.
3. Create a new point on the segment.
4. Create a new point off the segment and try to drag it onto the segment.
5. Create two segments that intersect and use the intersection tool to construct a point at the intersection.
6. Create two segments that intersect and use the point tool to construct a point at the intersection.
7. Create a new point off the segments and try to drag it onto the intersection.
8. Always use the drag test by dragging objects around to make sure they behave like you want them to.

Enter in your (paper or digital) copy of this document a summary of what you and your team noticed and wondered during your work on this Topic. You can also chat about these thoughts with your team. This will be a valuable record of your work. You may want to come back and think more about these entries later.

Extra: Dynamic Polygons

If you have time, try this. It shows different ways to construct figures with several sides (polygons). Use the "**drag test**" to check how the different methods make a difference in the dependencies of the lengths of the sides. Then use the three methods to make your own figure with six sides.



If you did not have enough time to finish exploring this section, you can always come back later to this chat room—either with your team or by yourself.

What we noticed:

What meaning we give to what we noticed:

What we wondered:

Copying Line Segments

In this topic, you will learn the important technique of copying a length in dynamic geometry using the **compass tool**. This technique is tricky, but it is used a lot in dynamic geometry. Make sure that everyone in your team can do it and that everyone understands what it does.

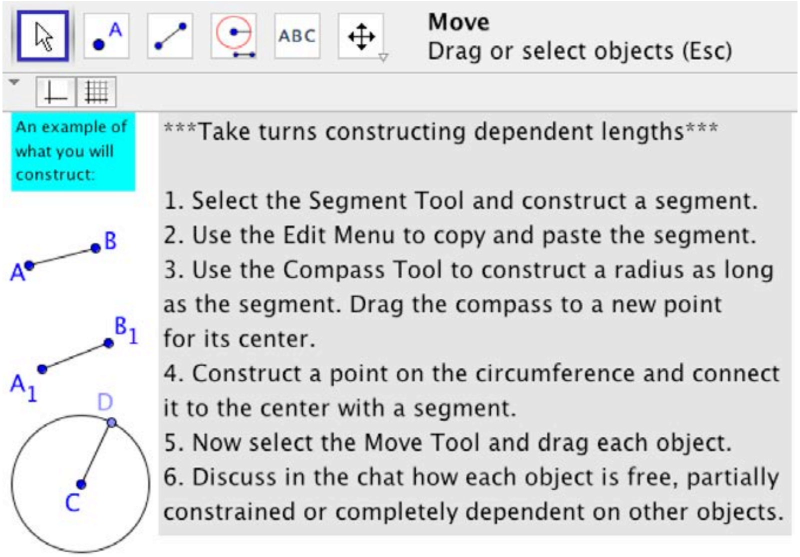
First, get a good introduction to using the GeoGebra ***compass tool*** *from this YouTube clip:*
http://www.youtube.com/watch?v=AdBNfEOEVco.

Also look through ***"Tour 3: VMT to Learn Together Online."*** *This will provide more details about using VMT to collaborate in your team.*

Copying Compass Circles versus Copy-and-Paste

Constructing one segment to be the same length as another segment is different from just **copying** the segment. The compass tool can be used to construct a segment whose length is **dependent** on the length of another segment.

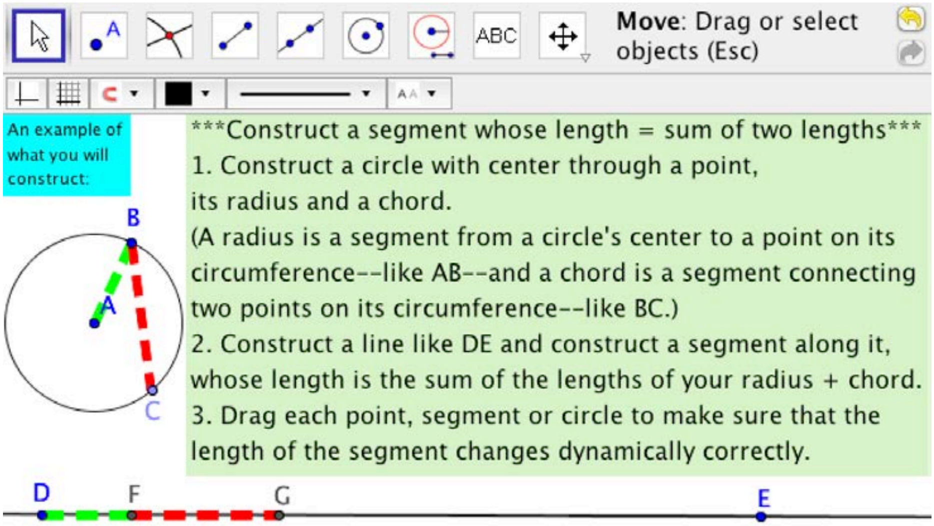
Compare copying a segment length with Copy-and-Paste to copying the same segment using the compass tool. First construct a segment like AB. Then select the segment and use the Edit menu to copy it and paste it to create a copy like A_1B_1. Next use the compass tool, first define its radius (by clicking on points A and B) and then locate a center C for it. Drag point A to change the length of segment AB. Does the copy A_1B_1 made with copy-and-paste change its length automatically? Does the radius CD change its length automatically when AB is changed? Why do you think this happens?



Adding Segment Lengths

In dynamic geometry, you can construct figures that have complicated dependencies of some objects on other objects. Here you will construct one segment whose length is dependent on the length of two other segments.

Use the compass tool to copy the lengths of the line segments. Using the compass tool requires practice. Creating line segment length DG = AB + BC provides a good visual image when you drag point B.



Discuss with your team how to do each step, especially step 2. Do you see how you can use the compass tool to lay out segments with given lengths (like AB and BC) along a given line (like DE)? Discuss with your teammates the difference between the circle tool and the compass tool and let them add their ideas.

Enter below a summary of what you and your team noticed and wondered during your work on this Topic. Chat about these thoughts with your team.

What we noticed:

What meaning we give to what we noticed:

What we wondered:

Constructing an Equilateral Triangle

The construction of an equilateral triangle illustrates some of the most important ideas in dynamic geometry. With this topic, you will explore that construction.

*Before working on this topic with your team, it could be helpful to watch a brief **YouTube clip** that shows clearly how to construct an equilateral triangle:*

http://www.youtube.com/watch?v=ORIaWNQSM_E

*Also look through **"Tour 4: GeoGebra Videos and Resources."***

Here are some suggestions that may help your team collaborate:

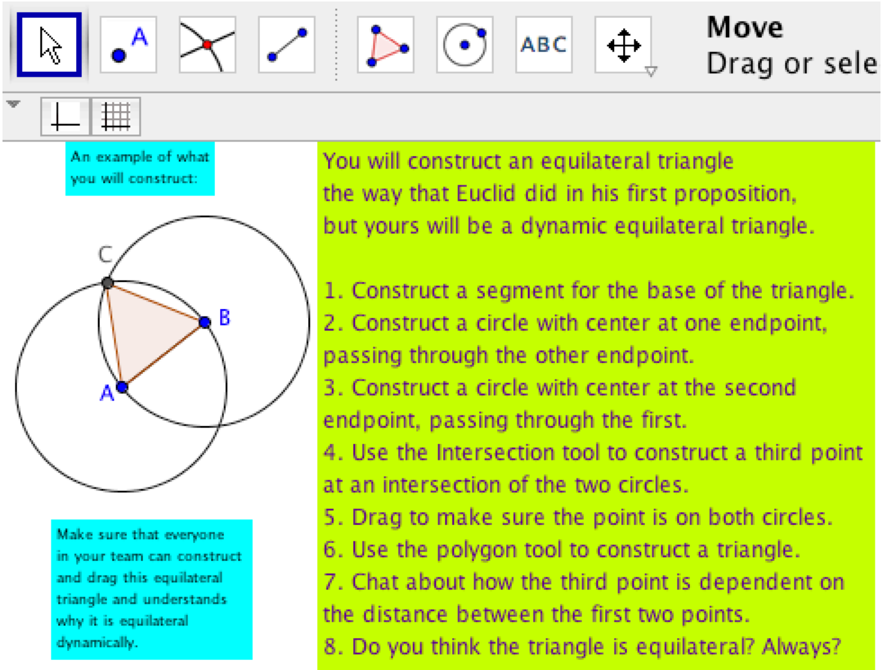
- Discuss things and ask questions.
- Include everyone's ideas.
- Ask what your team members think and what their reasons are.
- Cooperate to work together.
- Listen to each other.
- Agree before deciding.
- Make sure all of the ideas are on the table.
- Try out the ideas put forth, no matter how promising or relevant.
- Voice all doubts, questions and critiques.
- Ensure everyone's contributions are valued.
- Decide what to focus on, have ways of keeping track of and returning to other ideas or questions and use multiple approaches.
- Be sure that each team member knows how to construct figures in each task.

Constructing an Equilateral Triangle

When Euclid organized ideas and techniques of geometry 2,300 years ago, he started with this construction of an equilateral triangle, whose three sides are constrained to always be the same lengths as each other and whose three angles are always equal to each other. This construction can be considered the starting point of Euclidean and dynamic geometry.

Have everyone in your team work on this. It shows—in one simple but beautiful example—the most important features of dynamic geometry. Using just a few points, segments and circles (strategically related), it constructs a triangle whose sides are always equal no matter how the points, segments or circles are dragged. Using just the basic definitions of geometry—like that the points on a circle are all the same distance from the center—it proves that the triangle must be equilateral (without even measuring the sides or the angles).

Your team should construct an equilateral triangle like the one already in the tab. Drag the one that is there first to see how it works. Take turns controlling the GeoGebra tools.



Euclid argued that both of the circles around centers A and B have the same radius, namely AB. The three sides of triangle ABC are all radii of these two circles. Therefore, they all have the same length. Do you agree with this argument (proof)? Are you convinced that the three sides of ABC have equal lengths – *without having to* *measure them*? If you drag A, B, or C and change the lengths of the sides are they *always* still equal? Do they *have* to be equal? Why or why not?

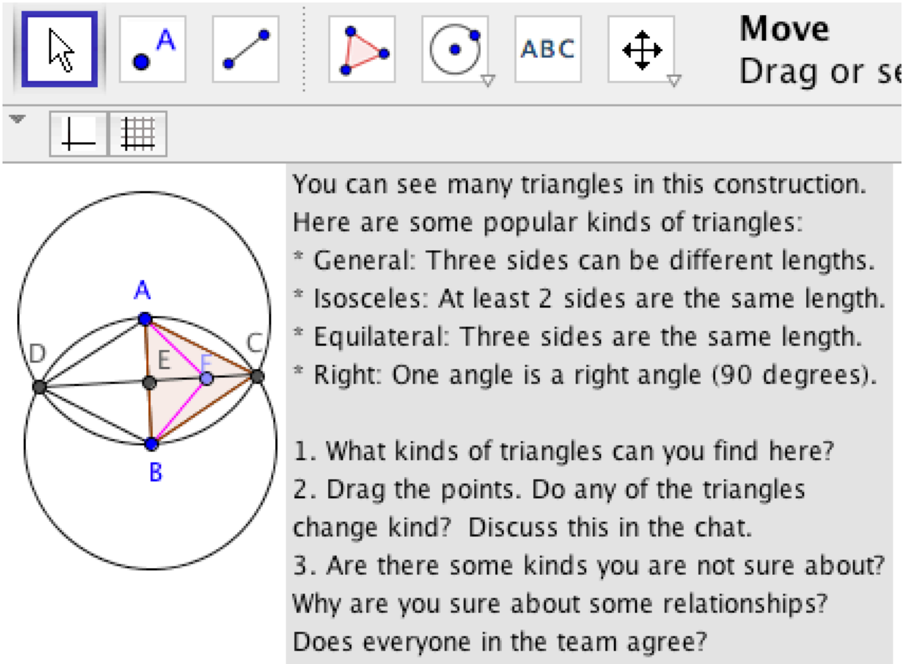
Euclid **designed** the construction so that the triangle ABC would necessarily be equilateral. He designed it so point C would be **dependent** on points A and B in a way that the distances from C to A and from C to B would both always be the same as the distance from A to B, making the side lengths all equal. He did that by locating point C at the intersection of circles centered on A and B with radii equal to the length of AB. Were you able to construct a triangle with that constraint? Do you see any other points that are equal distances from A and B?

Where Are Perpendicular and Right Angles?

Let us look more closely at the relationships that are created in the construction of the equilateral triangle. In the next tab, more lines are drawn in. Explore some of the relationships that are created among line segments in this more complicated figure. What line segments do you think are equal length – without having to measure them? What angles do you think are equal without having to measure them? Try dragging different points; do these equalities and relationships stay dynamically? Can you see how the construction of the figure made these segments or angles equal?

Can you find different kinds of triangles in this construction? If a triangle always has a certain number of sides or angles equal, then it is a special kind of triangle. We know the construction of this figure defined an equilateral triangle, ABC. What other kinds of triangles did it define?

What kinds of angles can you find? Are there right angles? Are there lines perpendicular to other lines? Are they always that way? Do they have to be? Can you explain why they are?



What we noticed:

What meaning we give to what we noticed:

What we wondered:

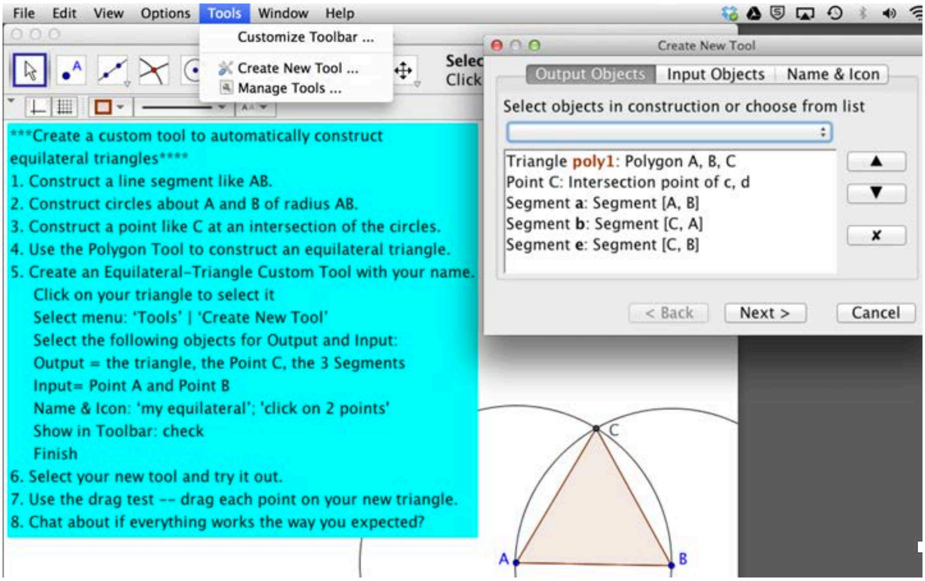
Programming Custom Tools

For constructing geometric figures and for solving typical problems in geometry, it is useful to have tools that do things for you, like construct midpoints of segments, perpendiculars to lines, and parallel sets of lines. GeoGebra offers about 100 tools that you can use from the tool bar, input bar or menu. However, you can also create your own custom tools to do additional things—like copy an angle, construct an isosceles triangle or locate a center of a triangle. Then you can build your own mini-geometry using a set of your own custom tools—like defining a complicated figure using custom tools for several of its parts.

Furthermore, programming your own tools gives you a good idea about how GeoGebra's standard tools were created and why they work the way they do. By building up tools from the basic point, line and circle tools, you will understand better how geometry and its procedures work.

Creating an Equilateral-Triangle Tool

Create a specialized custom tool for quickly generating equilateral triangles with the menu "**Tools**" | "**Create New Tool …**"



GeoGebra makes programming a tool easy. However, it takes some practice to get used to the procedure.

To program a tool, you have to define the Outputs you want (the points, lines, etc. that will be created by the custom tool) and then the Inputs that will be needed (the points, lines, etc. that a person will have to create to use the tool). For instance, to create a triangle using this custom tool, you will first select the custom tool as the active tool. Next, you will construct or select two points to define the base of the triangle, AB, as inputs to the custom tool. Then the triangle line will appear automatically (as the output of the custom tool).

Hint: You can identify the output objects by selecting them with your cursor before or after you go to the menu "Tools" | "Create New Tool …". Hold down the Command key (on a Mac) or the Control key (in Windows) to select more than one object. You can also identify the output objects from the pull-down list in the Output Objects tab. Similarly, you can identify input objects in the Input Objects tab by selecting them with the cursor or from the pull-down list. GeoGebra might identify most of the necessary input objects automatically. Give the tool a name that will help you to find it later and check the "Show in Toolbar" box so your tool will be included on the toolbar.

Important: To be able to use your custom tool in another tab or another chat room later, you have to save it now. Save your custom tool as a **.ggt** file on your desktop. Take control and use the GeoGebra menu "Tools" | "Manage Tools…" | "Save As." Save your custom tool to your computer desktop, to a memory stick, or somewhere that you can find it later and give it a name like "Pat's_Equilateral.ggt" so you will know what it is. When you want to use your custom tool in another tab or topic, take control, use the GeoGebra menu "File" | "Open…" then find and open the .ggt file that you previously saved. You should then be able to select your custom tool from the toolbar or from the menu "Tools" | "Manage Tools." When your custom tool is available to you, it will also be available to your teammates when they are in that tab.

Hint: If a custom tool does not appear on your tool bar when you think it should be available, use the menu "Tools" | "Customize Toolbar", find the custom tool in the list of Tools, and insert it on the toolbar list where you want it (highlight the group or the tool you want it to be listed after).

Note: The custom tools in GeoGebra have some limitations, unfortunately. Not all outputs of custom tools are completely dynamic. For instance, if you define a right-triangle tool, you will not be able to freely drag the new vertex of the right triangles that are created with this tool.

Hint: The GeoGebra help page for custom tools at http://wiki.geogebra.org/en/Custom_Tools has more information about defining and using custom tools. Most of that information applies when using VMT-with-GeoGebra as well.

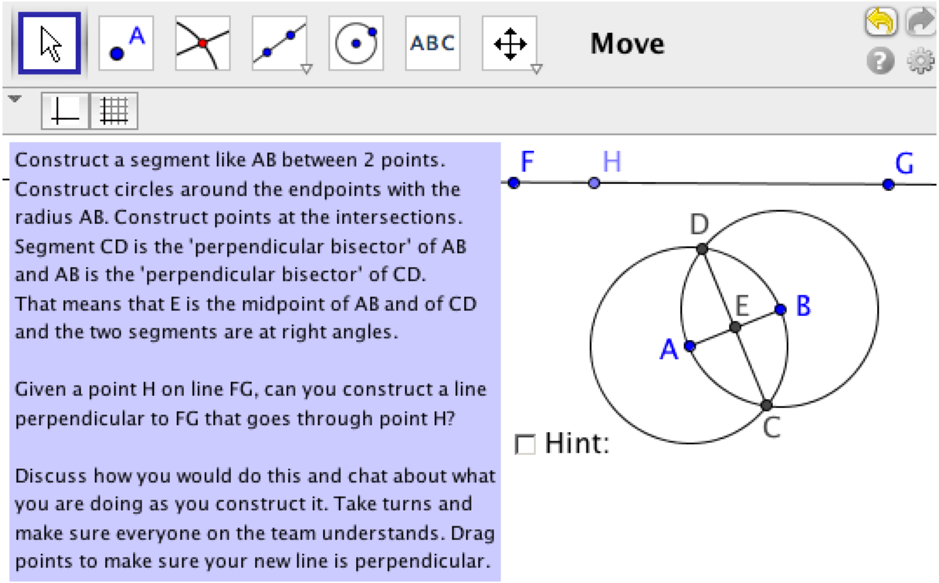
Creating a Bisector Tool

The procedure used to construct an equilateral triangle can be used to locate the midpoint of a segment and to construct a perpendicular to that segment, passing through the midpoint.

As you already saw, the construction process for an equilateral triangle creates a number of interesting relationships among different points and segments. In this tab, points A, B and C form an equilateral triangle. Segment AB crosses segment CD at the exact **midpoint** of CD and the angles between theses two segments are all right angles (90 Degrees). We say that AB is the "**perpendicular bisector**" of CD—meaning that AB cuts CD at its midpoint, evenly in two sectors, and that AB is perpendicular (meaning, at a right angle) to CD.

Can your team create a custom tool to find a midpoint of a segment?

For many geometry constructions, it is necessary to construct a new line perpendicular to an existing line (like line FG). In particular, you may need to have the perpendicular go through the line at a certain point (like H). Can your team figure out how to do that?



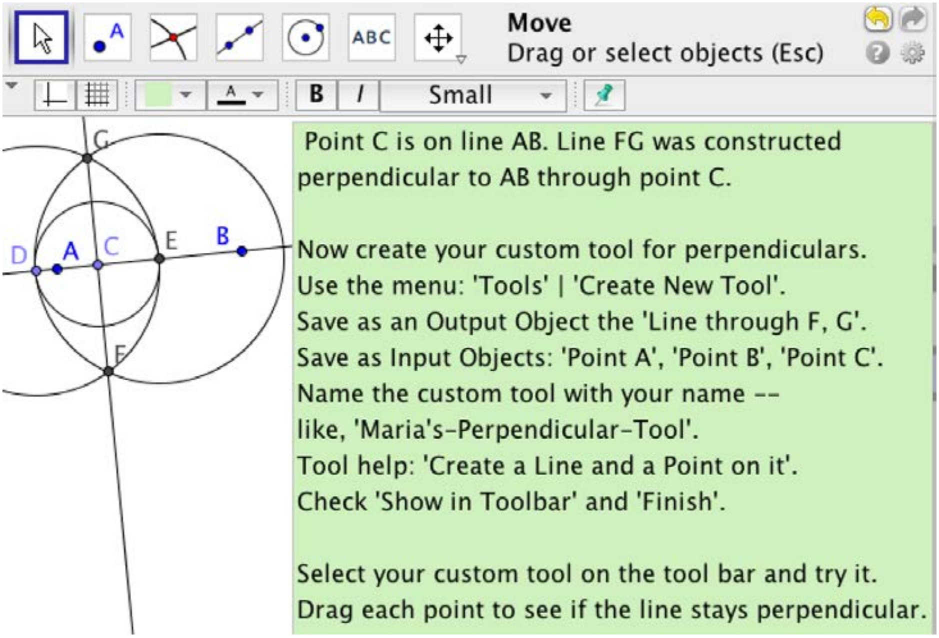
Creating a Perpendicular Tool

Now that you know how to construct a perpendicular, you can automate this process to save you work next time you need a perpendicular line. Create a custom tool to automatically construct a perpendicular to a given line through a given point by following the directions in the tab.

Members of the team should each create their own custom perpendicular tool. One person could create a tool to construct a perpendicular bisector through the midpoint of a given line (no third point would be needed as an input for this one). Another person could create a perpendicular through a given point on the line. A third person could create a perpendicular through a given point that is not on the line. Everyone should be able to use everyone else's custom tool in this tab. Do the three tools have to be different? Does GeoGebra have three different tools for this? Do your custom tools work just like the GeoGebra standard perpendicular tools? Are there other cases for constructing perpendiculars?

GeoGebra has a perpendicular tool that works like your custom tool. If you just used the standard tool, you would not be aware of the hidden circles that determine the dependencies to keep the lines perpendicular during dragging. Now that you understand these dependencies, you can use either the standard tool or your custom tool. You will not see the hidden circles maintaining the dependencies of lines that are dynamically perpendicular, but you will know they are there, working in the background.

The tools of GeoGebra extend the power of dynamic geometry while maintaining the underlying dependencies. By defining your own custom tools, you learn how dynamic geometry works "under the hood." In addition, you can extend its power yourself in new ways that you and your team think of.

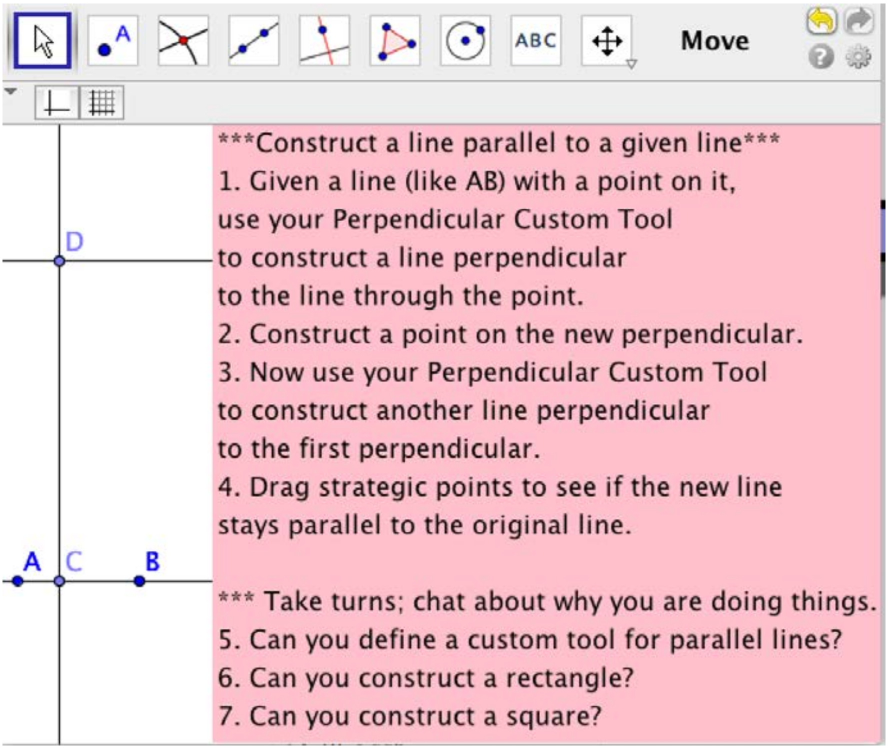


Can you explain why points D and E were constructed? They are designed to form the base of equilateral triangles DEF and DEG, such that line FG goes through point C. How are points D and E dependent on C in a way that makes point C the midpoint of segment DE? Is this all necessary to construct a perpendicular to AB through C? What would have to be done differently if point C was not on line AB? What could be done differently if you wanted the perpendicular to go through the midpoint of segment AB?

Creating a Parallel Line Tool

One member of the group should create a custom perpendicular tool (like you did in the previous tab) in this tab or load a custom tool from the previous tab's work. Now use this custom perpendicular tool (or the standard perpendicular tool) to create a custom parallel tool. See how tools can build on each other to create a whole system of new possible activities.

Note that if line CD is perpendicular to line AB, then any line perpendicular to line CD will be parallel to line AB. Imagine a rectangle (with all right angles). Its opposite sides are parallel to each other and perpendicular to the other sides, so the opposite sides are perpendicular to their perpendiculars.



What we noticed:

What meaning we give to what we noticed:

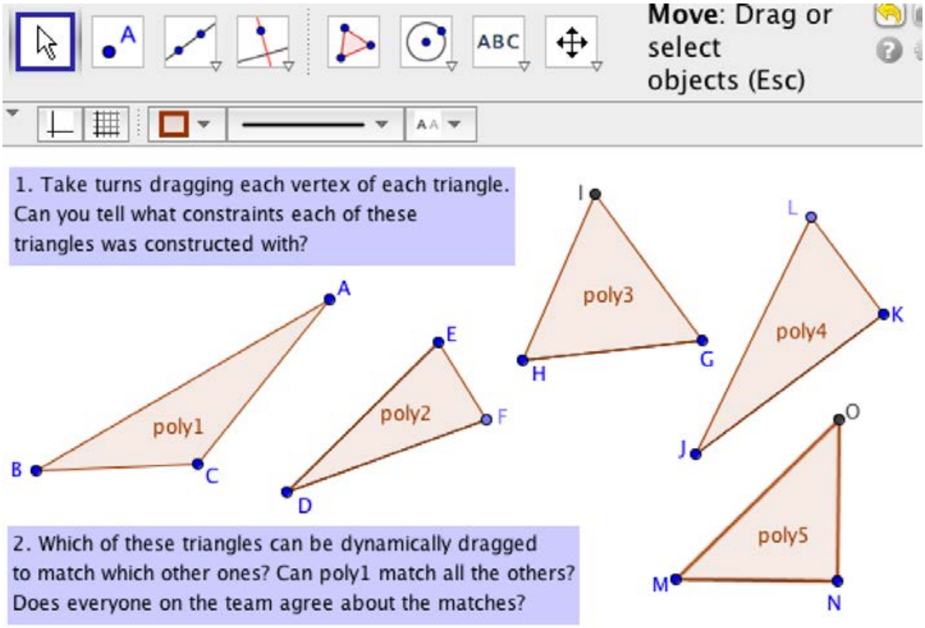
What we wondered:

Constructing Other Triangles

A triangle is a relatively simple geometric construction: simply join three segments at their endpoints. Yet, there are many surprising and complex relationships possible in triangles with different dependencies designed into them.

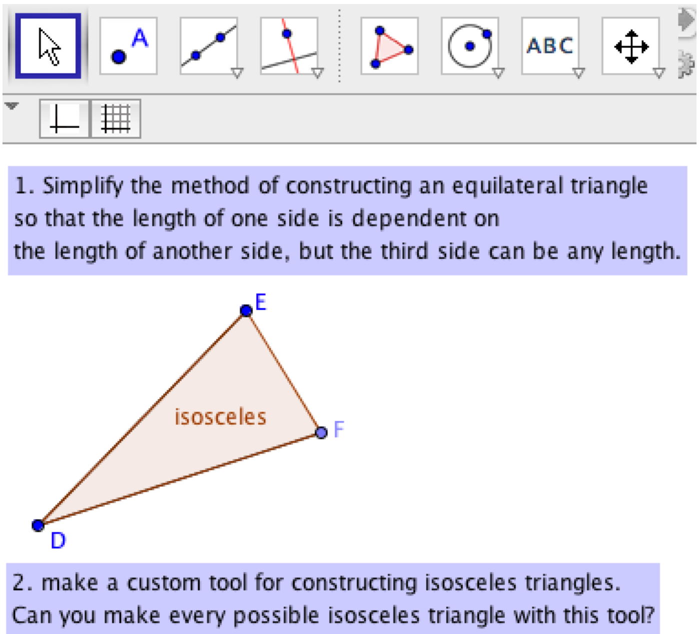
Triangles with dependencies

Here are some triangles constructed with different dynamic constraints. See if your team can figure out which ones were constructed to always have a certain number of equal sides, a certain number of equal angles and/or a certain number of right angles. In dynamic geometry, sometimes one triangle can be dragged to look like another type of triangle, but it does not have the same necessary relationships. For instance, a scalene triangle (with no special constraints) can be dragged to look like an equilateral triangle, but it does not have the lengths of its sides dependent on each other.



Constructing an Isosceles Triangle

An isosceles triangle has two sides that are always the same length. Can you simplify the method of constructing an equilateral triangle so that the length of one side is dependent on the length of another side, but the third side can be any length? Can your team make a custom tool for constructing isosceles triangles? Can you make every possible isosceles triangle with this tool?



Constructing a Right Triangle and a Right-Isosceles

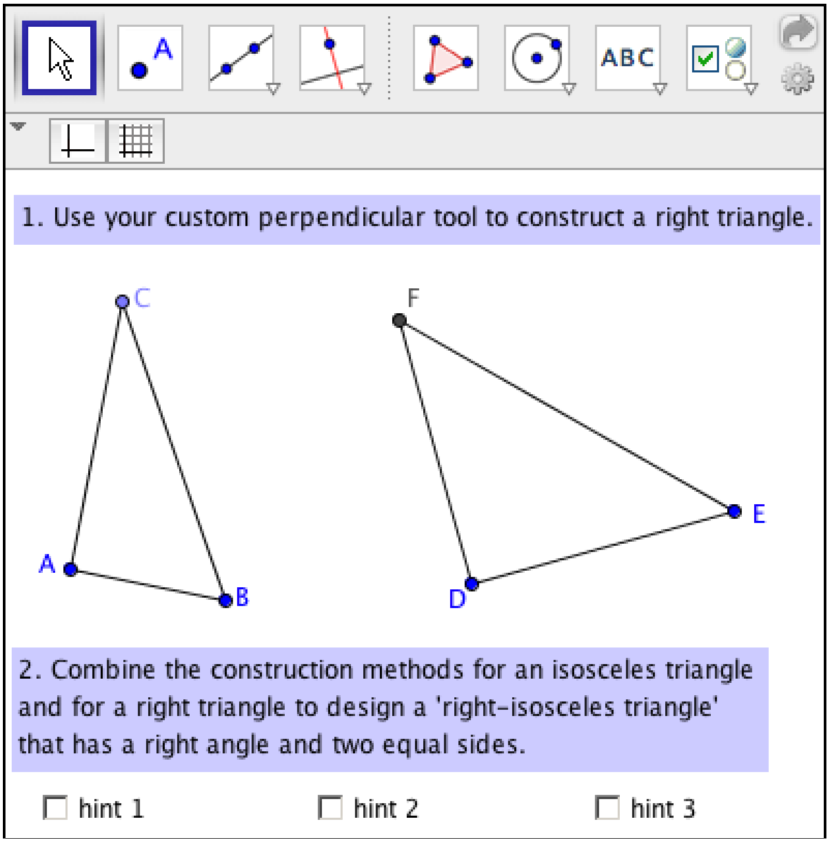
Use your custom perpendicular tool to construct a right triangle.

Then combine the construction methods for an isosceles triangle and for a right triangle to design a triangle that has a right angle and two equal sides.

Be sure to take turns controlling the GeoGebra tools and chat about what you are doing. Work together—do not try to solve something yourself and then explain it to your team mates. Talk it out in the chat:

- Discuss what to do for each step in the chat.
- Do it in the GeoGebra tab.

- Discuss what you did and how it worked.
- Make sure everyone in the team understands what was done and can do it themselves.
- Discuss what you noticed and what you wondered about the construction.
- Discuss why it worked: why is the triangle you constructed always isosceles or always right-angled no matter how you drag it?



What we noticed:

What meaning we give to what we noticed:

What we wondered:

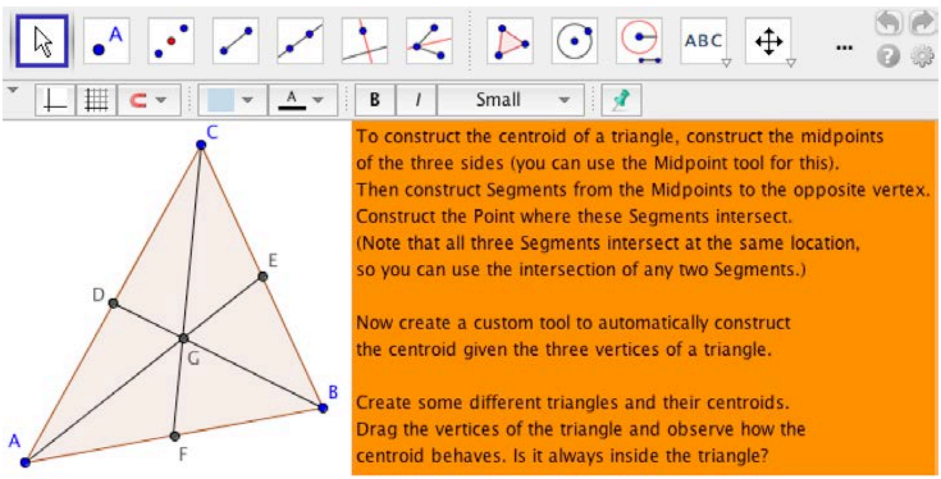
Constructing Tools for Triangle Centers

There are a number of interesting ways to define the "center" of a triangle, each with its own interesting properties.

The Centroid of a Triangle

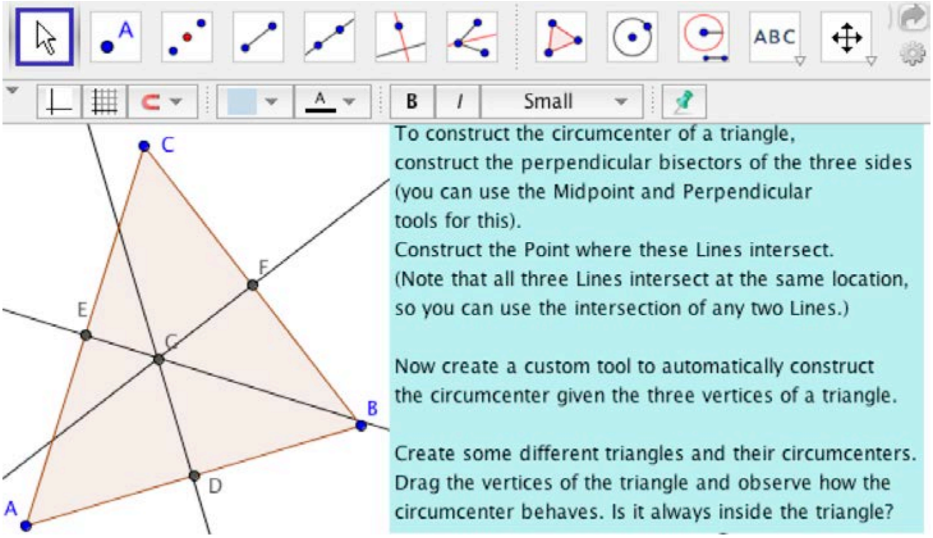
The "centroid" of a triangle is the meeting point of the three lines from the midpoints of the triangle's sides to the opposite vertex. Create a custom centroid tool.

Take control and use the GeoGebra menu "Tools" | "Manage Tools…" | "Save As." Save your custom tool to your computer desktop or somewhere that you can find it later and give it a name like "Tasja's_Centroid.ggt" so you will know what it is. When you want to use your custom tool in another tab or topic, take control, use the GeoGebra menu "File" | "Open…" then find and open the .ggt file that you saved. You should then be able to select your custom tool from the tool bar or from the menu "Tools" | "Manage Tools." When your custom tool is available to you, it will also be available to your teammates when they are in that tab.



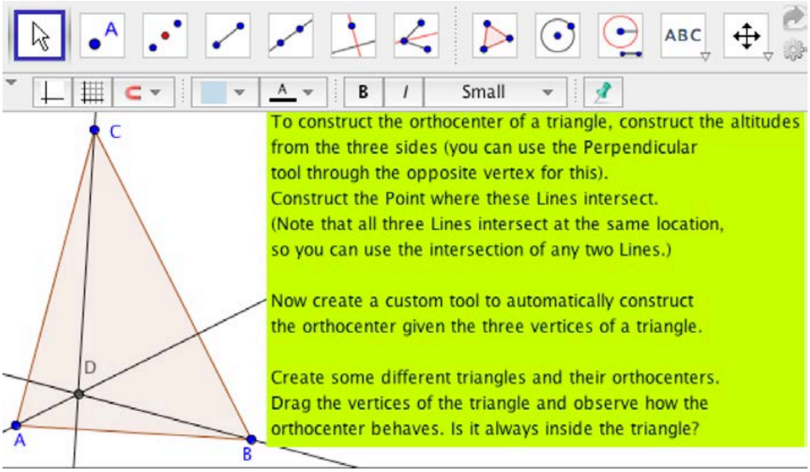
The Circumcenter of a Triangle

The "circumcenter" of a triangle is the meeting point of the three perpendicular bisectors of the sides of the triangle. Create a custom circumcenter tool and save it.



The Orthocenter of a Triangle

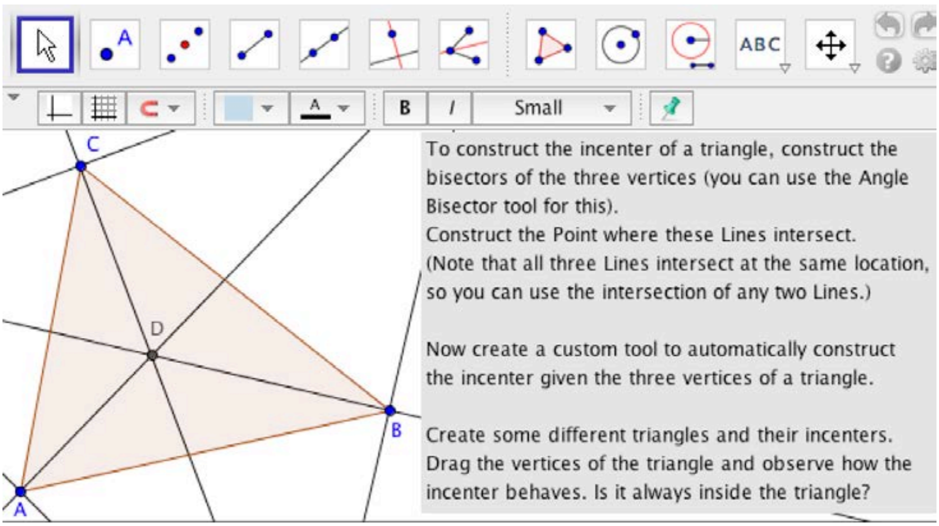
The "orthocenter" of a triangle is the meeting point of the three altitudes of the triangle. An "altitude" of a triangle is the segment that is perpendicular to a side and goes to the opposite vertex. Create a custom orthocenter tool and save it.



The Incenter of a Triangle

The "incenter" of a triangle is the meeting point of the three angle bisectors of the angles at the triangle's vertices. Create a custom incenter tool and save it.

Note: The incenter of a triangle is the center of a circle inscribed in the triangle. A radius of the inscribed circle is tangent to each side of the triangle, so you can construct a perpendicular from the incenter to a side to find the inscribed circle's point of tangency – and then use this point to construct the inscribed circle.



What we noticed:

What meaning we give to what we noticed:

What we wondered:

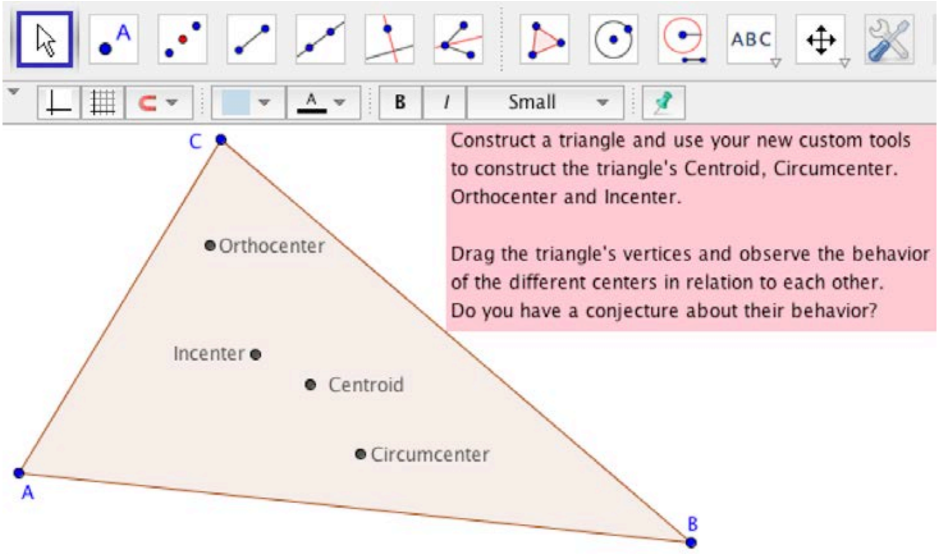
Exploring the Euler Segment and Circle

The relationships presented in this topic seem quite surprising. However, they are results of the dependencies imposed in the triangle by the constructions of the different centers. Sometimes combinations of complex dependencies have surprising results.

The Euler Segment of a Triangle

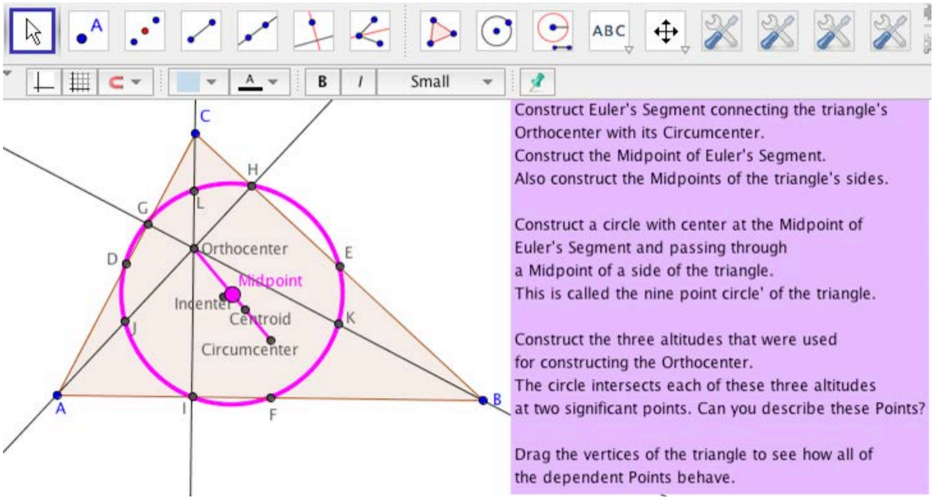
A Swiss mathematician named Euler (his last name is pronounced "oiler") discovered a relationship among three of the centers that you created custom tools for. Can you discover what he did? He did this in the 1700s—without dynamic-geometry tools. Euler's work renewed interest in geometry and led to many discoveries beyond Euclid's.

Note: Take turns to re-create a custom tool for each of the triangle's special points: centroid, circumcenter, orthocenter and incenter, as done in the previous tabs. Or else, load the custom tools you created before using the GeoGebra menu "File" | "Open…" then find and open the .ggt files that you saved. You should then be able to select your custom tools from the tool bar or from the menu "Tools" | "Manage Tools." When your custom tools are available to you, they will also be available to your teammates in that tab.



The Nine-Point Circle of a Triangle

You can construct a circle that passes through a number of special points in a triangle. First construct custom tools for the four kinds of centers or open them from your .ggt files that you saved. Connect the orthocenter to the circumcenter: this is "Euler's Segment." The Centroid lies on this segment. A number of centers and related points of a triangle are all closely related by Euler's Segment and its Nine-Point Circle for any triangle. Create an Euler Segment and its related Nine-Point Circle, whose center is the midpoint of the Euler Segment.



You can watch a six-minute video of this segment and circle at: www.khanacademy.org/math/geometry/triangle-properties/triangle_property_review/v/euler-line.

The video shows a hand-drawn figure, but you can drag your dynamic figure to explore the relationships more accurately and dynamically.

Are you amazed at the complex relationships that this figure has? How can a simple generic triangle have all these special points with such complex relationships? Could this result from the dependencies that are constructed when you define the different centers in your custom tools?

What we noticed:

What meaning we give to what we noticed:

What we wondered:

Visualizing Congruent Triangles

Two triangles are called "congruent" if all their corresponding angles and sides are equal size. However, you can constrain two triangles to be congruent by just constraining 3 of their corresponding parts to be equal – for certain combinations of 3 parts. Dynamic geometry helps you to visualize, to understand and to remember these different combinations.

In this topic, you will have to copy angles. You can copy an angle in dynamic geometry using the compass tool, but it is a bit more complicated than copying a segment.

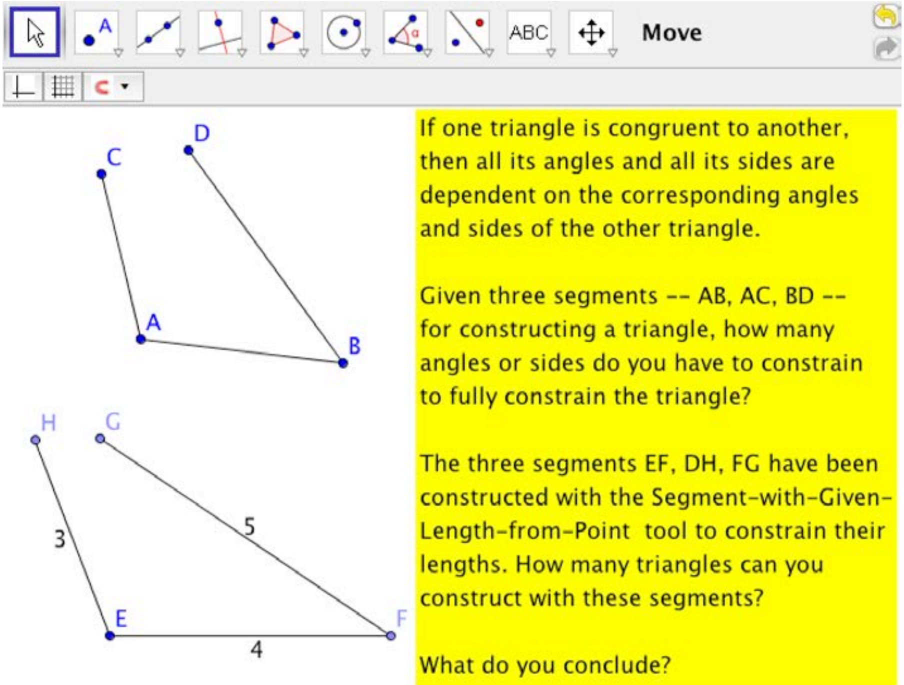
Watch this ***YouTube clip:*** *http://www.youtube.com/watch?v=qngWUFgyyHc. It gives a clear view of how to use GeoGebra to copy an angle using tools that are equivalent to traditional geometry construction with a physical straightedge (for making line segments) and compass (for making circles).*

Corresponding Sides and Angles of Congruent Triangles

What constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle? Two congruent triangles have 6 corresponding parts equal (3 sides and 3 angles), but you do not have to constrain all 6 parts to be equal in order to make sure the triangles are congruent.

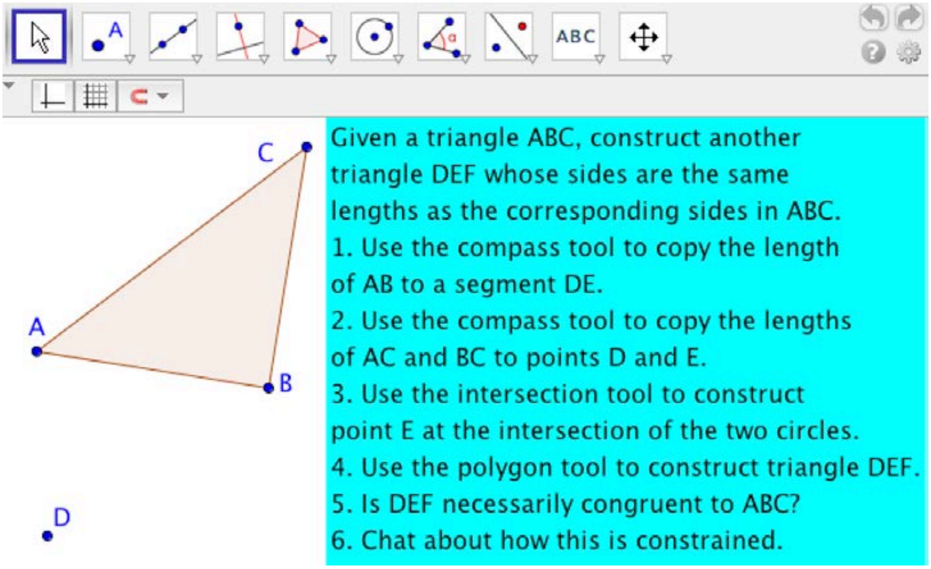
For instance, two triangles with their corresponding 3 angles equal are called "similar" but they are not "congruent." You can drag one of them to be larger that the other one. They will still have the same shape, but the corresponding side lengths of one will all be larger than the side lengths of the other triangle.

If two triangles have their corresponding angles constrained to be equal and then you constrain two corresponding sides to be equal length, will the triangles necessarily be congruent? Suppose you only constrained one of the corresponding sides to be equal? Explore different combinations of 4 or 5 of the 6 corresponding triangle parts being constrained to be equal. Which combinations guarantee that the triangles are congruent?



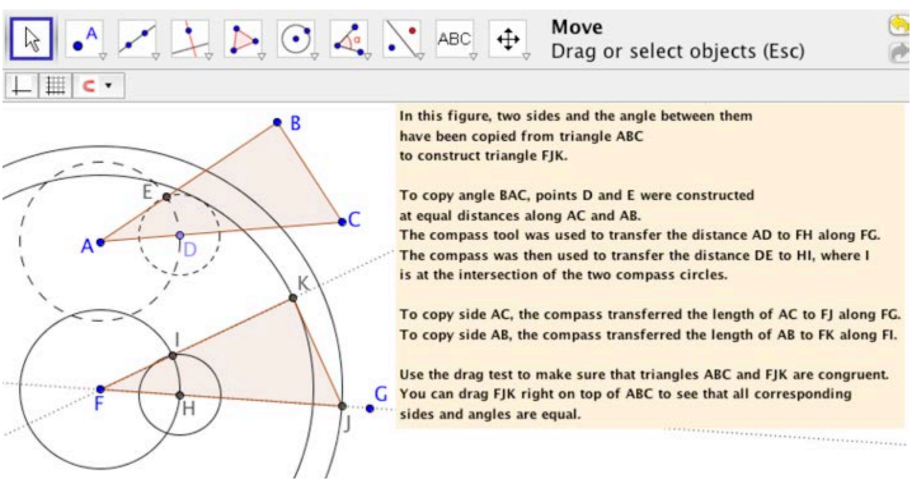
Side-Side-Side (SSS)

If all three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent. This is called the "Side-Side-Side" (or "SSS") rule.



Side-Angle-Side (SAS)

If two sides and the angle **between them** of one triangle are equal to the corresponding sides and angle **between them** of another triangle, then the two triangles are congruent.



What we noticed:

What meaning we give to what we noticed:

What we wondered:

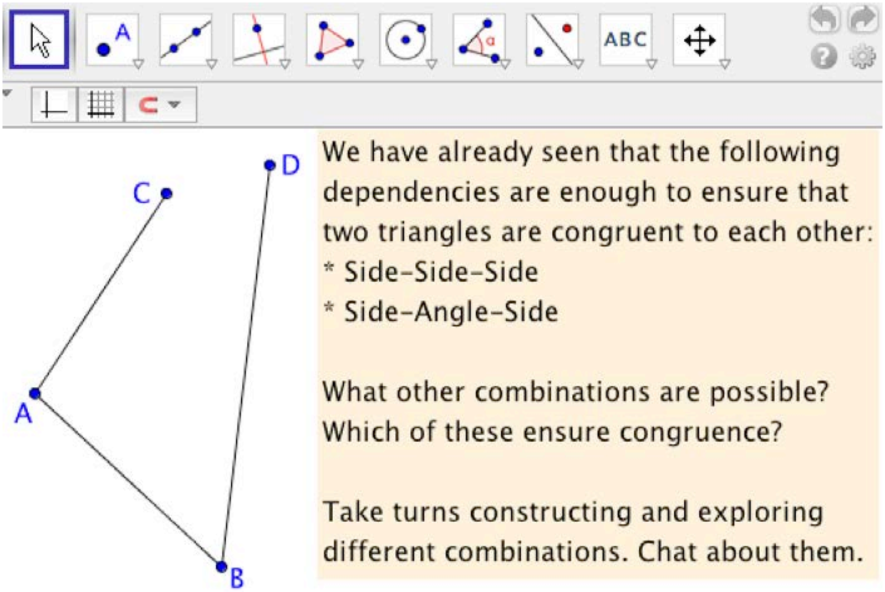
Constraining Congruent Triangles

Use dynamic geometry to explore and visualize the different cases of constraints on triangles, in addition to SSS and SAS.

Combinations of Sides and Angles

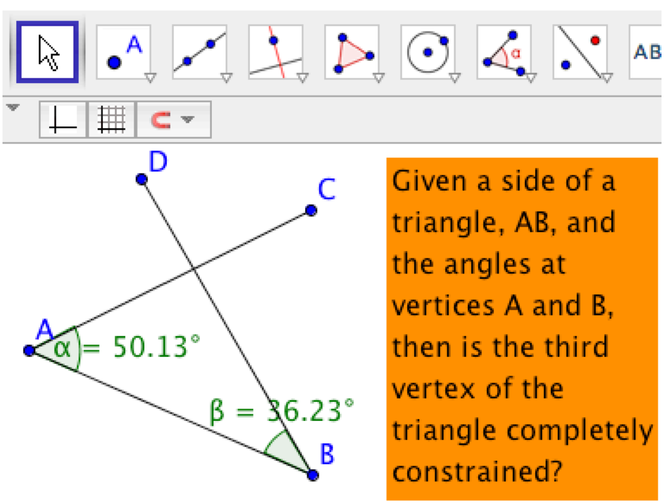
You can constrain two dynamic triangles to be congruent using a number of different combinations of equal corresponding sides and/or angles.

What combinations of constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle?



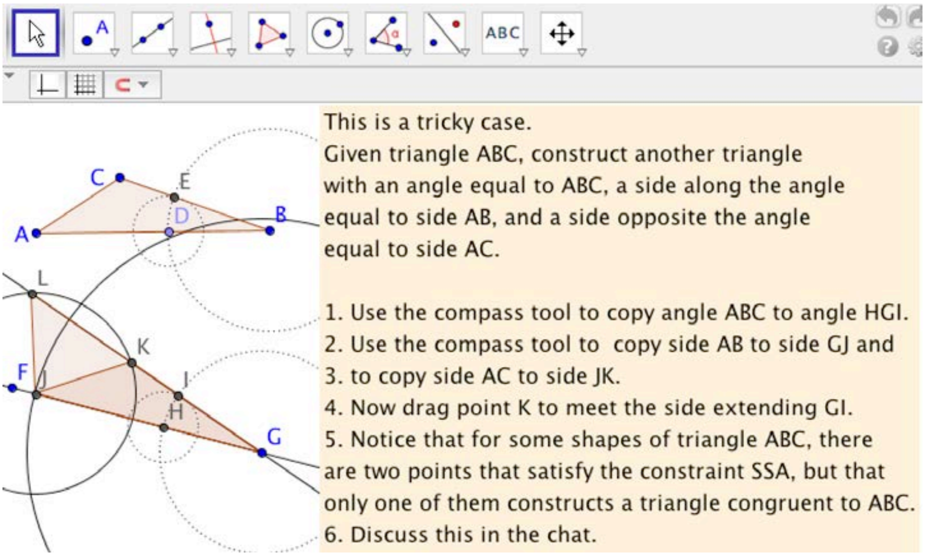
Angle-Side-Angle (ASA)

If two angles and the side included **between them** of one triangle are equal to the corresponding two angles and side **between them** of another triangle, then the two triangles are congruent. This is called the Angle-Side-Angle or ASA rule.



Side-Side-Angle (SSA)

What if two corresponding sides and an angle are equal, but it is not the angle included between the two sides?



What we noticed:

What meaning we give to what we noticed:

What we wondered:

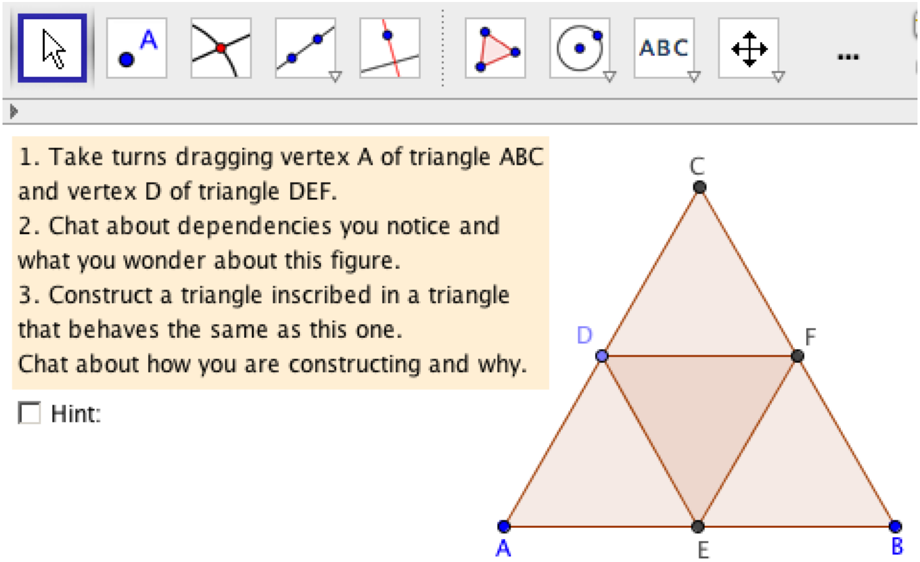
Inscribing Triangles

This topic presents a challenging problem for your team to construct.

It also includes a chance to prove that you succeeded. The proof uses what you learned about congruent triangles.

The Inscribed Triangles Problem

Construct a pair of inscribed triangles. First, explore the given figure. Note the dependencies in the figure. Then construct your own pair of inscribed triangles that behaves the same way.

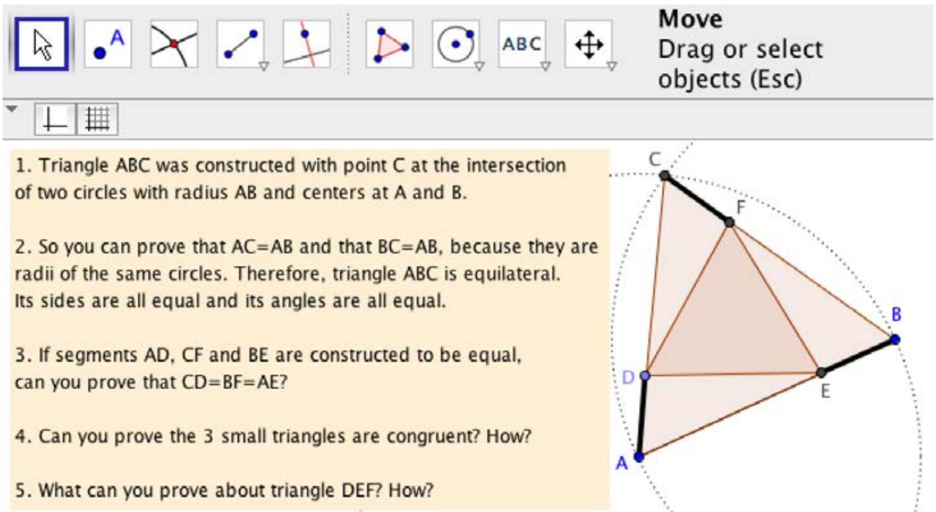


If you solved this, did you construct the lengths of the sides of the outer triangle to be directly dependent upon each other? Did you construct the lengths of the sides of the inner triangle to be directly dependent upon each other? Do you think that the triangles are equilateral? How do you know?

Proofs about Triangles

Constructing figures in dynamic geometry—like the inscribed triangles—requires thinking about dependencies among points, segments and circles. You can talk about these dependencies in the form of proofs, which explain why the relationships among the points, segments and circles are always, necessarily true, even when any of the points are dragged around.

Many proofs in geometry involve showing that some triangles are congruent to others. You can prove that the inscribed triangles are equilateral by proving that certain triangles are congruent to each other.



Chat about what you can prove and how you know that certain relationships are necessarily true in your figure. Explain your proof to your team. Does everyone agree?

What we noticed:

What meaning we give to what we noticed:

What we wondered:

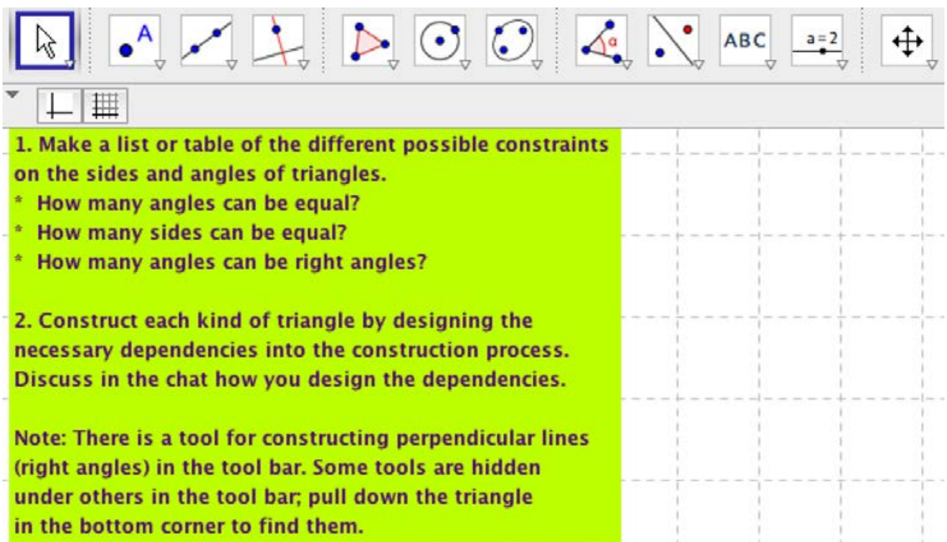
Building a Hierarchy of Triangles

Now you can use GeoGebra tools (or your own custom tools) to construct triangles with different constraints. How are the different kinds of triangles related to each other? Which ones are special cases of other kinds?

Constructing Constrained Triangles

How many different kinds of dynamic triangles can your team construct? You might want to make a list or table of the different possible constraints on the sides and angles of triangles. How many angles can be equal? How many sides can be equal? How many angles can be right angles?

Can your team construct each kind of triangle by designing the necessary dependencies into the construction process?



The Hierarchy of Triangles

How are the different kinds of triangles related to each other?

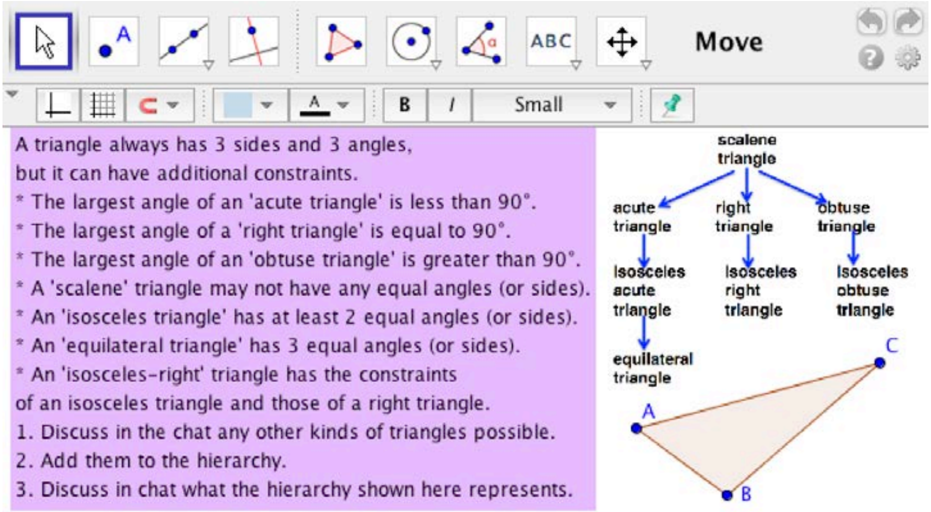
There are different ways of thinking about how triangles are related in dynamic geometry.

In dynamic geometry, a generic or "scalene" triangle with no special constraints on its sides or angles may be dragged into special cases, like a right triangle or an equilateral triangle. However, it does not have the constraints of a right angle vertex or equal sides built into it by its construction, so it will not necessarily retain the special-case characteristics when it is dragged again.

You can think of a hierarchy of kinds of triangles: an equilateral triangle can be viewed as a special case of an isosceles acute triangle, which can be viewed as a special case of an acute triangle, which can be viewed as a special case of a scalene triangle.

Can your team list all the distinct kinds of triangles?

Can your team connect them in a hierarchy diagram? Create a hierarchy diagram like the one shown in the tab. Add more kinds of triangles to it if you found some. You may want to reorganize the structure of the diagram.



When you study quadrilaterals (figures with four sides) in geometry, you should create a hierarchy of the different types of quadrilaterals with different constraints: How many angles can be equal? How many sides can be equal? How many angles can be right angles? How many pairs of sides can be parallel? There are many types of constrained quadrilaterals, many without any common names.

What we noticed:

What meaning we give to what we noticed:

What we wondered:

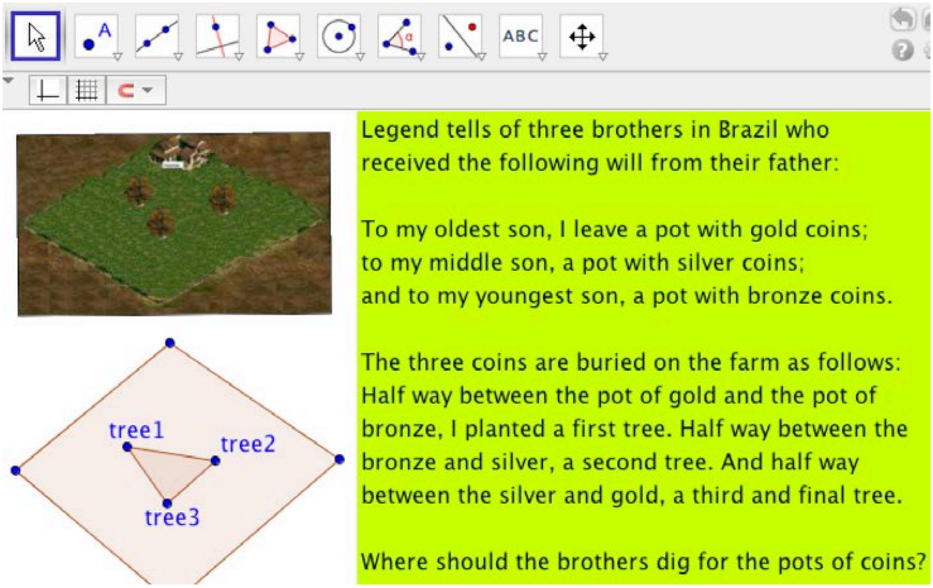
Solving Geometry Problems

Here is a set of challenge problems for your team. Have fun!

If the team does not solve them during its session, try to solve them on your own and report your findings in the next team session.

Treasure Hunt

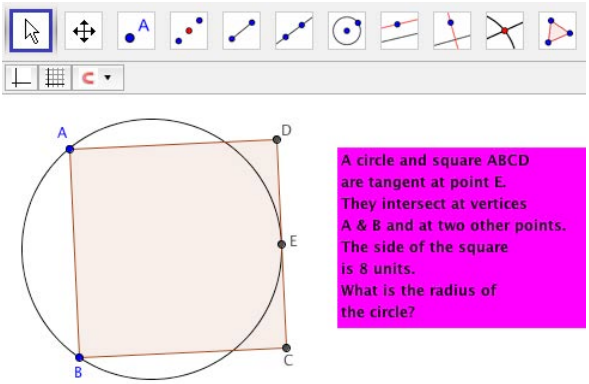
Can you discover the pot of gold in this tale told by Thales de Lelis Martins Pereira, a high school teacher in Brazil? You might want to construct some extra lines in the tab.



Square and Circle

Can you determine the radius of the circle? You should not have to measure. What if the side of the square is "s" rather than "8"?

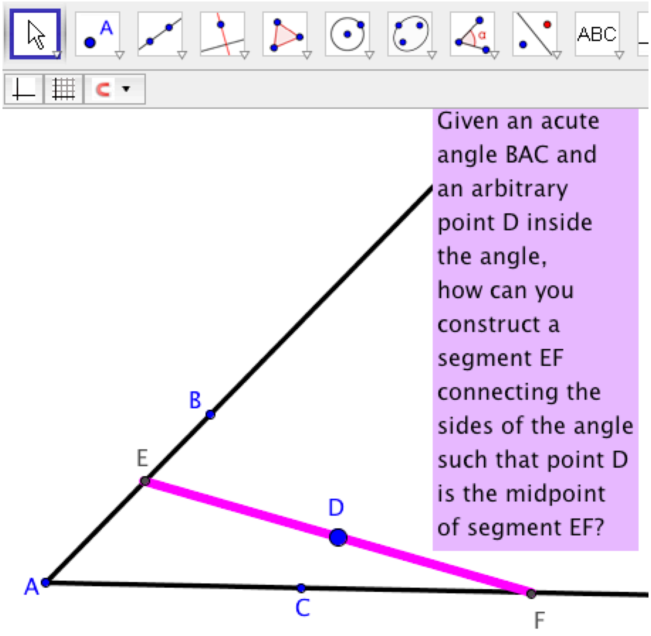
Hint: To solve this kind of problem, it is usually useful to construct some extra lines and explore triangles and relationships that are created. If you know basic algebra, you might set up some equations based on the relationships in the figure.



Crossing an Angle

Can you construct the segment crossing the angle with the given midpoint?

Hint: This is a challenging problem. Try to add some strategic lines and drag the figure in the tab to see what would help to construct the segment EF at the right place.



Into the Future

Congratulations on completing "Explore Dynamic Geometry Together"!

If your virtual math team has successfully completed the previous topics, then you are ready to go off on your own—either as a team or individually—to explore geometry using GeoGebra. Actually, GeoGebra also supports algebra, 3-d geometry, trigonometry, pre-calculus, calculus and other forms of mathematics. So you will be able to use it for the rest of your life as a tool for better understanding mathematical concepts and relationships.

You are welcome to continue to use VMT for collaborative events. You can create your own rooms and invite people to them to explore topics that you define. You can also download GeoGebra from www.GeoGebra.org to your computer or iPad to use on your own.

Enjoy!

What we noticed:

What meaning we give to what we noticed:

What we wondered:

3. Topics in Dynamic Geometry for Virtual Math Teams

Introduction

Topics in Dynamic-Geometry for Virtual Math Teams is a set of topic statements for use with the Virtual Math Teams with GeoGebra (VMTwG) collaboration software.[2]

Dynamic geometry is a new form of mathematics—and you can be a pioneer in it, exploring, discovering and creating new insights and tools. Dynamic geometry realizes some of the potential that has been hidden in geometry for thousands of years: by constructing dynamic-geometry figures that incorporate carefully designed mathematical relationships and dependencies, you can drag geometric objects around to investigate their general properties.

In addition, the approach of these topics allows you to take advantage of the power of *collaboration*, so that your team can accomplish more than any one of you could on your own—by chatting about what you are doing, and why, as well as discussing what you notice and wonder about the dynamic figures. Working in a team will prevent you from becoming stuck. If you do not understand a geometry term or a task description, someone else in the team may have a suggestion. If you cannot figure out the next step in a problem or a construction, discuss it with your team. Decide how to proceed as a team.

The topics have been designed for *everyone* interested in geometry. Students who have not yet studied any geometry can use the topics to prepare them for thinking geometrically. Students who are in the middle of a geometry course or have already completed one can use the topics to gain a deeper appreciation of geometry. Even experienced geometry teachers can use the topics to gain a new perspective on an

[2] The latest version of ***Topics in Dynamic Geometry for Virtual Math Teams*** is always available at: www.GerryStahl.net/elibrary/topics/topics.pdf

ancient subject.

To harness the truly awesome power of *collaborative dynamic geometry* requires patience, playfulness and persistence. It will pay off by providing skills, tools and understanding that will be useful for a lifetime. You will need to learn how to construct complicated figures; this will be tricky and require practice. You will need to think about the hidden dependencies among dynamic points, which make geometry work; this may keep you up at night. You will even create your own custom construction tools to extend GeoGebra; this will put you in control of mathematics.

Collaborative problem solving is central to this set of topics. The topics have been selected to offer you the knowledge, skills and hands-on experience to solve typical geometry problems, to explain your solution to others and to think more like a mathematician.

These topics present a special *approach* to dynamic geometry, which may be quite different from approaches to learning geometry that you are accustomed to.[3] They stress the importance of understanding how dependencies are constructed into geometric figures. Dragging points around in pre-constructed GeoGebra apps can help you to discover and visualize relationships within a figure. However, such apps also hide the dependencies that maintain the relationships. You should know how to construct those dependencies yourself. This will give you a deeper understanding of geometry as a mathematical system. This approach may take more time and thought, but it will also be more fun and more rewarding.

The approach in this document is built around a set of 13 *core topics* that provide a coherent experience of collaborative dynamic geometry. In addition, there is an introductory topic for individuals to do on their own to get started and then a transition topic for individuals to do at the end to start to explore the rest of GeoGebra. There are also extensions to some of the core topics for individuals or groups to explore farther what they are interested in. Finally, several open-ended advanced topics present new areas of mathematics beyond the core topics.

Each topic is designed for an online team to work on together for about one hour. The sequence of topics introduces student teams to GeoGebra and guides them in the exploration of dynamic geometry, including triangles, quadrilaterals and transformations. Ideally, everyone should spend 13 hour-long, online, synchronous, collaborative sessions working on the core topics. An instructor might want to select which of these topics to use and which to skip or make optional if it is not possible to do all 13 or if there is time available to do more than 13. A virtual math team might choose which topics they want to explore.

Several "tours" are included at the end of this document. They provide *tutorials* in important features of the VMT and GeoGebra software. Teachers and students can

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[3] The approach is discussed in detail from various perspectives in ***Translating Euclid: Designing a Human-Centered Mathematics***. See www.GerryStahl.net/elibrary/euclid

take the tours when they want more information on using the software. To work on topics outside of a team, individual teachers or students can download the single-user desktop version of GeoGebra and then download selected .ggb files. [4]

The pages of this document can be distributed to students as *worksheets* to keep their notes on and in case the instructions in the chat room tabs become erased. The whole document can be distributed to team members to serve as a journal for students to take notes on their sequence of topics and to provide access to the tutorials. It is helpful if everyone has a printed or electronic version of this booklet on their physical or digital desktop when they are working on the topics.

To get started, it is important that everyone do the *Individual Warm-Up Topic* on the computer that they will be using—well before the first collaborative session. This checks that your computer is properly set up and that you are able to enter VMT chat rooms. It also provides a valuable introduction to GeoGebra.

The topics and tours have been designed for a *broad range of users*. Students who have not previously studied much geometry should focus on the main points of each topic and make sure that all team members understand the constructions. Teams of experienced math teachers may engage in more in-depth discussion on implications, conceptualizations and pedagogy of the dynamic-geometry topics. If something is unclear in the topic instructions, discuss it in your team and decide as a group how to proceed. Pace your team to try to complete all the core parts of a topic in the time you have together.

Here is a *general procedure* you might want to follow: Before the time assigned for a group session, read the topic description in this document. It might suggest watching a brief video or taking a tour at the end of this document as important preparation. Think about the topic on your own. Then, in the group session, discuss the topic and work together on the various tasks. Discuss what you are doing in the chat and respond to the questions posed in the topic. Mark down in your copy of this document what you noticed that surprised you and what you wondered about that you want to think more about later. Do not just rush though the topic; discuss what is important in it. The point is to learn about collaborative dynamic geometry, not just to get through the topic steps. When the session is over, try to work on your own on any parts that your team did not get to. Report to your team what you discover. Maybe the team can get together for an extra session on the rest of the topic.

This set of topics offers a unique opportunity to experience deeply topics that have

[4] The standard single-user version of GeoGebra is available for download at: www.GeoGebra.org The GeoGebra files to be loaded into GeoGebra tabs in VMT chat rooms can be downloaded. For instance, download the .ggb file for Topic 7.3 at: www.GerryStahl.net/vmt/topics/7c.ggb as well as from the "VMT channel" of GeoGebraTube at: www.geogebratube.org/collection/show/id/4531

fascinated people since the beginning of civilization. Take advantage. Be creative. Collaborate. Explore. Chat. Enjoy!

Individual Warm-up Activity

You can start on this Warm-up activity by yourself at any time to explore the Virtual Math Teams with GeoGebra (VMTwG) environment for discovering collaborative dynamic geometry.

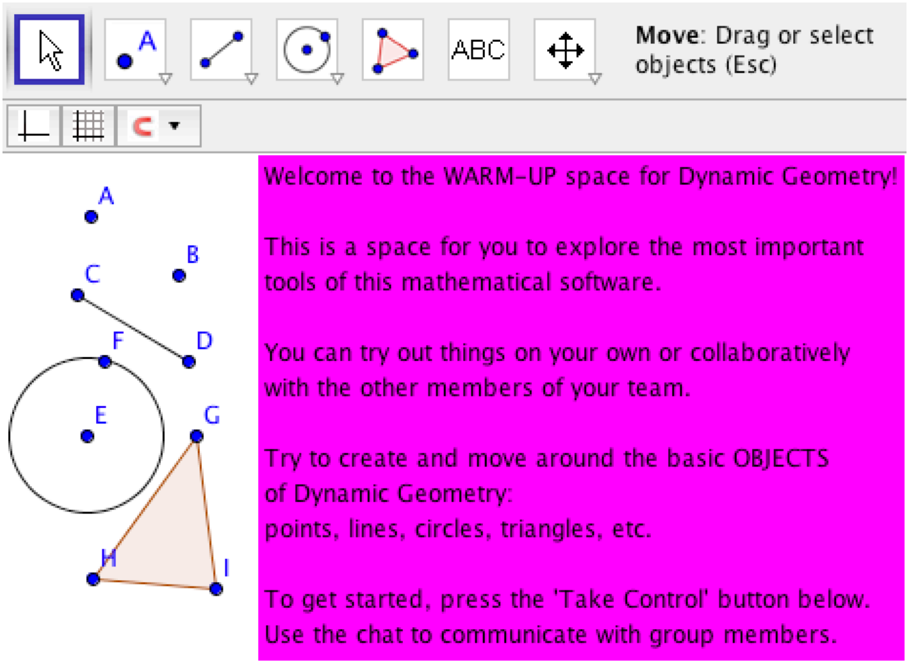
First, you can get a good introduction to using the GeoGebra software from this ***YouTube clip****: http://www.youtube.com/watch?v=2NqblDIPl38. It gives a clear view of how to use GeoGebra. It also provides tips on GeoGebra tools that are equivalent to traditional geometry construction with a physical straightedge (for making line segments) and compass (for making circles).*

Then, read ***"Tour 1: Joining a Virtual Math Team."*** *This will introduce you to using the VMT software to register, login, find a chat room, etc. The Tours are all found toward the end of this booklet, after all the Topics.*

Also look through ***"Tour 2: GeoGebra for Dynamic Math."*** *This will provide more details about the GeoGebra system for dynamic geometry. For instance, it will describe many of the buttons shown in the Tool Bar. You can always go back to these tours if you become stuck using VMT or GeoGebra.*

Welcome!

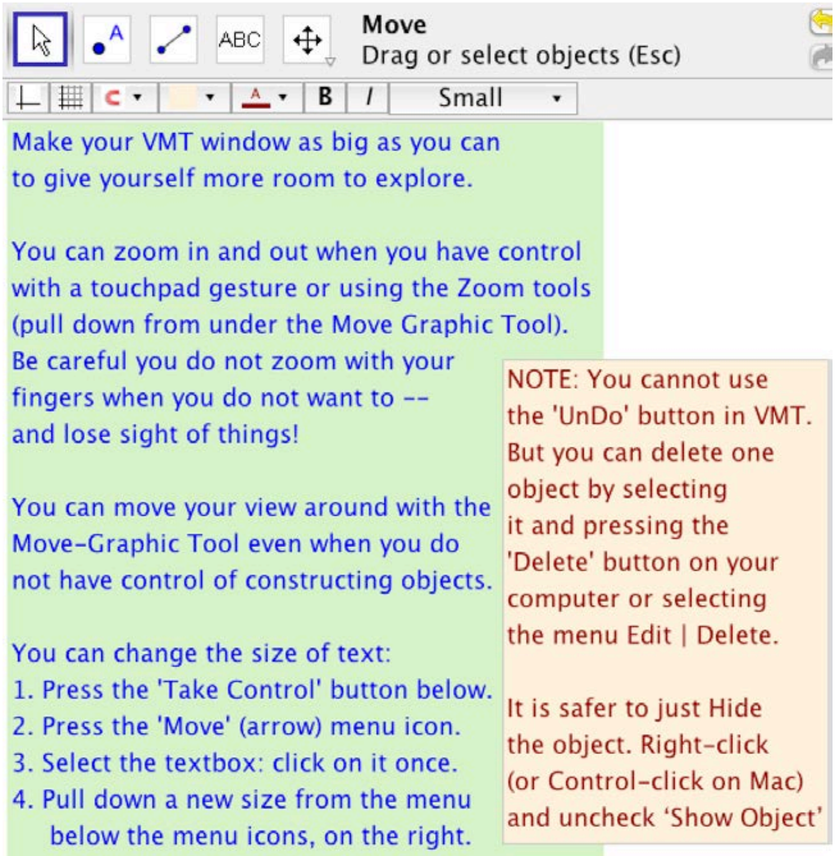
Start in the Welcome tab (shown below) of the Warm-Up "Topic 01" chat room by reading the instructions and then dragging the objects shown (press the "Take Control" button at the bottom, select the "Move Tool" arrow and click on a point to drag it). Then create your own similar objects using the buttons in the Tool Bar (near the top). You can change to one of the other tabs, like "#1 Team Member" to have an empty GeoGebra workspace to create your own points and lines. Try out each of the tools in the Tool Bar.



If there is not an unused tab available to create your points and lines in, you can create a new tab with the "+" button in the upper right corner of the VMTwG window. You can use the "ABC" text tool to construct a text box with your name or comments in it: Select the text tool and then click in the tab about where you want to have the text box. A form will open for you to enter your text.

Helpful Hints

Here are some hints for taking advantage of the limited space on your computer. Read these hints and adjust the size of your VMT window and its GeoGebra tab.



There are a number of Zoom Tools, which can be pulled down from the Move Graphic Tool (the crossed arrows on the right end of the Tool Bar).

If you are using a touchpad on your computer, you can zoom with a two-finger touchpad gesture. **Caution**: It is easy to move things around without wanting to as your fingers move on the touchpad. Things can quickly zoom out of sight. This can happen even when you do not have control of construction. Then you will not see the same instructions and figures that other people on your team see.

Note: Most Zoom Tools will only effect what **you** see on your computer screen, **not** what your teammates see on theirs. It is possible for you to create points on an area of your screen that your teammates do not see. If this happens, ask everyone if they want you to adjust everyone's screen to the same zoom level and then select the menu item "GeoGebra" | "Share Your View."

When you work in a team, it is important to have everyone agree before changes are made that affect everyone. If you want to delete a point or other object – especially one that you did not create – be sure that everyone agrees it should be deleted.

Be very careful when **deleting** points or lines. If any objects are dependent on them, those objects will also be deleted. It is easy to unintentionally delete a lot of the group's work.

Instead of deleting objects, you should usually **hide** them. Then the objects that are dependent on them will still be there and the dependencies will still be in effect. Use control-click (on a Mac) or right-click (in Windows) with the cursor on a point or line to bring up the context-menu. Select "Show Object" to unclick that option and hide the object. You can also use this context-menu to hide the object's label, rename it, etc.

You may want to **save** the current state of your GeoGebra tab to a .ggb file on your desktop sometimes so you can load it back if things are deleted. Use the menu "File" | "Save" to save your work periodically. Use the menu "File" | "Open" to load the latest saved version back into the current tab. Check with everyone in your team because this will change the content of the tab for everyone.

If the instructions in a tab are somehow erased, you can look them up in this document. You can also scroll back in the history of the tab to see what it looked like in the past. Finally, you can have your whole team go to a different room if one is available in the VMT Lobby that is not assigned to another team.

If you have **technical problems** with the chat or the figures in the tabs not showing properly, you should probably close your VMT window and go back to the VMT Lobby to open the room again.

You should use this booklet to **keep notes** on what you learn or wonder, either on paper or on your computer. Here is a space for notes on Topic 01:

Notes:

Messing Around with Dynamic Geometry

Dynamic geometry is an innovative form of mathematics that is only possible using computers. It is based on traditional Euclidean geometry, but has interesting objects, tools, techniques, characteristics and behaviors of its own. Understanding dynamic geometry will help you think about other forms of geometry and mathematics.

In this topic, you will practice some basic skills in dynamic geometry. There are several tabs to work through; try to do them all with your team. Pace yourselves. If the team becomes stuck on one tab during a session, move on and come back to it later, maybe on your own after the team session.

Make sure that everyone in your team understands the important ideas in a tab and then have everyone move to the next tab.

Dynamic Points, Lines & Circles

Geometry begins with a simple point. A point is just the designation of a particular location. In dynamic geometry, a point can be dragged to another location.

Everything in geometry is built up from simple points. For instance, a line segment is made up of all the points (the "locus") along the shortest (direct, straight) path between two points (the endpoints of the segment). A circle is all the points ("circumference," "locus") that are a certain distance ("radius") from one point ("center").

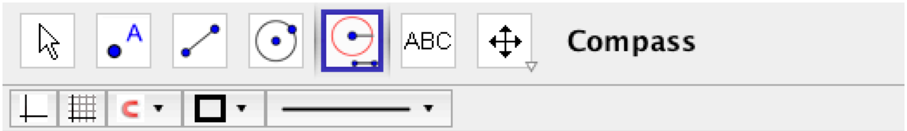
In this tab, create some basic dynamic-geometry objects and drag them to observe their behavior. Take turns taking control and creating objects like the ones you see.

When you drag point J between the two points, the locus of a line segment will be colored in. (This locus will only appear on your computer screen, so everyone in the team has to try it themselves).

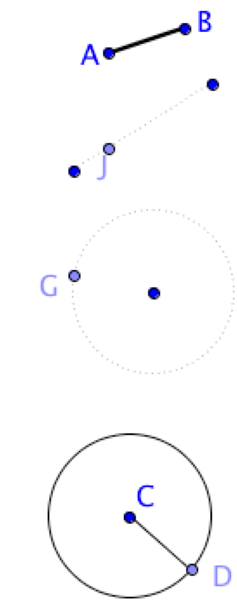
The same for dragging point G around the locus of the circle.

With these simple constructions, you are starting to build up, explore and understand the system of dynamic geometry.

Don't forget that you have to press the "Take Control" button to do actions in GeoGebra. Chat about who should take control for each step. Be sure you "Release Control" when you are done so someone else can take control.



Here is an example of what you will construct:



Dynamic point:
A point can be at any single location at a time
1. Someone select the New Point Tool.
2. Click at two locations to construct two points.

Dynamic line:
A line segment is the locus of all points along the shortest path between two points
3. Someone select the Segment Tool.
4. Click on your two points to construct a segment.
5. Select the Move Tool and drag example point J.

Dynamic circle:
A circle is the locus of all points the same distance from a center point
6. Someone select the Circle Tool; click on a point for the center and on a point on the circumference.
7. Select the Move Tool and drag example point G.

Constructing a segment dependent on another segment:
8. Someone select the Compass Tool. Click on your two points to construct a portable circle with radius equal to the distance between those points.
9. Drag that circle to a new location and click to create it.
10. Select the Point Tool; put a point on the circumference.
11. Select the segment tool, connect that point to the center.

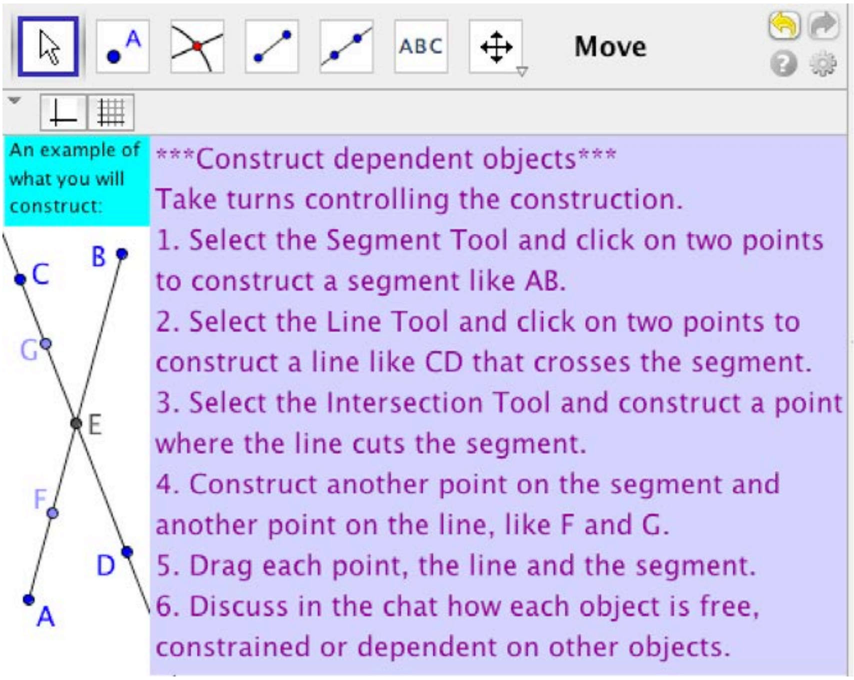
Dynamic drag test:
12. Select the Move Tool and drag the points you have created.
13. Chat about what you notice. What surprised you?

Note: You can change the "properties" of a dynamic-geometry object by first Taking Control and then control-clicking (on a Mac computer: hold down the "control" key and click) or right-clicking (on a Windows computer) on the object. You will get a pop-up menu. You can turn the Trace (locus) on/off, show/hide the object (but its constraints still remain), show/hide its label information, change its name or alter its other properties (like color and line style). Try these different options.

Dynamic Dragging

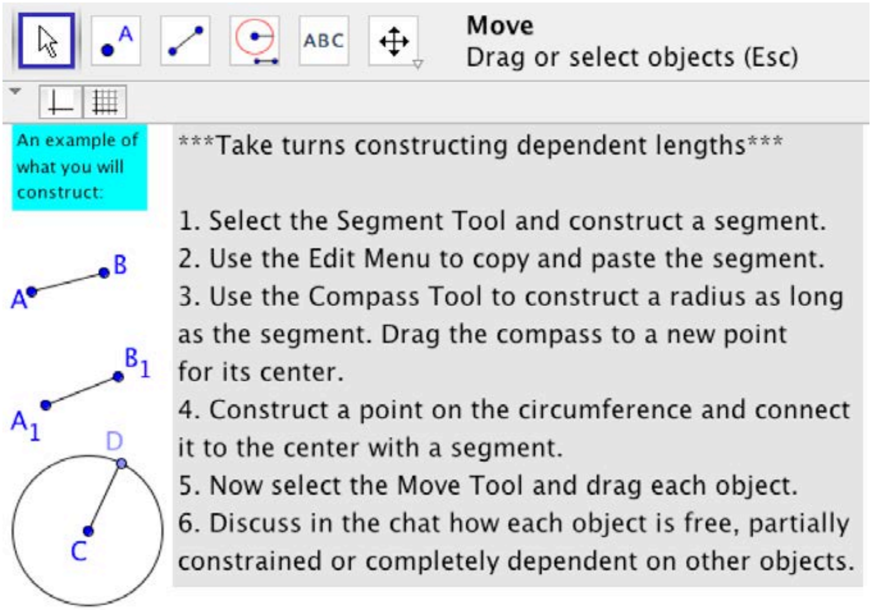
When you construct a point to be on a line (or segment, or ray, or circle) in dynamic geometry, it is constrained to stay on that line; its location is dependent upon the location of that line, which can be dragged to a new location. Use the "drag test" to check if a point really is constrained to the line: select the Move tool (the first tool in the Tool Bar, with the arrow icon), click on your point and try to drag it; see if it stays on the line.

Take control and construct some lines and segments with some points on them, like in the example shown in the tab. Notice how some points can be dragged freely, some can only be dragged in certain ways (we say they are partially "constrained") and others cannot be dragged directly at all (we say they are fully "dependent").



Constructing Segment Lengths

Constructing one segment to be the same length as another segment is different from just copying the segment. The compass tool can be used to construct a segment whose length is dependent on the length of another segment. To use the compass tool, first define its radius and then locate a center for it. Drag point A to change the length of segment AB. Does the copy made with copy-and-paste change its length automatically? Does the radius CD change its length automatically when AB is changed? Why do you think this happens?



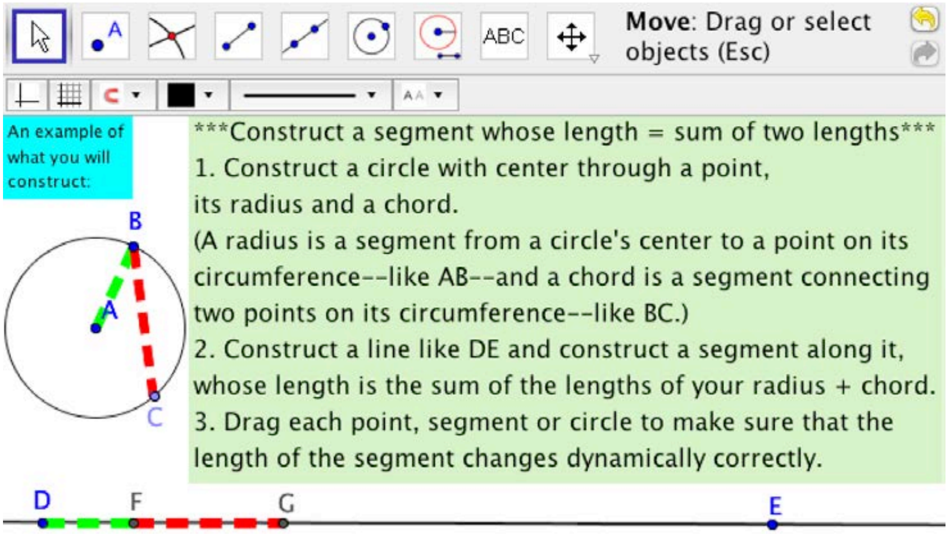
The compass tool can be tricky to use. Here is a brief YouTube video that shows how to use it:

http://www.youtube.com/watch?v=AdBNfEOEVco

Adding Segment Lengths

In dynamic geometry, you can construct figures that have complicated dependencies of some objects on other objects. Here you will construct a segment whose length is dependent on the length of two other segments. Copying a length like this so that the length of the copy is always the same as the length of the original (even when the original is dragged to a different length) is one of the most important operations in dynamic geometry. Make sure that everyone in your team understands how to do this.

Use the compass tool to copy the lengths of the line segments. Using the compass tool requires practice. Creating line segment length DG = AB + BC provides a good visual image when you drag point B.



Discuss with your team how to do each step, especially step 2. Do you see how you can use the compass tool to lay out segments with given lengths (like AB and BC) along a given line (like DE)? Explain to your teammates the difference between the circle tool and the compass tool and let them add their ideas.

Enter in your (paper or digital) copy of this document a summary of what you and your team noticed and wondered during your work on this Topic 02. This will be a valuable record of your work. You may want to come back and think more about these entries later.

What we noticed:

What we wondered:

Visualizing the World's Oldest Theorems

Scientific thinking in the Western world began with the ancient Greeks and their proofs of theorems in geometry. Thales lived about 2,600 years ago (c. 624–546 BCE). He is often considered the first philosopher (pre-Socratic), scientist (predicted an eclipse) and mathematician (the first person we know of to prove a mathematical theorem deductively). Pythagoras came 30 years later and Euclid (who collected many theorems of geometry and published them in his geometry book called *Elements*) came 300 years later. Thales took the practical, arithmetical knowledge of early civilizations—like Egypt and Babylonia—and introduced a new level of theoretical inquiry into it. With dynamic-geometry software, you can take the classic Greek ideas to yet another level.

Before working on this topic with your team, it would be good to read ***Tour 3: "VMT to Learn Together Online"*** *near the end of this booklet. It will show you how to use the Referencing Tool and other communication techniques.*

Visualize the Theorem of Thales

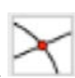
Thales took a "conjecture" (a mathematical guess or suspicion) about an angle inscribed in a semi-circle and he proved why it was true. You can use dynamic geometry to *see* that it is true for all angles all along the semi-circle. Then you can *prove* that it is always true.

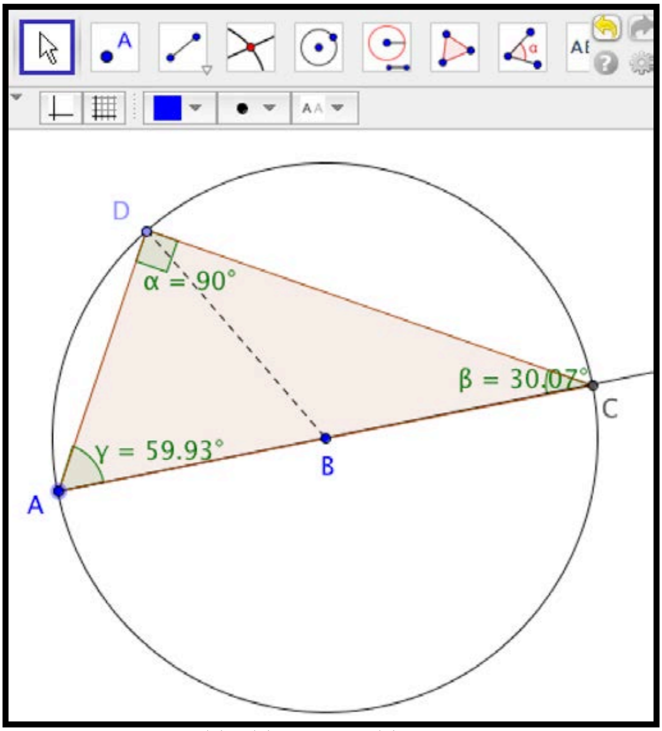
Construction Process

Take control and follow these steps to construct an angle inscribed in a semi-circle like the one already in the tab. The letter labels on your team's new figure will be different than the letter labels listed below. You will be able to move the angle dynamically and see how things change.

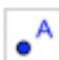
Step 1. Construct a ray  like AB.

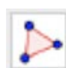
Step 2. Construct a circle  with center at point B and going through point A.

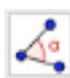
Step 3. Construct a point like point C at the intersection  of the line and the circle, forming the diameter of the circle, AC.



The Theorem of Thales.

Step 5. Construct a point  like D anywhere on the circumference of the circle.

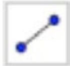
Step 6. Create triangle ADC with the polygon tool .

Step 7. Create the interior angles  of triangle ADC. (Always click on the three points forming the angle in clockwise order—otherwise you will get the measure of the outside angle.) In geometry, we still use the Greek alphabet to label angles: α, β, γ are the first three letters (like a, b, c), called "alpha," "beta," and "gamma."

Step 8. Drag  point D along the circle. What do you notice? Are you surprised? Why do you think the angle at point D always has that measure?

Challenge

Try to come up with a proof for this theorem.

Hint: To solve a problem or construct a proof in geometry, it is often helpful to construct certain extra lines, which bring out interesting relationships. Construct the radius BD as a segment .

Thales had already proven two theorems previously:

(1) The base angles of an isosceles triangle are equal. (An "isosceles" triangle is defined as having at least two equal sides.)

(2) The sum of the angles $\alpha+\beta+\gamma=180°$ in any triangle.

Can you see why $\alpha=\beta+\gamma$ in the figure, no matter how you drag point D? (Remember that all radii of a circle are equal by definition of a circle.) That means that $(\beta+\gamma)+\beta+\gamma=180°$. So, what does α have to be?

Visualization #1 of Pythagoras' Theorem

Pythagoras' Theorem is probably the most famous and useful theorem in geometry. It says that the length of the hypotenuse of a right triangle (side **c**, opposite the right angle) has the following relationship to the lengths of the other two sides, **a** and **b**:

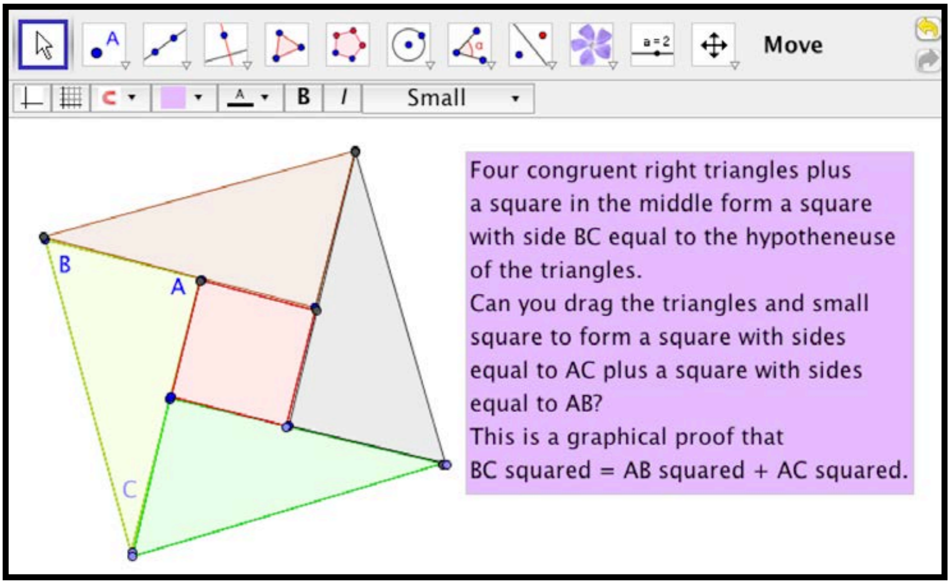
$$\mathbf{c^2 = a^2 + b^2}$$

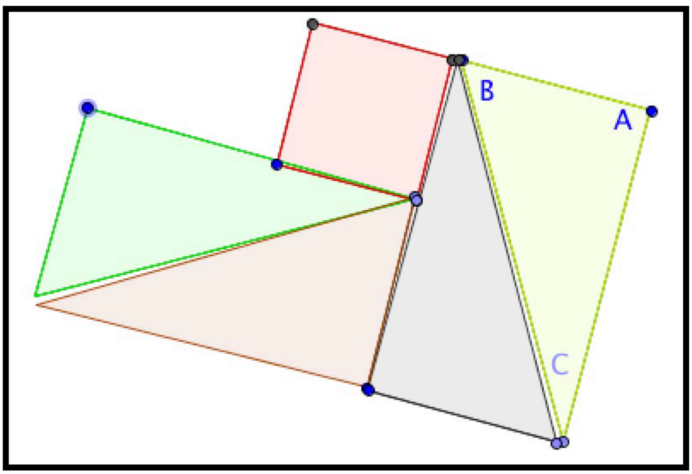
The figures in the rest of the tabs of this topic show ways to visualize this relationship. They involve transforming squares built on the three sides of the triangle to show that the sum of the areas of the two smaller squares is equal to the area of the larger square. The area of a square is equal to the length of its side squared, so a square whose side is **c** has an area equal to $\mathbf{c^2}$.

Explain what you see in these two visualizations. Can you see how the area of the $\mathbf{c^2}$ square is rearranged into the areas $\mathbf{a^2}$ and $\mathbf{b^2}$ or vice versa?

Notice that these are geometric proofs. They do not use numbers for the lengths of sides or areas of triangles. This way they are valid for any size triangles. In the GeoGebra tab, you can change the size and orientation of triangle ABC and all the relationships remain valid. Geometers always made their proofs valid for any sizes, but with dynamic geometry, you can actually change the sizes and see how the proof is still valid (as long as the construction is made with the necessary dependencies).

It is sometimes helpful to see the measures of sides, angles and areas to help you make a conjecture about relationships in a geometric figure. However, these numbers never really prove anything in geometry. To prove something, you have to explain why the relationships exist. In dynamic geometry, this has to do with how a figure was constructed—how specific dependencies were built into the figure. In this figure, for instance, it is important that the four triangles all remain right triangles and that they have their corresponding sides the same lengths (**a**, **b**, and **c**). If these lengths change in one triangle, they must change exactly the same way in the others. Can you tell what the side length of the square in the center has to be?

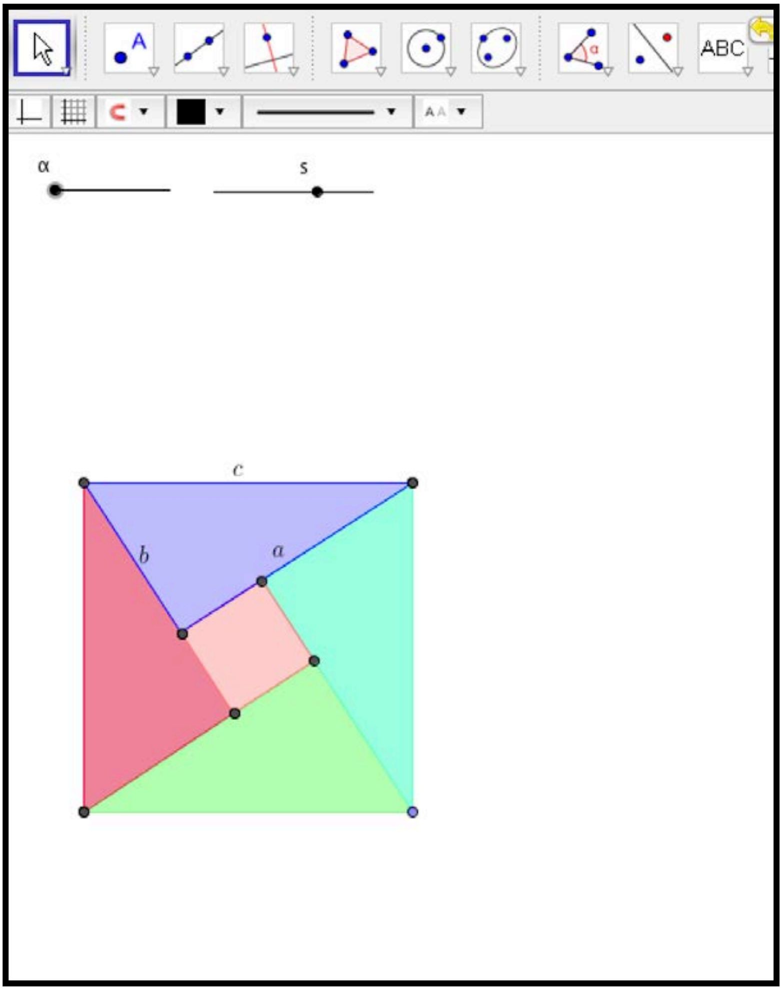


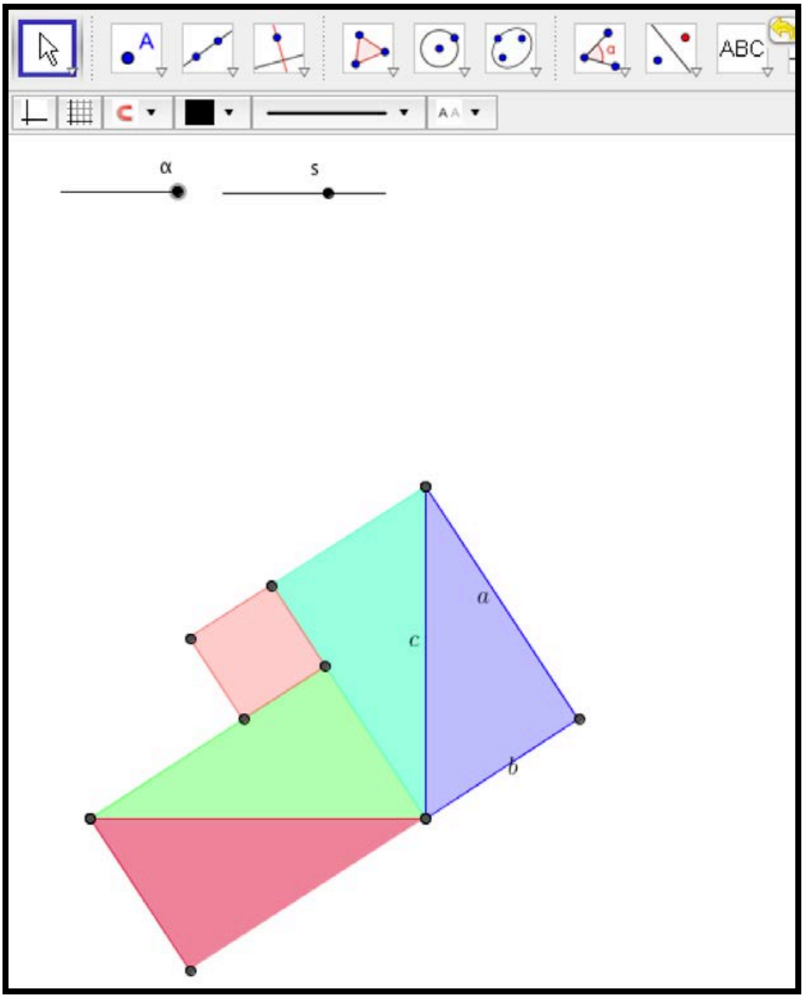


Visualization #1 of Pythagoras' Theorem.

Visualization #2 of Pythagoras' Theorem

The next figure automates the same proof of Pythagoras' Theorem with GeoGebra sliders. Try it out. Do not forget, you have to "Take Control" before you can move the sliders. Move the sliders for α and s to see what they change.

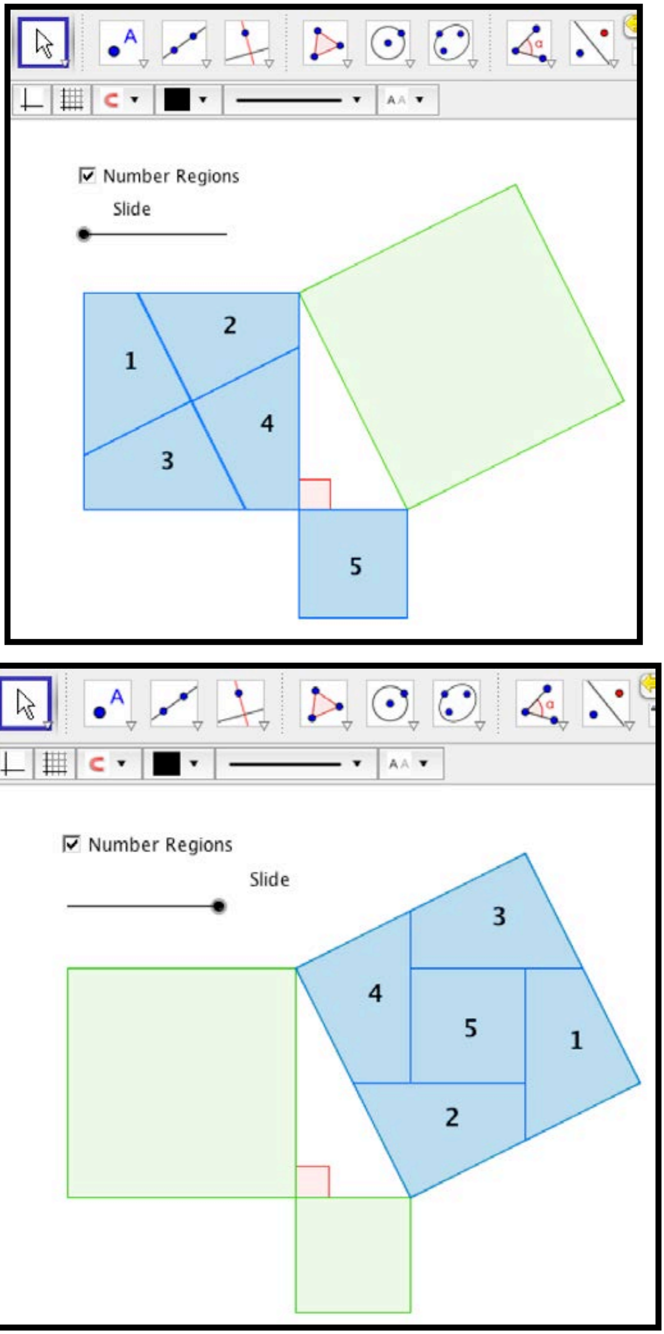

ABC
α
s
c
b
a



Visualization #2 of Pythagoras' Theorem.

Visualization #3 of Pythagoras' Theorem

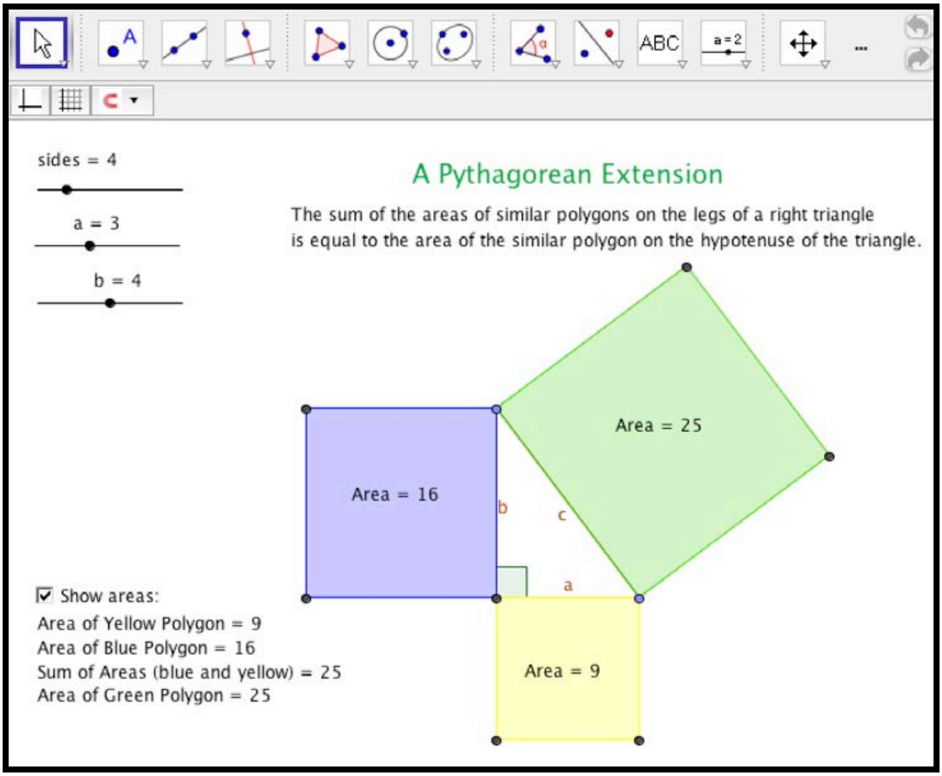
The next figure shows another way to visualize the proof of Pythagoras' Theorem. Slide the slider. Is it convincing?



Visualization #3 of Pythagoras' Theorem.

Visualization #4 of Pythagoras' Theorem

The next figure shows an interesting extension of the proof of Pythagoras' Theorem:

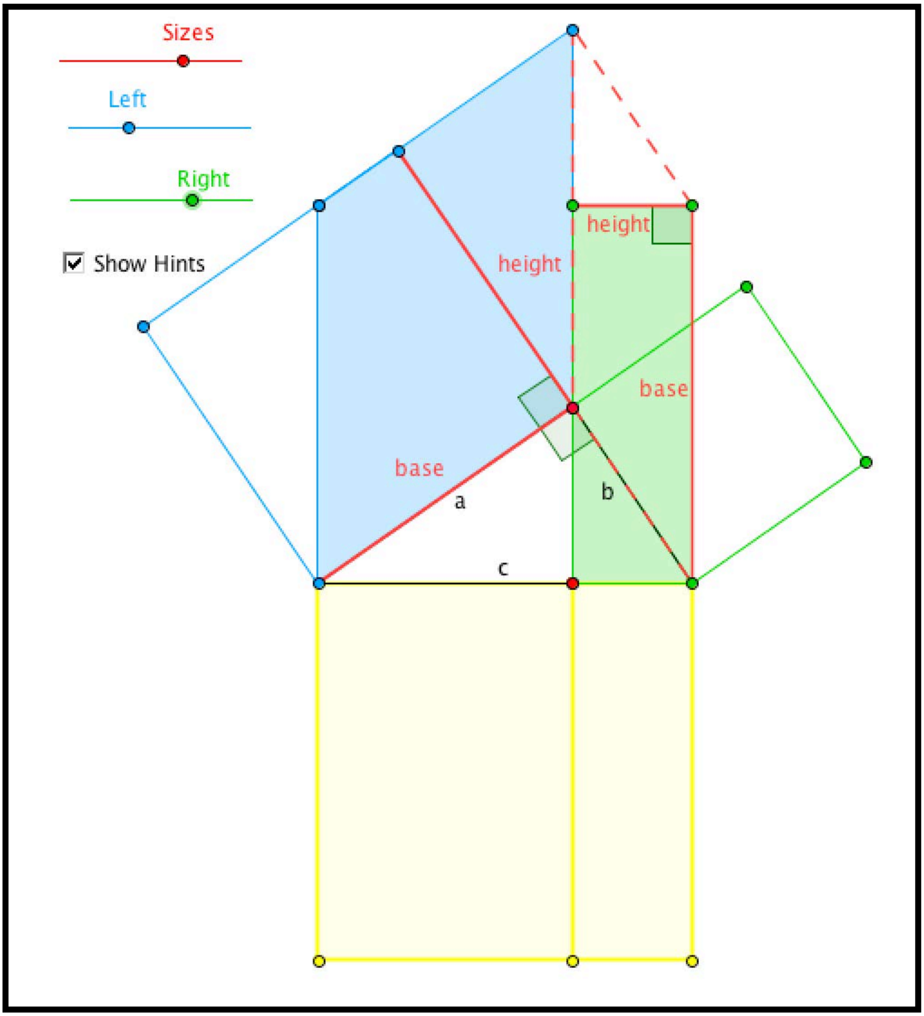


Visualization #4 of Pythagoras' Theorem.

Can you explain to each other why it works for all regular polygons if it works for triangles?

Visualization #5 of Pythagoras' Theorem

Finally, here is Euclid's own proof of Pythagoras' Theorem in his 47^{th} proposition. It depends on some relationships of quadrilaterals, which you may understand better after completing the topics in this workbook. Drag the sliders in this GeoGebra figure slowly and watch how the areas are transformed.



Visualization #5 of Pythagoras' Theorem.

There are many other visual, geometric and algebraic proofs of this famous theorem. Which do you find most elegant of the ones you have explored here?

What we noticed:

What we wondered:

Constructing Triangles

A triangle is a relatively simple geometric construction: simply join three segments at their endpoints. Yet, there are many surprising and complex relationships inside triangles.

Before working on this topic with your team, it could be very helpful to watch two brief ***YouTube clips*** *that show clearly how to copy a segment and to construct an equilateral triangle:*

http://www.youtube.com/watch?v=AdBNfEOEVco

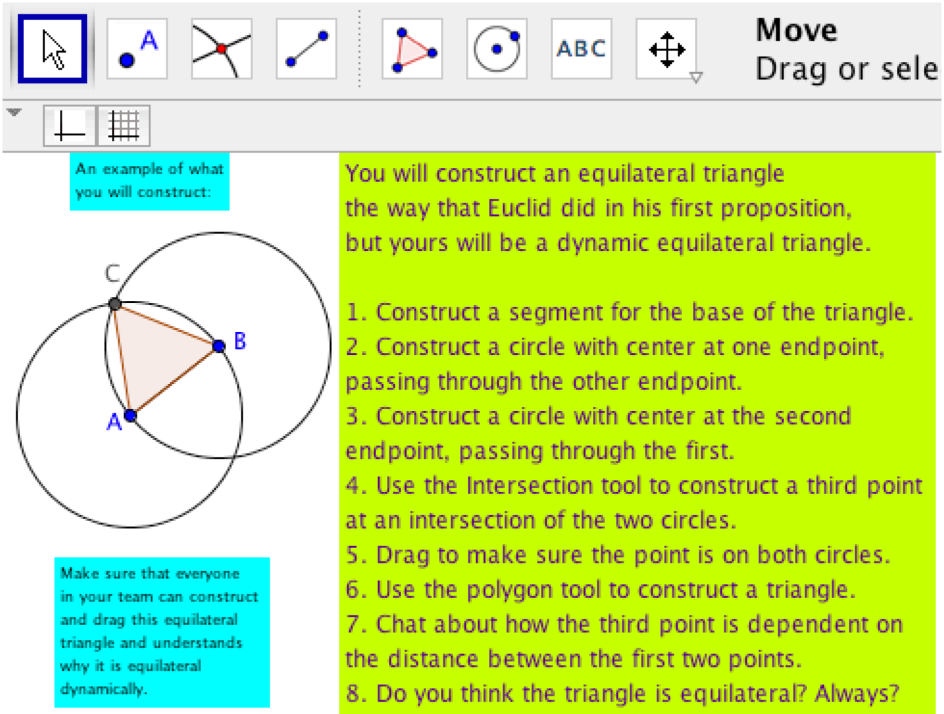
http://www.youtube.com/watch?v=ORIaWNQSM_E

Constructing an Equilateral Triangle

When Euclid organized ideas and techniques of geometry 2,300 years ago, he started with this construction of an equilateral triangle, whose three sides are constrained to always be the same lengths as each other. This construction can be considered the starting point of Euclidean and dynamic geometry.

Have everyone in your team work on the first tab for this topic. It shows—in one simple but beautiful example—the most important features of dynamic geometry. Using just a few points, segments and circles (strategically related), it constructs a triangle whose sides are always equal no matter how the points, segments or circles are dragged. Using just the basic definitions of geometry—like the points of a circle are all the same distance from the center—it proves that the triangle must be equilateral (without even measuring the sides).

Your team should construct an equilateral triangle like the one already in the tab. Drag the one that is there first to see how it works. Take turns controlling the GeoGebra tools.

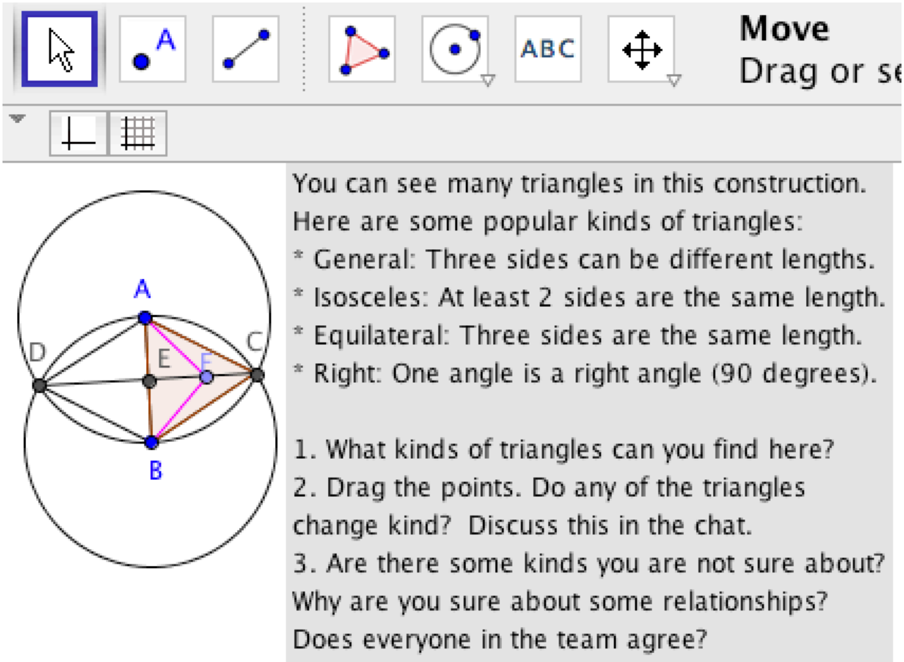


Euclid argued that both of the circles around centers A and B have the same radius, namely AB. The three sides of triangle ABC are all radii of these two circles. Therefore, they all have the same length. Do you agree with this argument (proof)? Are you convinced that the three sides of ABC have equal lengths – *without having to measure them*? If you drag A, B, or C and change the lengths of the sides are they always still equal?

Where's Waldo?

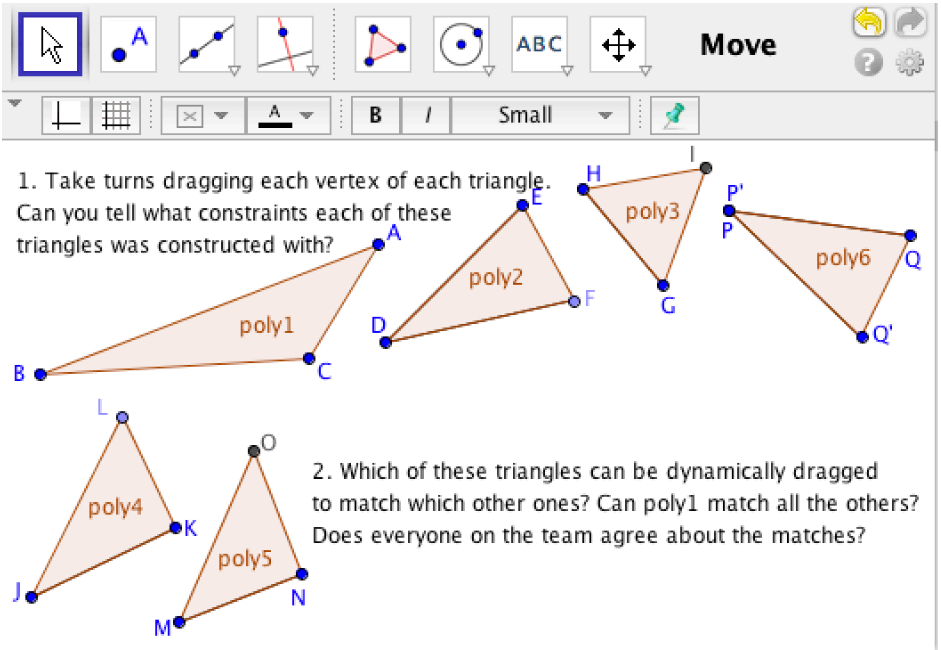
Let us look more closely at the relationships that are created in the construction of the equilateral triangle. In this tab, more lines are drawn in. Explore some of the relationships that are created among line segments in this more complicated figure. What line segments do you think are equal length – without having to measure them? What angles do you think are equal without having to measure them? Try dragging different points; do these equalities and relationships stay dynamically? Can you see how the construction of the figure made these segments or angles equal?

Can you find different kinds of triangles in this construction? If a triangle always has a certain number of sides or angles equal, then it is a special kind of triangle. We know the construction of this figure defined an equilateral triangle, ABC. What other kinds of triangles did it define?



Exploring Different Triangles

Here are some triangles constructed with different dynamic constraints. See if your team can figure out which ones were constructed to always have a certain number of equal sides, a certain number of equal angles or a right angle. Which triangles can be dragged to appear the same as which other triangles?



What we noticed:

What we wondered:

Programming Custom Tools

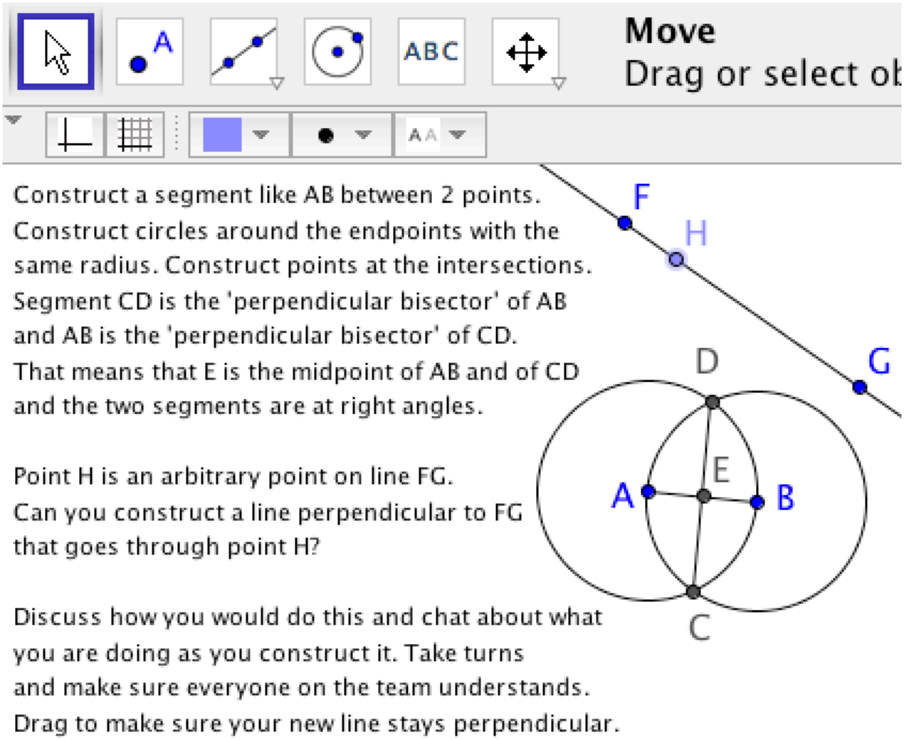
For constructing geometric figures and for solving typical problems in geometry, it is useful to have tools that do things for you, like construct midpoints of segments, perpendiculars to lines, and parallel sets of lines. GeoGebra offers about 100 tools that you can use from the tool bar, input bar or menu. However, you can also create your own custom tools to do additional things—like copy an angle, construct an isosceles triangle or locate a center of a triangle. Then you can build your own mini-geometry using a set of your own custom tools—like defining a “nine-point circle” using custom tools for several centers of a triangle and for an “Euler segment” connecting them (see 0). Programming your own tools can be fun once you get the hang of it. Furthermore, it gives you a good idea about how GeoGebra’s standard tools were created and why they work.

Constructing a Perpendicular Bisector

The procedure used to construct an equilateral triangle can be used to locate the midpoint of a segment and to construct a perpendicular to that segment, passing through the midpoint.

As you already saw, the construction process for an equilateral triangle creates a number of interesting relationships among different points and segments. In this tab, points A, B and C form an equilateral triangle. Segment AB crosses segment CD at the exact midpoint of CD and the angles between theses two segments are all right angles (90 Degrees). We say that AB is the “perpendicular bisector” of CD—meaning that AB cuts CD at its midpoint, evenly in two sectors, and that AB is perpendicular (meaning, at a right angle) to CD.

For many geometry constructions, it is necessary to construct a new line perpendicular to an existing line (like line FG). In particular, you may need to have the perpendicular go through the line at a certain point (like H). Can your team figure out how to do that?



Creating a Perpendicular Tool

Now that you know how to construct a perpendicular, you can automate this process to save you work next time you need a perpendicular line. Create a custom tool to automatically construct a perpendicular to a given line through a given point by following the directions in the tab.

Members of the team should each create their own custom perpendicular tool. One person could create a tool to construct a perpendicular bisector through the midpoint of a given line (no third point would be needed as an input for this one). Another person could create a perpendicular through a given point on the line. A third person could create a perpendicular through a given point that is not on the line. Everyone should be able to use everyone else's custom tool in this tab. Do the three tools have to be different? Does GeoGebra have three different tools for this? Do your custom tools work just like the GeoGebra standard perpendicular tools? Are there other cases for constructing perpendiculars?

GeoGebra makes programming a tool easy. However, it takes some practice to get used to the procedure. To program a tool, you have to define the Outputs you want (the points, lines, etc. that will be created by the custom tool) and then the Inputs that will be needed (the points, lines, etc. that a person will have to create to use the tool). For instance, to create a perpendicular to line AB through point C on it using this custom tool, a person would first select the custom tool as the active tool. Next, they would construct or select three points to define A, B and C (the line and a point on it as inputs to the custom tool). Then the perpendicular line would appear automatically (as the output of the custom tool).

Hint: You can identify the output objects by selecting them with your cursor before or after you go to the menu "Tools" | "Create New Tool …". Hold down the Command key (on a Mac) or the Control key (in Windows) to select more than one object. You can also identify the output objects from the pull-down list in the Output Objects tab. Similarly, you can identify input objects in the Input Objects tab by selecting them with the cursor or from the pull-down list. GeoGebra might identify most of the necessary input objects automatically. Give the tool a name that will help you to find it later and check the "Show in Toolbar" box so your tool will be included on the toolbar.

To be able to use your custom tool in another tab or another chat room later, you have to save it now. Save your custom tool as a **.ggt** file on your desktop. Take control and use the GeoGebra menu "Tools" | "Manage Tools…" | "Save As." Save your custom tool to your computer desktop, to a memory stick, or somewhere that you can find it later and give it a name like "Maria's_Perpendicular.ggt" so you will know what it is. When you want to use your custom tool in another tab or topic, take control, use the GeoGebra menu "File" | "Open…" then find and open the .ggt file that you previously saved. You should then be able to select your custom tool from the toolbar or from the menu "Tools" | "Manage Tools." When your custom tool is available to you, it will also be available to your teammates when they are in that tab.

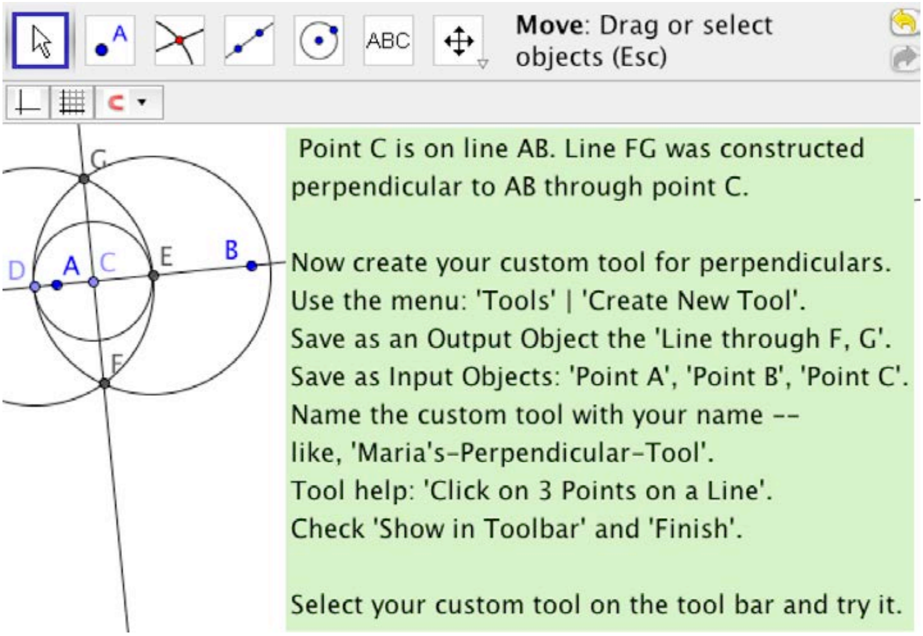
Hint: If a custom tool does not appear on your tool bar when you think it should be available, use the menu "Tools" | "Customize Toolbar", find the custom tool in the list of Tools, and insert it on the toolbar list where you want it (highlight the group or the tool you want it to be listed after).

Note: The custom tools in GeoGebra have some limitations, unfortunately. As noted in the help page, Outputs of custom tools are not moveable, even if they are defined as `Point[<Path>]`; if you need moveable output, you can define a list of commands and use it with Execute Command. For instance, if you define a right-triangle tool, you will not be able to freely drag the new vertex of the right triangles that are created with this tool; they will not be dynamic.

Hint: The GeoGebra help page for custom tools at http://wiki.geogebra.org/en/Custom_Tools has more information about defining and using custom tools. Most of that information applies when using VMT-with-GeoGebra as well.

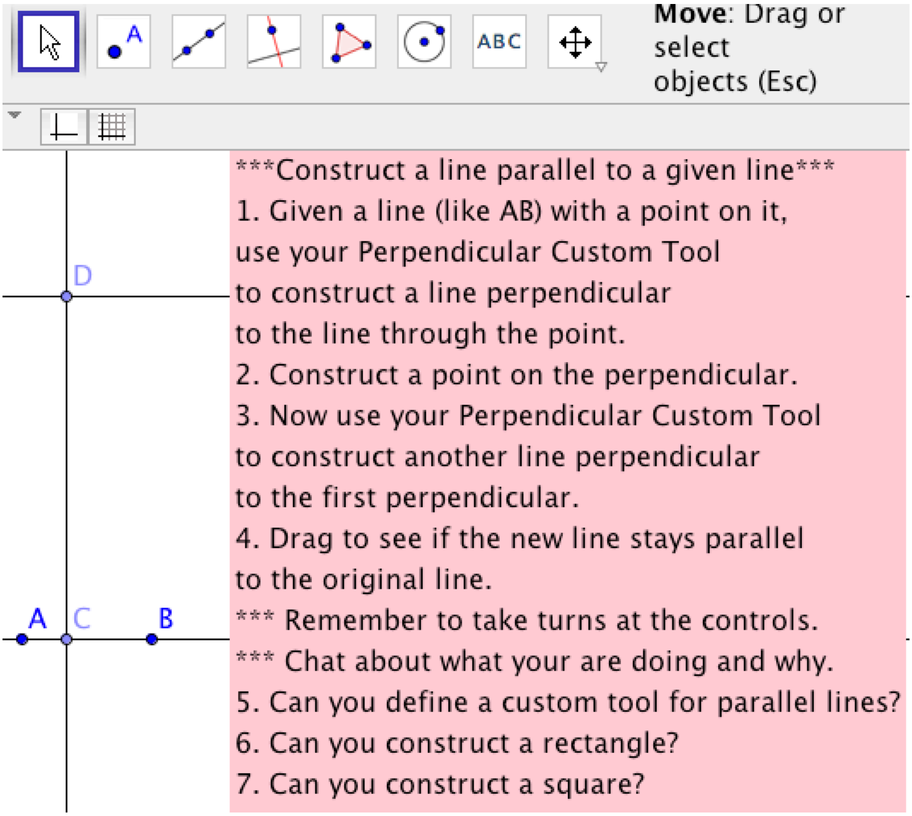
GeoGebra has a perpendicular tool that works like your custom tool. If you just used the standard tool, you would not be aware of the hidden circles that determine the dependencies to keep the lines perpendicular during dragging. Now that you understand these dependencies, you can use either the standard tool or your custom tool. You will not see the hidden circles maintaining the dependencies of lines that are dynamically perpendicular, but you will know they are there, working in the background.

The tools of GeoGebra extend the power of dynamic geometry while maintaining the underlying dependencies. By defining your own custom tools, you learn how dynamic geometry works "under the hood." In addition, you can extend its power yourself in new ways that you and your team think of.



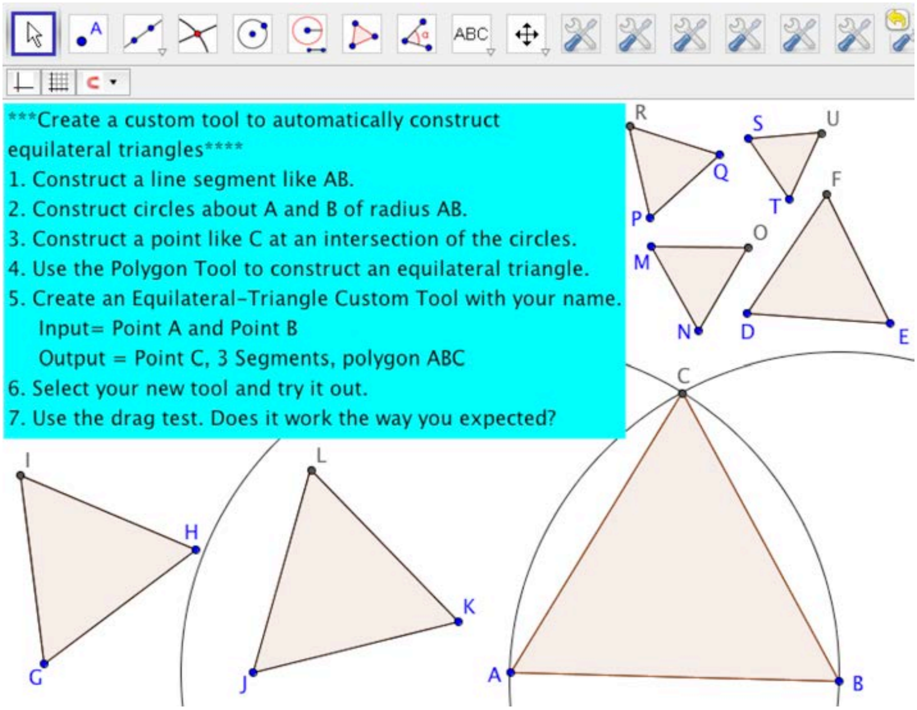
Creating a Parallel Line Tool

One member of the group should create a custom perpendicular tool (like you did in the previous tab) in this tab. Now use this custom perpendicular tool (or the standard perpendicular tool) to create a custom parallel tool. See how tools can build on each other to create a whole system of new possible activities.



Creating an Equilateral-Triangle Tool

Create a specialized custom tool for quickly generating equilateral triangles.



What we noticed:

What we wondered:

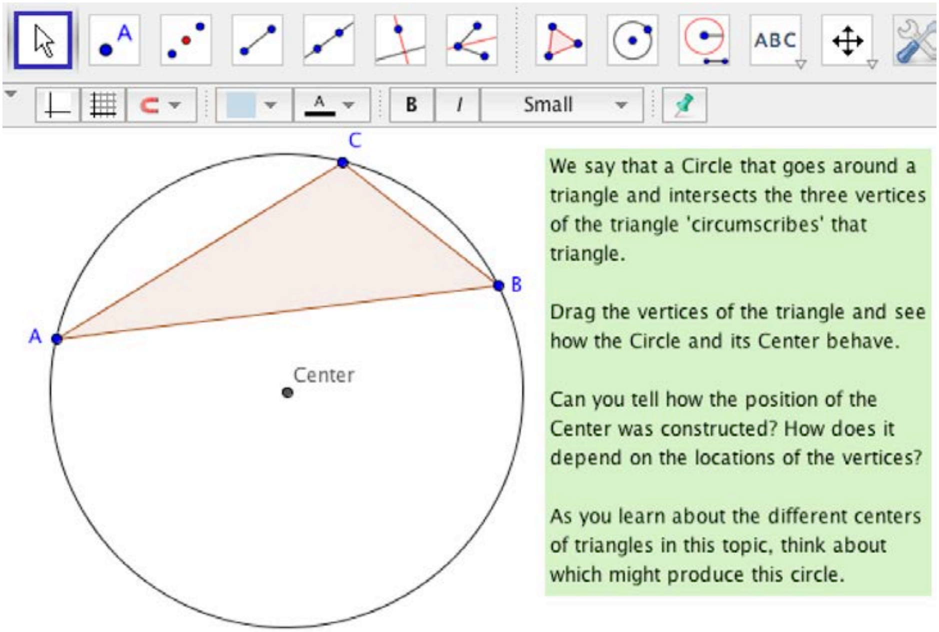
Finding Centers of Triangles

There are a number of interesting ways to define the "center" of a triangle, each with its own interesting properties.

(**Note**: *This is a long topic and may take a couple of sessions to complete. Or you can work on it more on your own after your team session works on the first several tabs.*)

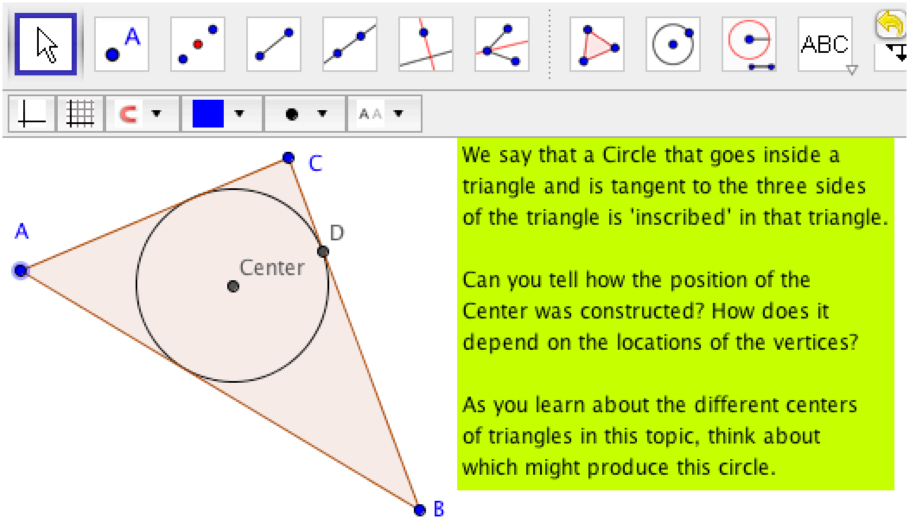
The Center for Circumscribing a Triangle

One "center" of a triangle is the point that is the center of a circle that circumscribes the triangle. Given a triangle ABC, how would you find and construct the center of a circle that goes through all three vertices of the triangle? When is this center inside of the triangle?



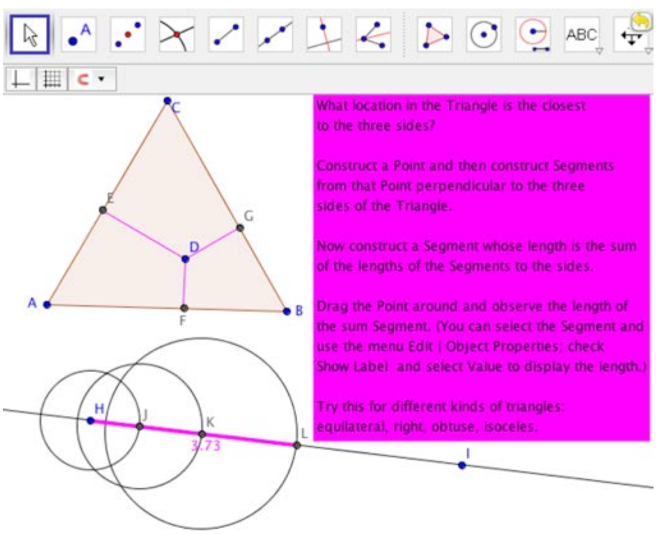
The Center for Inscribing a Triangle

Another center of a triangle is the point that is the center of a circle that is inscribed in the triangle. Drag the triangle in this tab to try to figure out how this center was constructed. When is this center inside of the triangle?



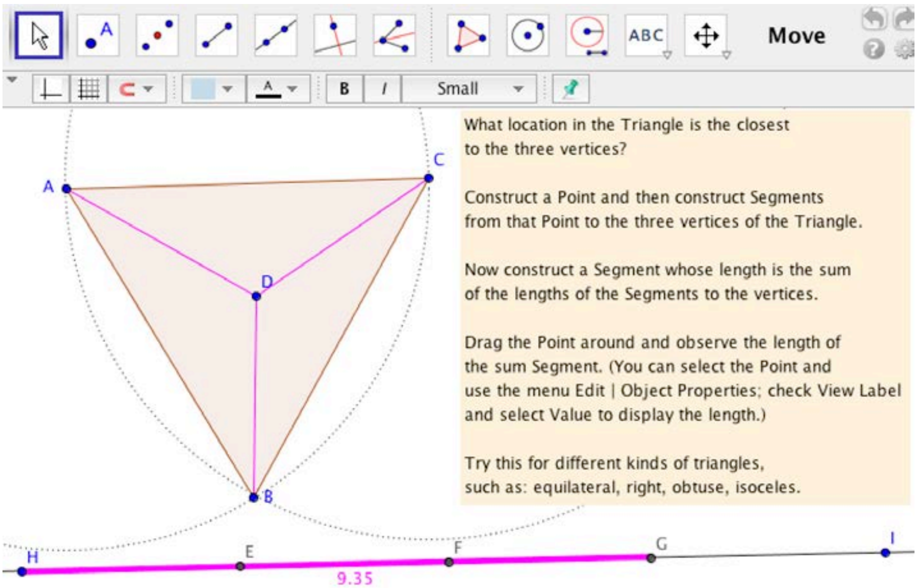
The Center Closest to The Sides of a Triangle

If you want to be as close as possible to all three sides of a triangle, where should you locate the center such that the sum of the distances to the three sides is as small as possible? Do you understand how line segment HL was constructed to display the sum of the lengths from the center to the sides? Are the segments DE, DF and DG perpendicular to the triangle sides? Why?



The Center Closest to the Vertices of a Triangle

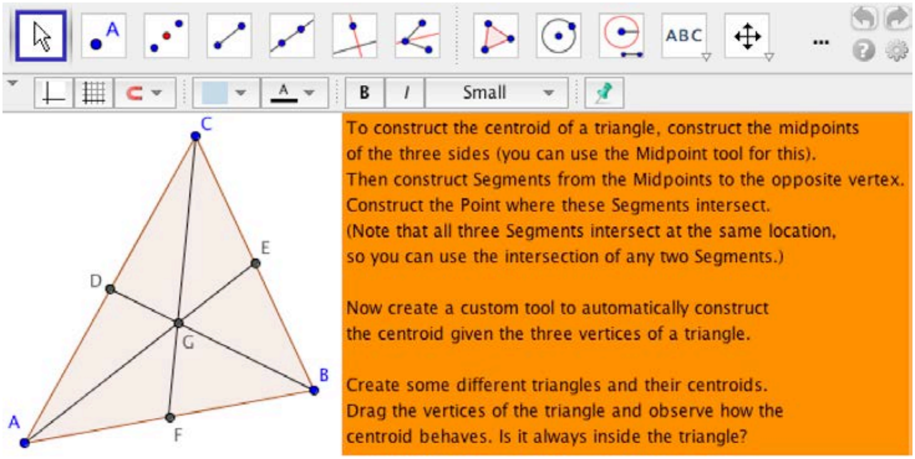
If you want to be as close as possible to all three vertices of a triangle, where should you locate the center such that the sum of the distances to the three vertices is as small as possible? Do you understand how line segment GH was constructed? Chat about this to make sure everyone in your team understands.



The Centroid of a Triangle

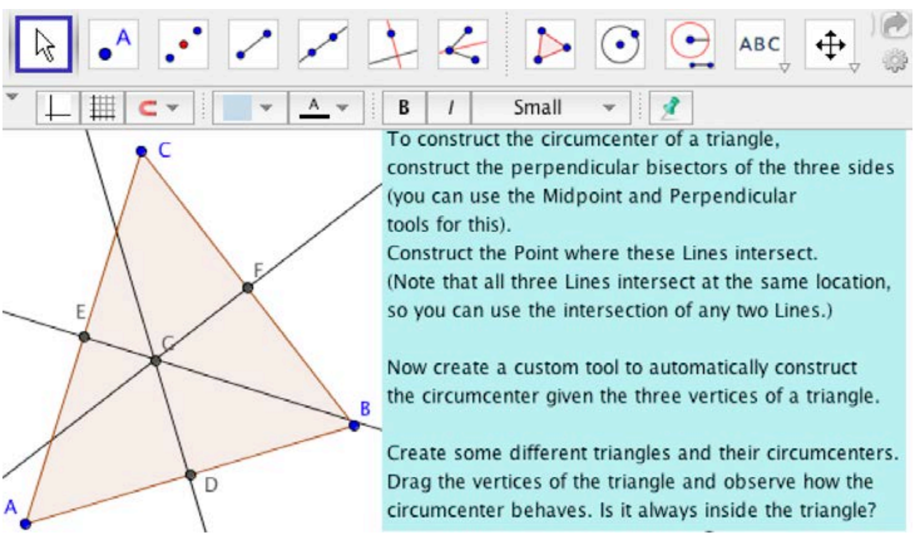
The “centroid” of a triangle is the meeting point of the three lines from the midpoints of the triangle’s sides to the opposite vertex. Create a custom centroid tool.

Take control and use the GeoGebra menu “Tools” | “Manage Tools…” | “Save As.” Save your custom tool to your computer desktop or somewhere that you can find it later and give it a name like “Tanya’s_Centroid.ggt” so you will know what it is. When you want to use your custom tool in another tab or topic, take control, use the GeoGebra menu “File” | “Open…” then find and open the .ggt file that you saved. You should then be able to select your custom tool from the menu “Tools” | “Manage Tools.” When your custom tool is available to you, it will also be available to your teammates when they are in that tab.



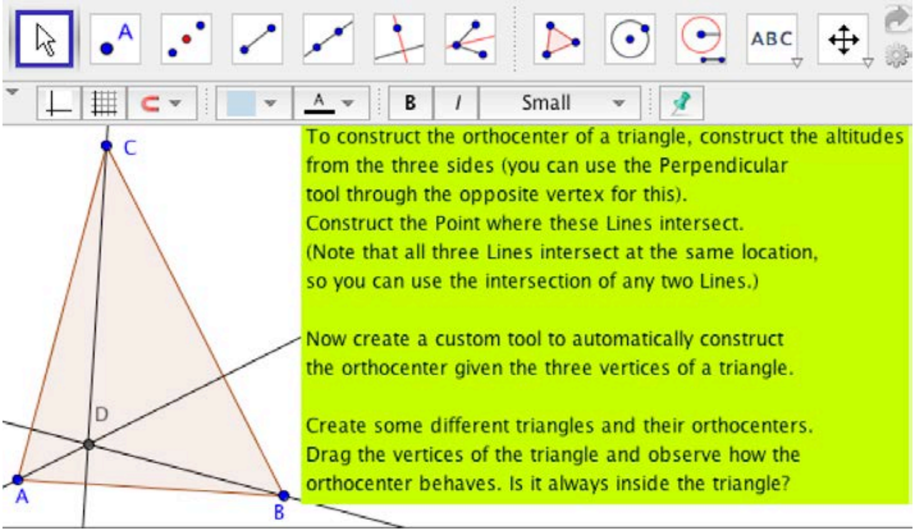
The Circumcenter of a Triangle

The "circumcenter" of a triangle is the meeting point of the three perpendicular bisectors of the sides of the triangle. Create a custom circumcenter tool and save it.



The Orthocenter of a Triangle

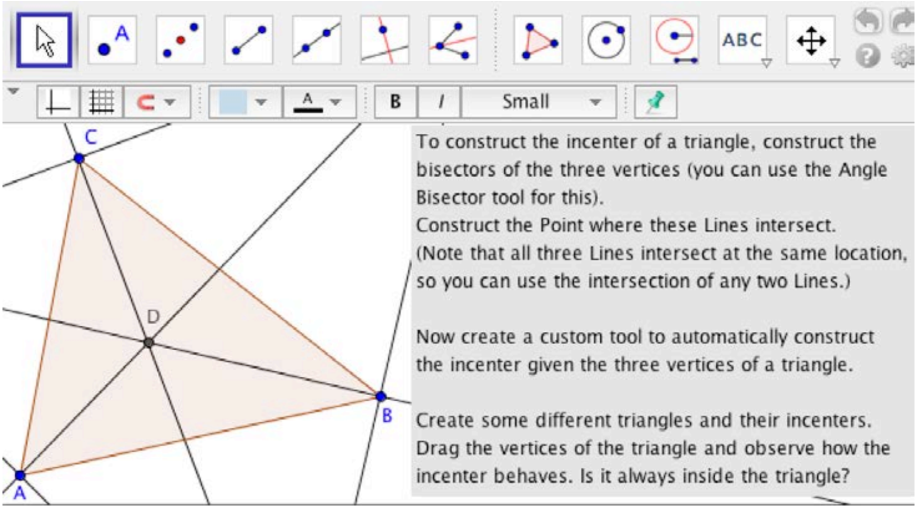
The "orthocenter" of a triangle is the meeting point of the three altitudes of the triangle. An "altitude" of a triangle is the segment that is perpendicular to a side and goes to the opposite vertex. Create a custom orthocenter tool and save it.



The Incenter of a Triangle

The "incenter" of a triangle is the meeting point of the three angle bisectors of the angles at the triangle's vertices. Create a custom incenter tool and save it.

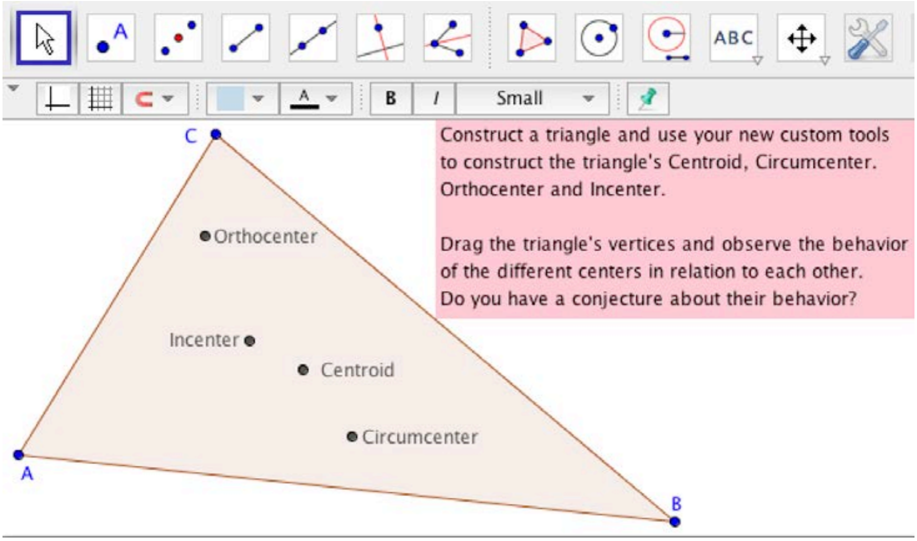
Note: The incenter of a triangle is the center of a circle inscribed in the triangle. A radius of the inscribed circle is tangent to each side of the triangle, so you can construct a perpendicular from the incenter to a side to find the inscribed circle's point of tangency – and then use this point to construct the inscribed circle.



The Euler Segment of a Triangle

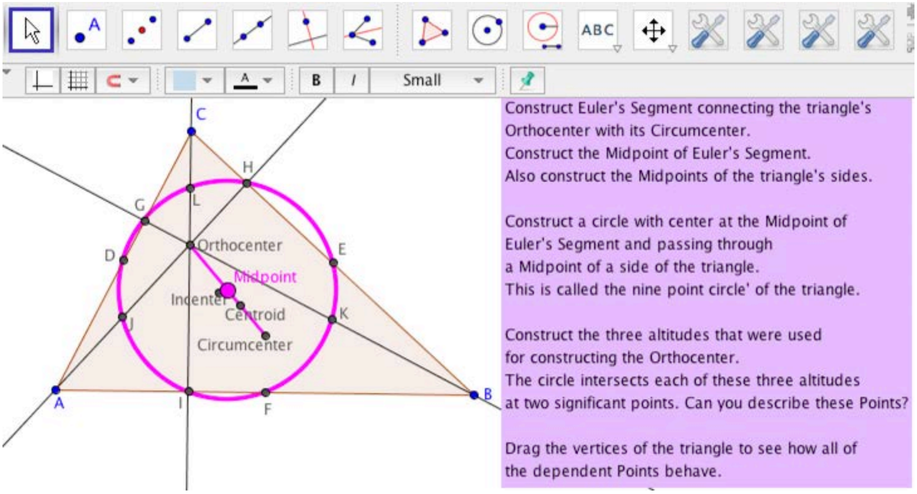
A Swiss mathematician named Euler discovered a relationship among three of the centers that you created custom tools for. Can you discover what he did? He did this in the 1700s—without dynamic-geometry tools. Euler's work renewed interest in geometry and led to many discoveries beyond Euclid.

Note: Take turns to re-create a custom tool for each of the triangle's special points: centroid, circumcenter, orthocenter and incenter, as done in the previous tabs. Or else, load the custom tools you created before using the GeoGebra menu "File" | "Open…" then find and open the .ggt files that you saved. You should then be able to select your custom tools from the menu "Tools" | "Manage Tools." When your custom tools are available to you, they will also be available to your teammates in that tab.



The Nine-Point Circle of a Triangle

You can construct a circle that passes through a number of special points in a triangle. First construct custom tools for the four kinds of centers or open them from your .ggt files that you saved in previous tabs of this topic. Connect the orthocenter to the circumcenter: this is "Euler's Segment." The Centroid lies on this segment. A number of centers and related points of a triangle are all closely related by Euler's Segment and its Nine-Point Circle for any triangle. Create an Euler Segment and its related Nine-Point Circle, whose center is the midpoint of the Euler Segment.



You can watch a six-minute video of this segment and circle at: www.khanacademy.org/math/geometry/triangle-properties/triangle_property_review/v/euler-line.

The video shows a hand-drawn figure, but you can drag your dynamic figure to explore the relationships more accurately and dynamically.

Are you amazed at the complex relationships that this figure has? How can a simple generic triangle have all these special points with such complex relationships? Could these result from the dependencies that get constructed when you define the different centers in your custom tools?

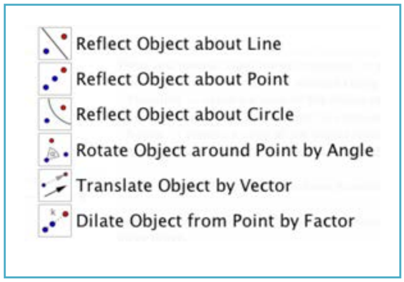
What we noticed:

What we wondered:

Transforming Triangles

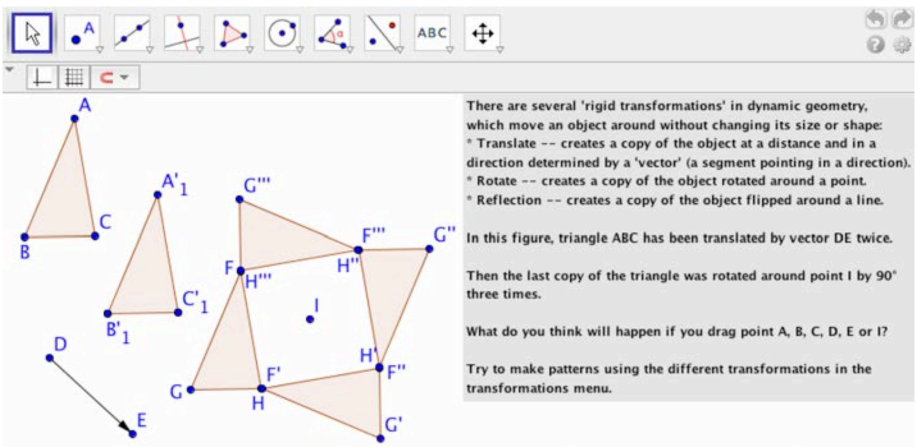
There are a number of "transformations" defined in dynamic geometry. They are useful for exploring and proving relationships among dynamic-geometry objects.

GeoGebra has a special menu of transformations. This topic introduces rotation, reflection and translation. Whole fields of mathematics could be based on transformation operations.



Rigid Transformations

There are "rigid transformations," which maintain the size and shape of any dynamic-geometry object, but translate, rotate or reflect the object from its original location. Drag the figures in this tab to discover how the dependencies work. Think up some new patterns and figure out how to construct them using GeoGebra's transformation tools.

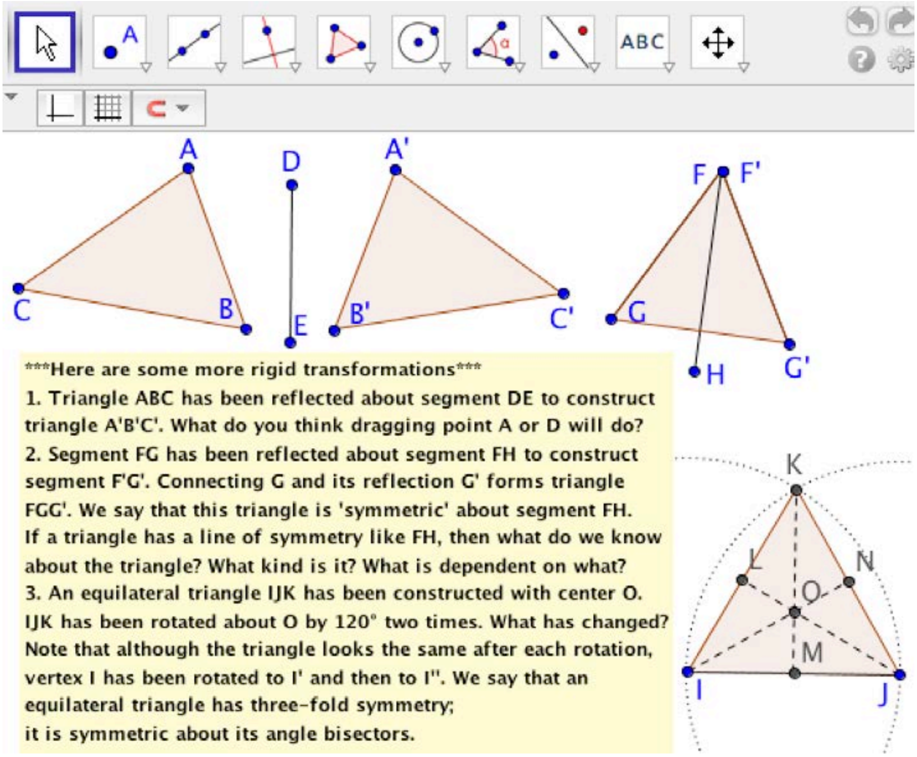


Angles of Symmetry

Here are some examples of symmetry in triangles.

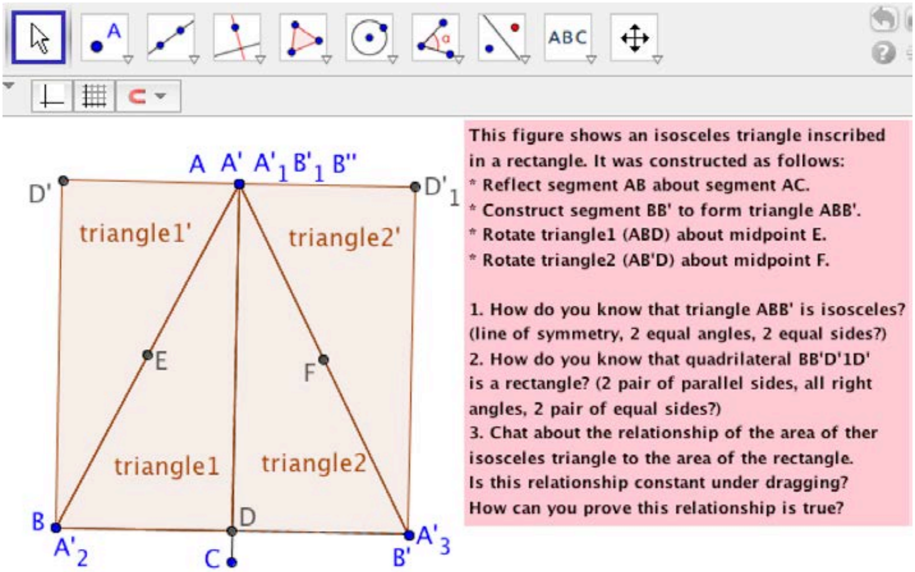
Construct an equilateral triangle like IJK with its angle bisectors. Try rotating the triangle around its center 120 degrees at a time. Try reflecting it about its angle bisectors. Symmetry means that certain transformations leave it looking the same.

What symmetries do you think a square has?



Areas of Triangles

You can use rigid transformations to demonstrate relationships among areas of dynamic-geometry objects.



What we noticed:

What we wondered:

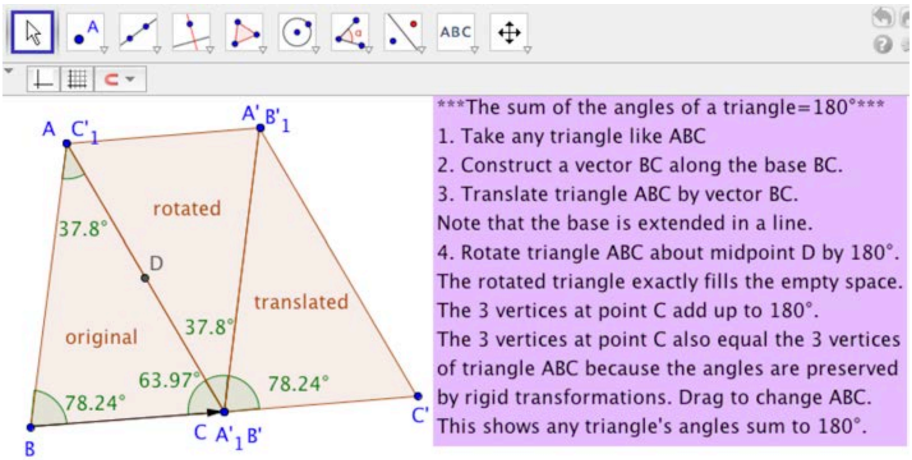
Exploring Angles of Triangles

There are a number of important theorems, propositions or proven facts about angles in triangles. They are basic theorems of geometry and are used often in solving typical geometry problems. Dynamic-geometry constructions can help you to understand why they are true. Constructing and dragging them will help you to remember, use and enjoy these facts. You will not have to memorize them, because you will understand them and be able to figure them out again if your forget them.

(**Note**: *This is a long topic. Try to explore some of the tabs on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session.*)

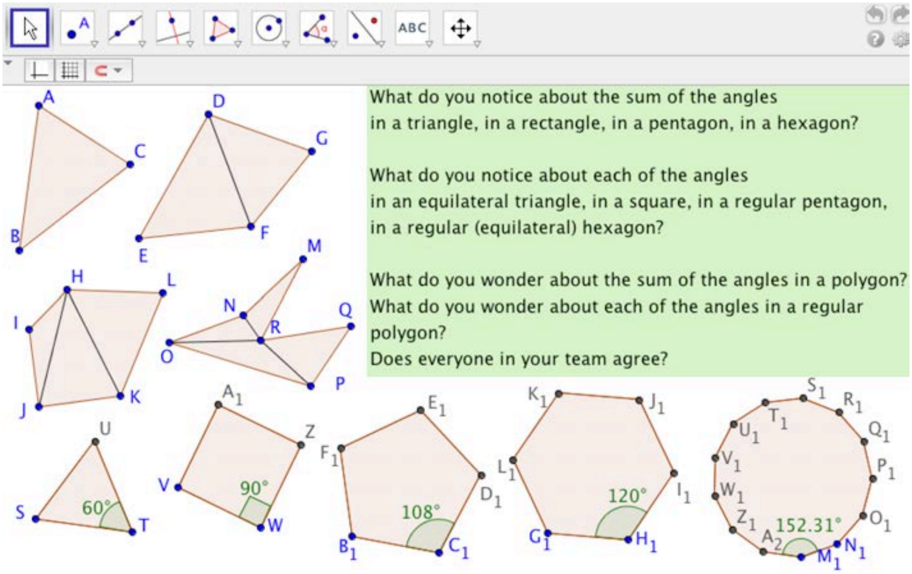
Sum of the Angles in a Triangle

The angle around a complete circle can be defined (arbitrarily, based on the ancient Babylonian system of mathematics) as 360 degrees. That makes the number of degrees in an angle that forms a straight line 180 degrees and the number of degrees in a right angle (formed by a perpendicular to a line) 90 degrees. You may have heard that the three angles in any triangle add up to 180 degrees. Why is that so?



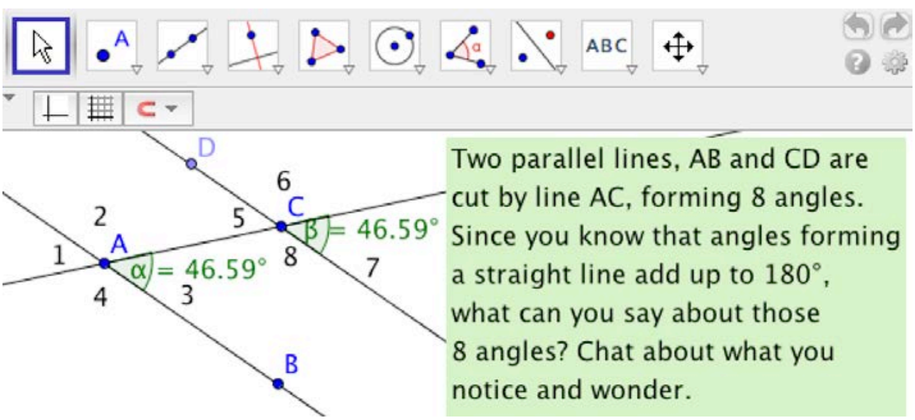
Sum of the Angles in a Polygon

Now that you know there are 180 degrees in any triangle, can you figure out how many are in other polygons?



Corresponding Angles

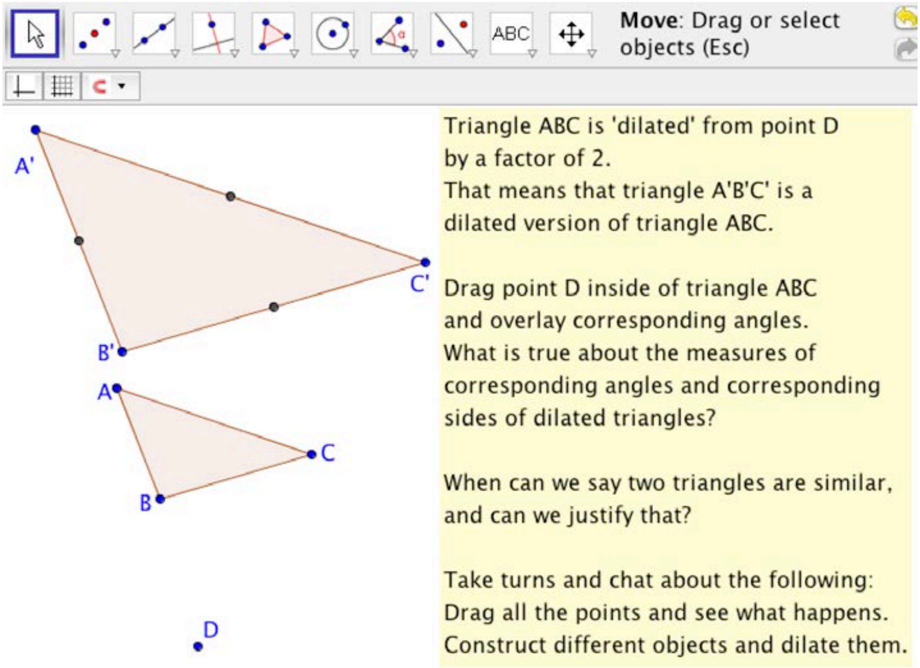
It is often useful when solving a geometry problem or applying geometry in the world to know which angles are equal to each other. Drag the lines in this tab around. Which angles stay equal to each other? Which pairs of angles always add up to 180 degrees?



Dilation of Triangles

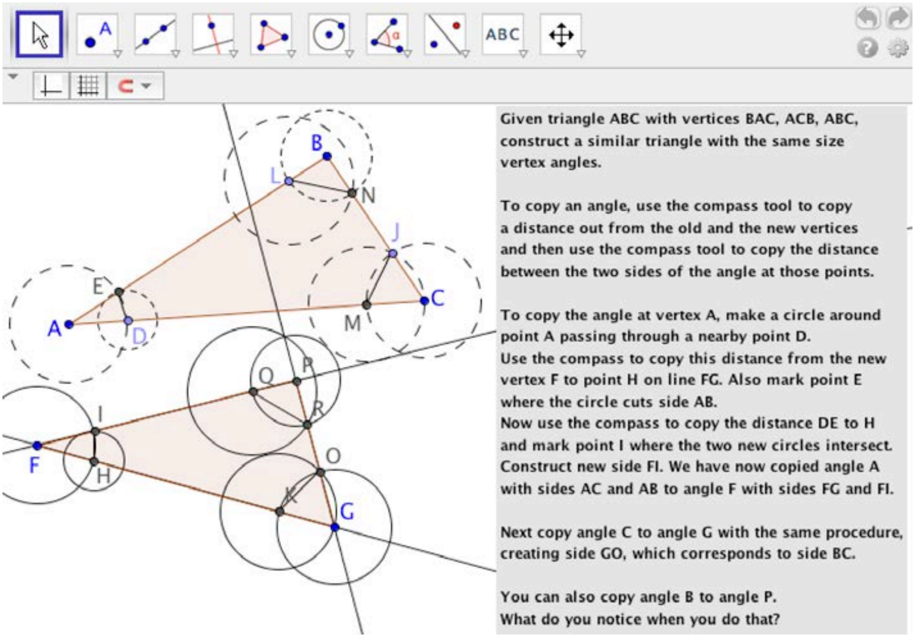
There is a transformation in dynamic geometry that is not rigid: "dilation." That means the object that is transformed can change size while remaining the same shape.

The term "similar" is a technical term in geometry. Two triangles are "similar" in dynamic geometry if they always have the same shape (i.e., all the same corresponding angles) as each other, even if they are different sizes.



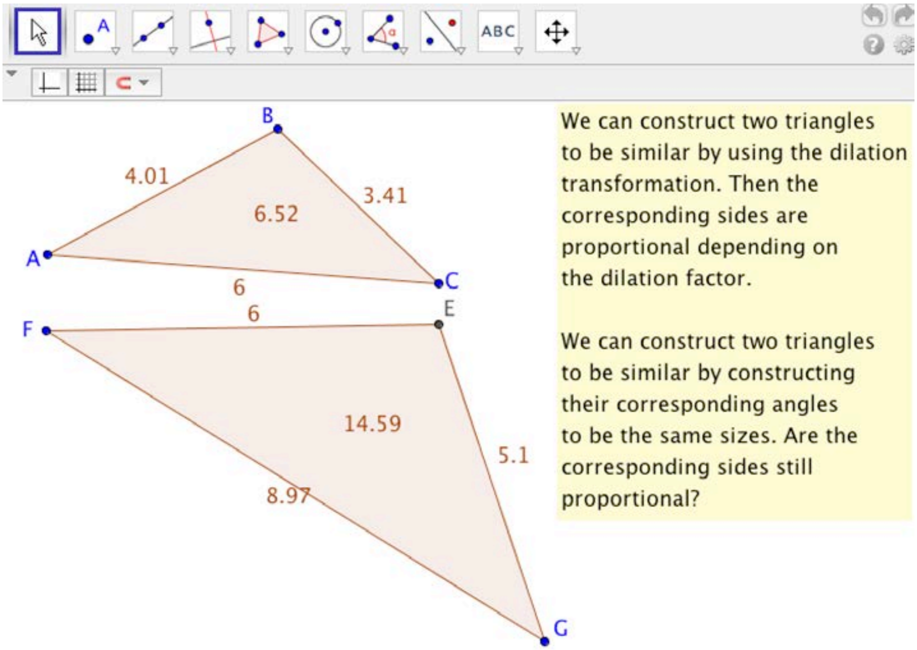
Angles of Similar Triangles (AAA)

Given a dynamic triangle ABC, if we construct another triangle that is constrained to have 1, 2 or 3 angles the same size as corresponding angles in ABC, will the triangles be similar? This is the Angle-Angle-Angle (or AAA) rule for similar triangles.



Sides of Similar Triangles

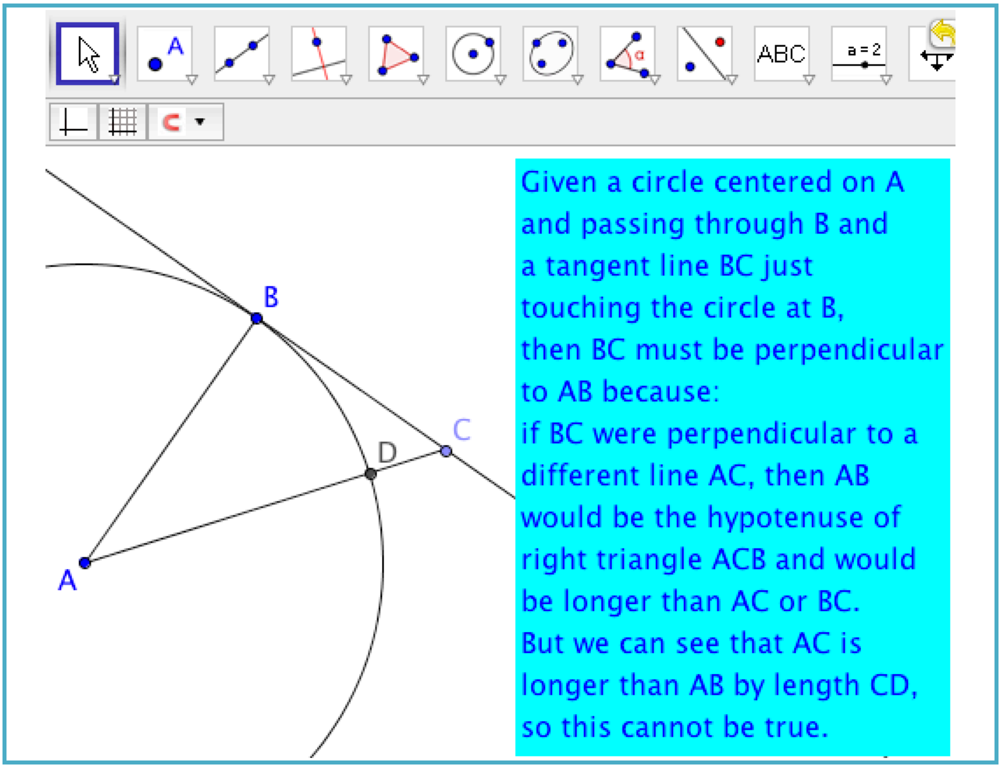
Given two similar triangles, what are the relationships among the corresponding sides of the triangles?



Tangent to a Circle

The tangent to a circle is perpendicular to the radius at the point of tangency. You can prove this using an interesting kind of "proof by contradiction" that Euclid often used. First, you assume that the conjecture is false – i.e., that there might be some other line that is perpendicular to the tangent. Then you show that would lead to a contradiction of the given conditions. Therefore, the assumption must be wrong and the conjecture must be right.

Explain each step in the proof in the chat. Does this make sense to everyone in the team?



What we noticed:

What we wondered:

Visualizing Congruent Triangles

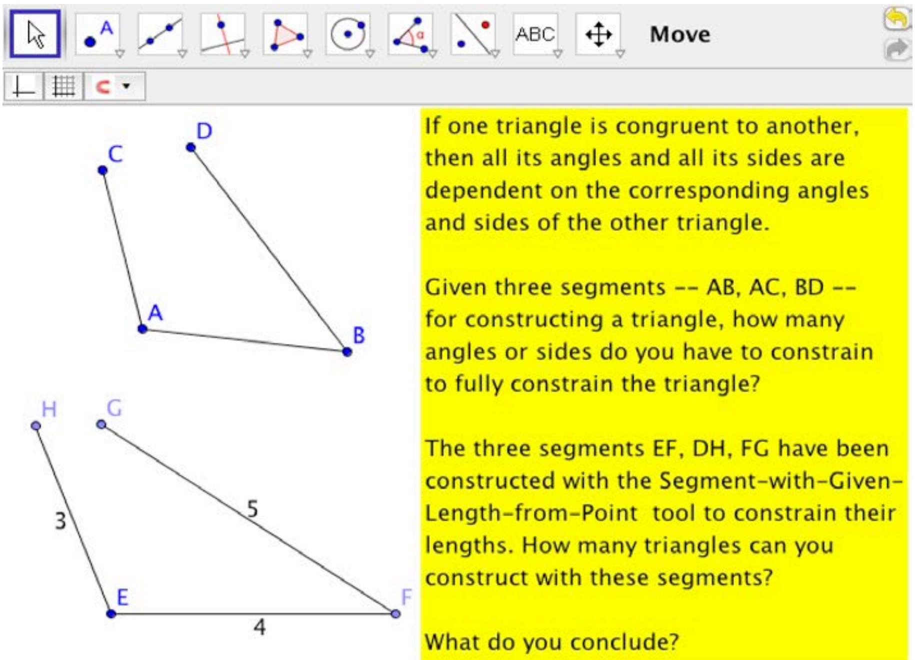
Before working on this topic with your team, you should read ***Tour 5: "VMT Logs and Replayer for Reflection"*** *near the end of this booklet. It will show you how to download logs of your chat and to replay your session. Then you can review your discussions and include excerpts of the chat log or screen shots of the session in your journal or reports.*

(**Note**: *This is a long topic. Try to explore some of the tabs on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session.*)

All the corresponding angles and sides of congruent triangles are equal. However, you can constrain two triangles to be congruent by just constraining 3 of their corresponding parts to be equal – for certain combinations of 3 parts. Dynamic geometry helps you to visualize, to understand and to remember these different combinations.

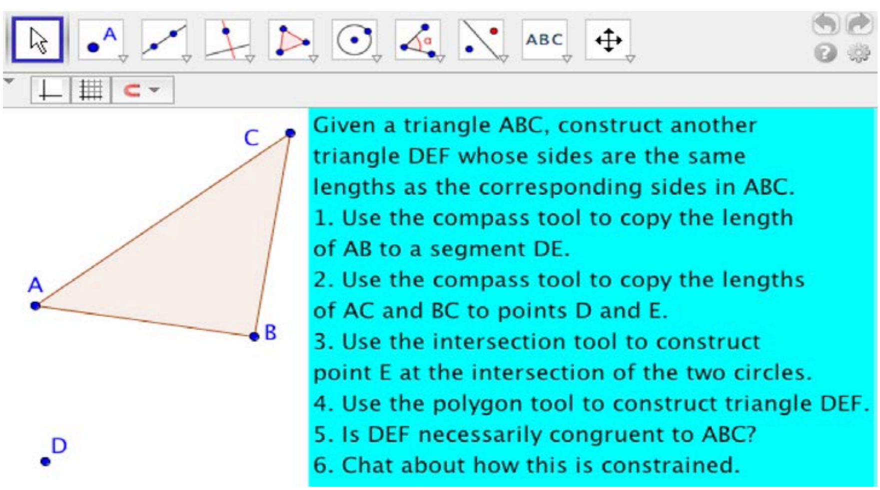
Corresponding Sides and Angles of Congruent Triangles

What constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle?



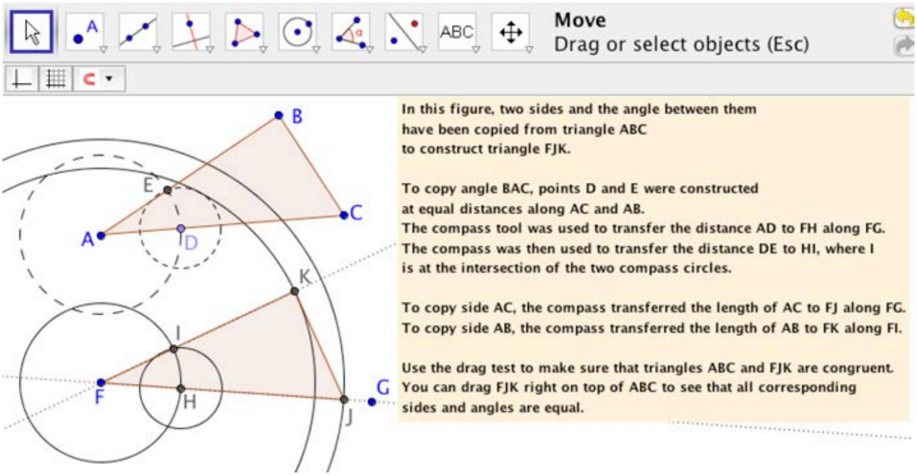
Side-Side-Side (SSS)

If all three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent. This is called the “Side-Side-Side” (or “SSS”) rule.



Side-Angle-Side (SAS)

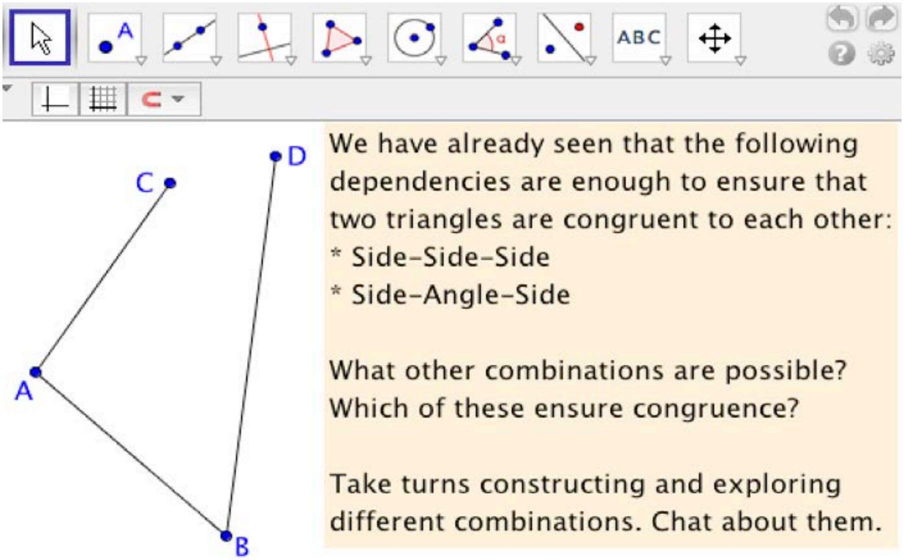
If two sides and the angle **between them** of one triangle are equal to the corresponding sides and angle **between them** of another triangle, then the two triangles are congruent.



Combinations of Sides and Angles

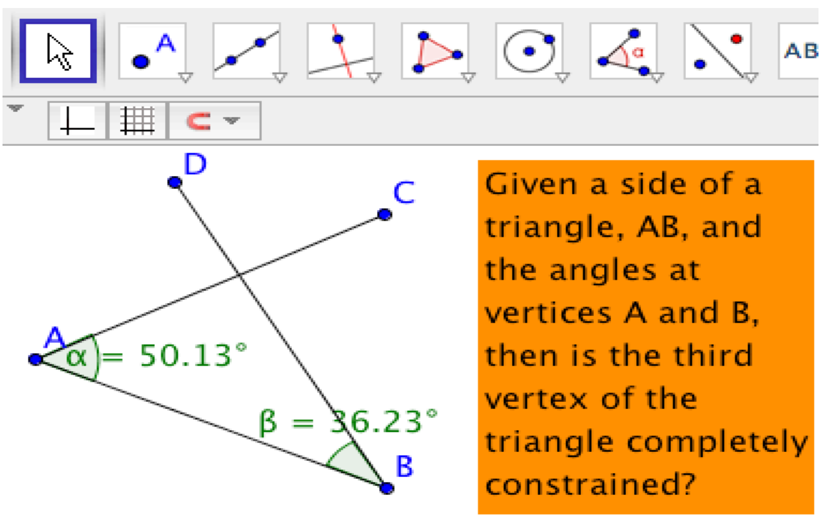
You can constrain two dynamic triangles to be congruent using a number of different combinations of equal corresponding sides and/or angles.

What combinations of constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle?



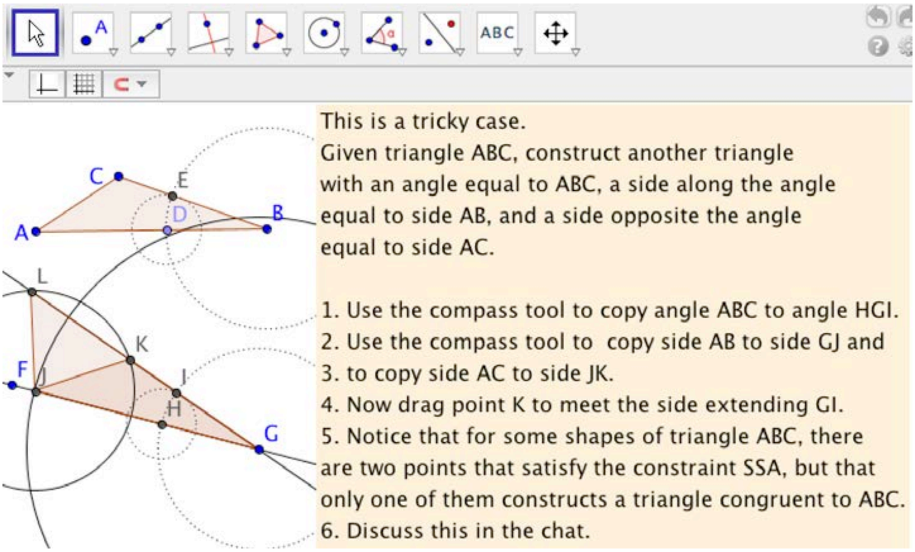
Angle-Side-Angle (ASA)

If two angles and the side included **between them** of one triangle are equal to the corresponding two angles and side **between them** of another triangle, then the two triangles are congruent. This is called the Angle-Side-Angle or ASA rule.



Side-Side-Angle (SSA)

What if two corresponding sides and an angle are equal, but it is not the angle included between the two sides?



What we noticed:

What we wondered:

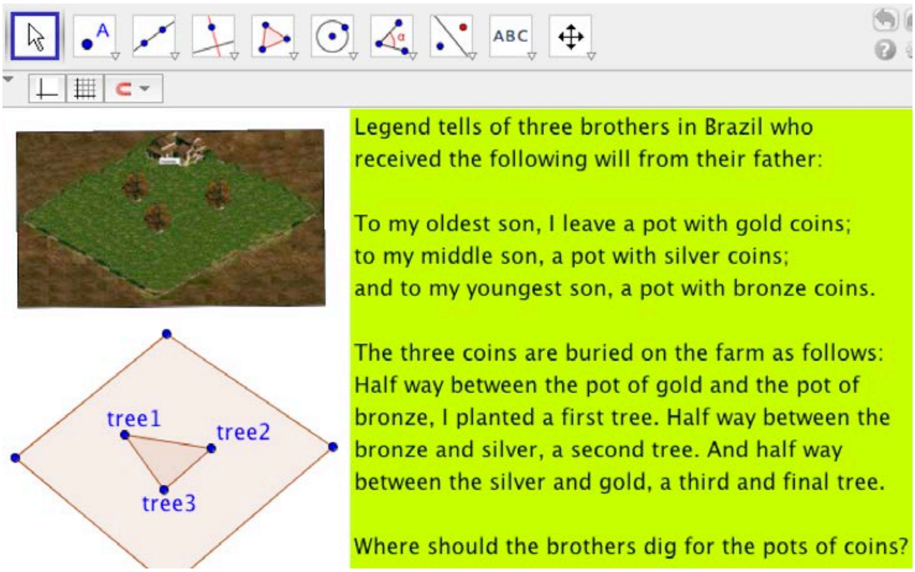
Solving Geometry Problems

Here is a set of challenge problems for your team.

If the team does not solve them during its session, try to solve them on your own and report your findings in the next team session.

Treasure Hunt

Can you discover the pot of gold in this tale told by Thales de Lelis Martins Pereira? You might want to construct some extra lines in the tab.



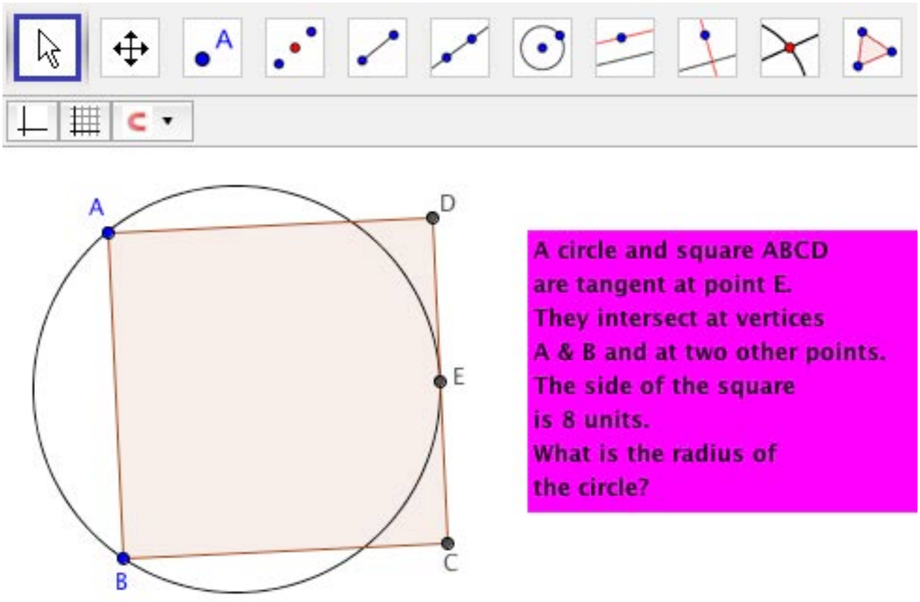
What we noticed:

What we wondered:

Square and Circle

Can you determine the radius of the circle? You should not have to measure. What if the side of the square is "s" rather than "8"?

Hint: To solve this kind of problem, it is usually useful to construct some extra lines and explore triangles and relationships that are created. If you know basic algebra, you might set up some equations based on the relationships in the figure.



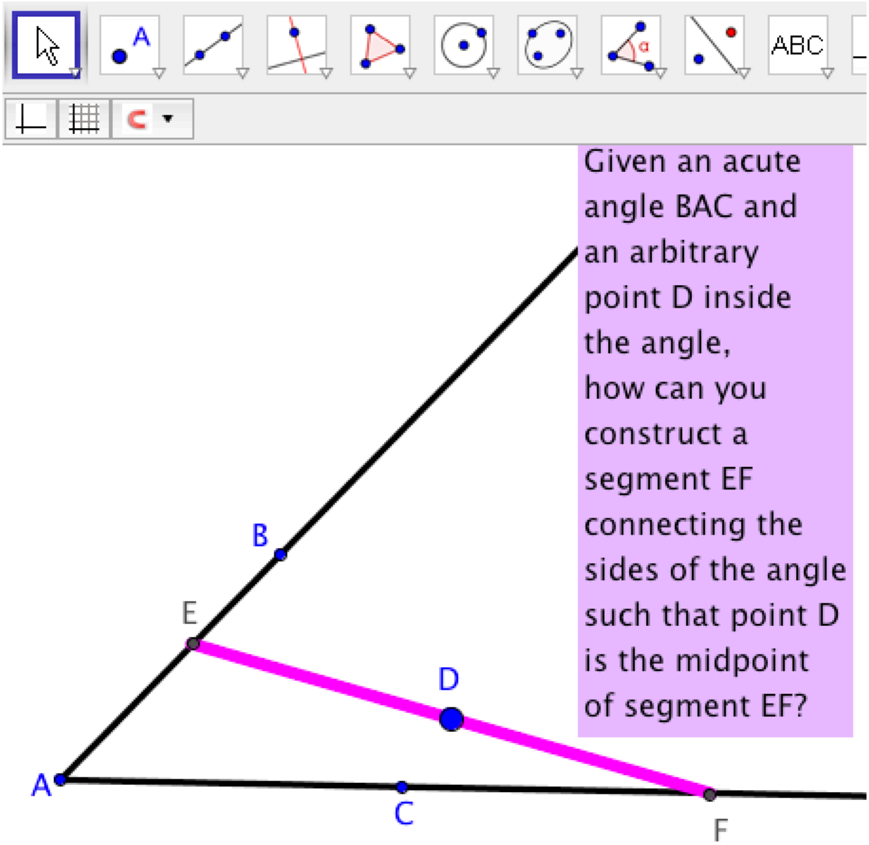
What we noticed:

What we wondered:

Crossing an Angle

Can you construct the segment crossing the angle with the given midpoint?

Hint: This is a challenging problem. Try to add some strategic lines and drag the figure in the tab to see what would help to construct the segment EF at the right place.



What we noticed:

What we wondered:

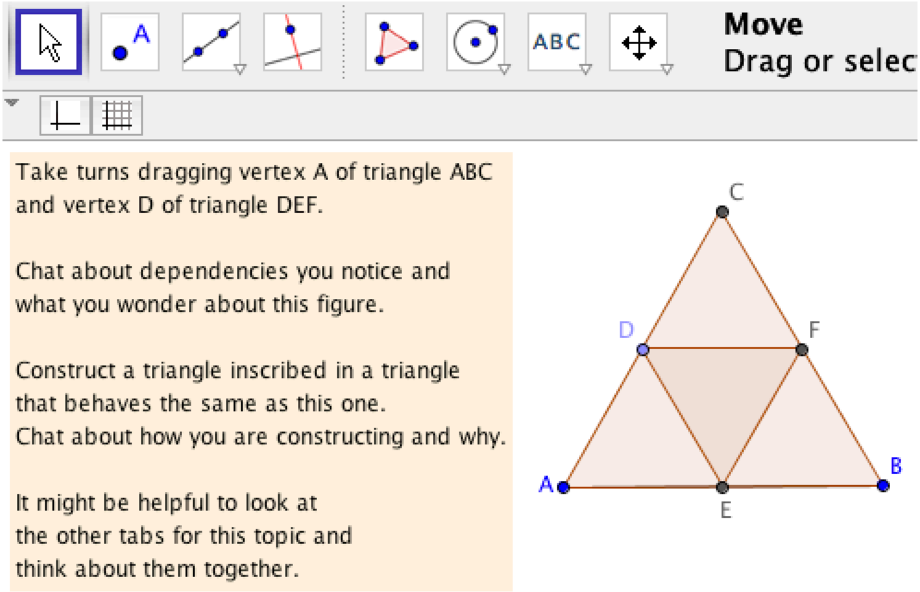
Inscribing Polygons

The following three tabs present an interesting set of figures. They illustrate how you can discover dependencies through dragging and create dependencies through construction. There are important similarities in the three tabs, suggesting a generalization.

If the team does not solve the three related problems during its session, try to solve them on your own and report your findings in the next team session.

Inscribed Triangles

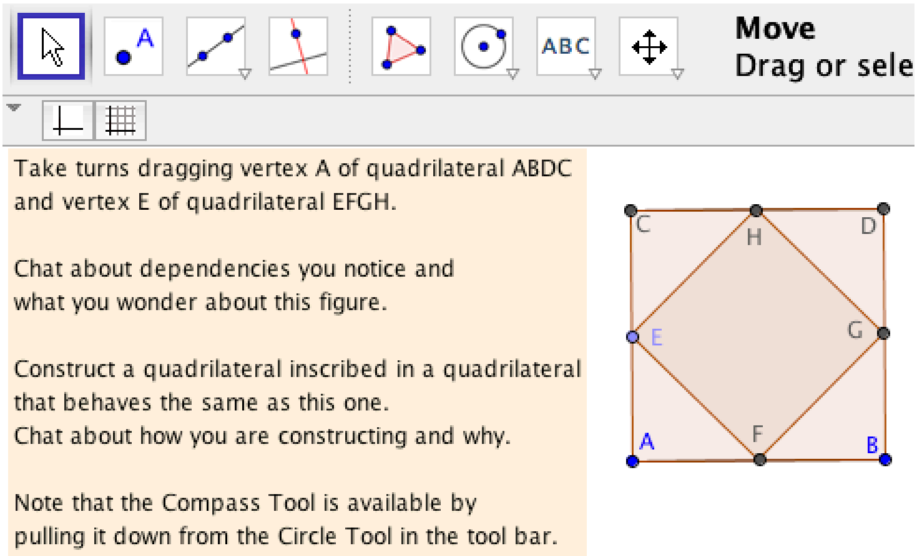
First, try to construct a pair of inscribed triangles.



If you solved this, did you construct the lengths of the sides of the outer triangle to be directly dependent upon each other? Did you construct the lengths of the sides of the inner triangle to be directly dependent upon each other? If not, do you think that the triangles are equilateral?

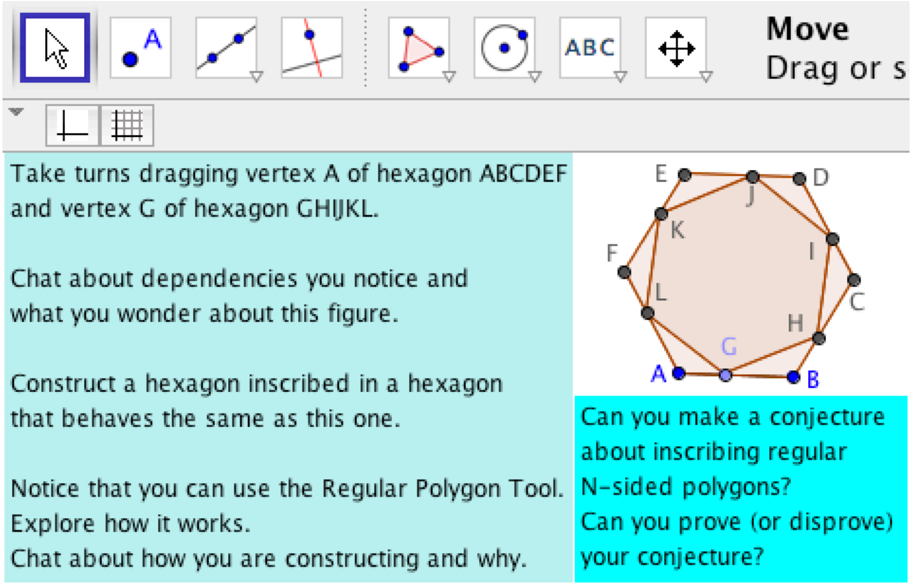
Inscribed Quadrilaterals

Can inscribed quadrilaterals behave the same way?



Inscribed Hexagons

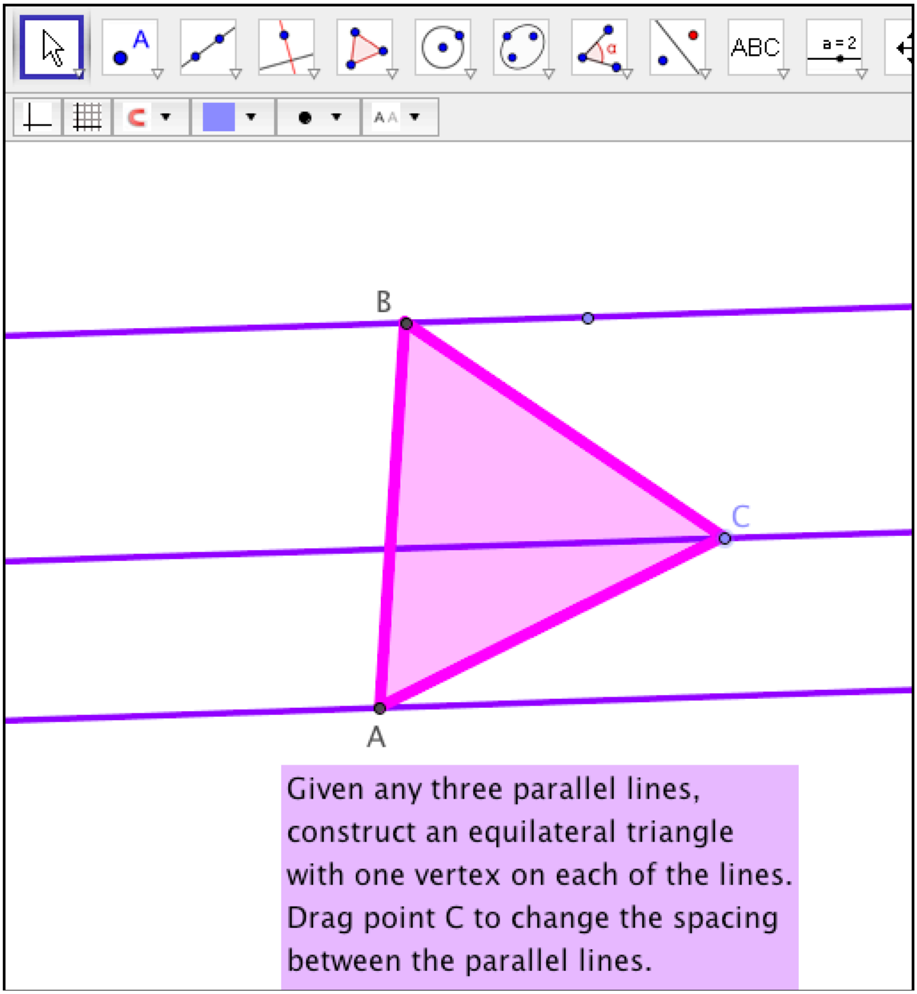
Can all polygons be inscribed the same way? Try it for a regular hexagon first. Do you see a general pattern between these three tabs: inscribed equilateral triangles, inscribed squares and inscribed regular hexagons? What is your conjecture about this? How might you prove it?



Triangle on Parallel Lines

Here is a challenge problem if you enjoyed the inscribed polygon problems:

Given any three parallel lines, can you construct an equilateral triangle such that it has one vertex on each of the parallel lines? Drag the example triangle to discover the dependencies as the line through point C takes different positions between the other two parallel lines. Then construct your own figure such that points B and C are equidistant from A and from each other no matter what position the line through point C takes.



What we noticed:

What we wondered:

Building a Hierarchy of Triangles

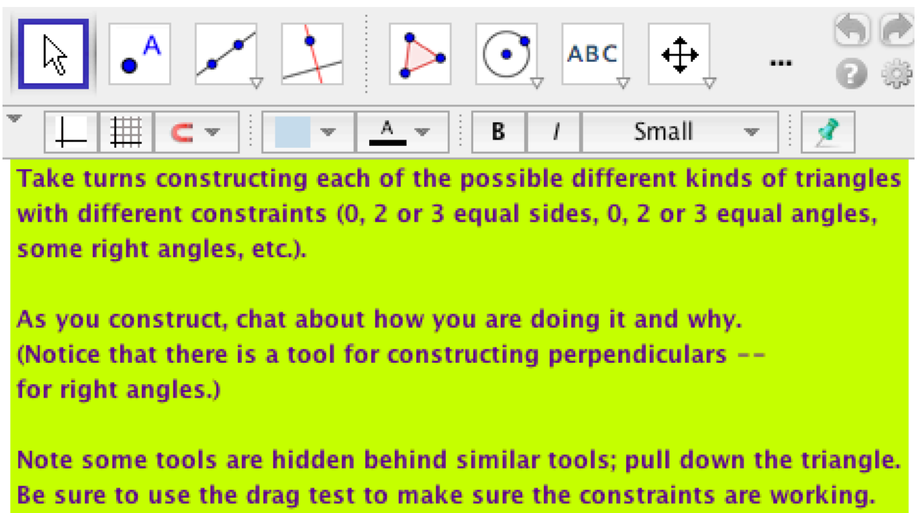
Now you can use GeoGebra tools (or your own custom tools) to construct triangles with different constraints. How are the different kinds of triangles related to each other? Which ones are special cases of other kinds?

Constructing Other Triangles

How many different kinds of dynamic triangles can your team construct?

Challenge: Note that there is no tool for copying an angle; how can you create an angle equal to an existing angle and can you make a custom tool for it?

You might want to make a list or table of the different possible constraints on the sides and angles of triangles.



The Hierarchy of Triangles

How are the different kinds of triangles related to each other?

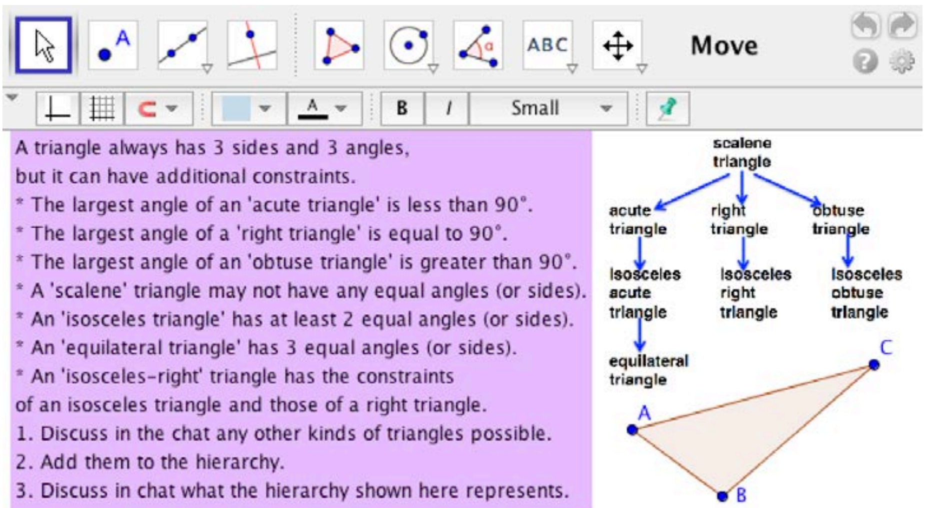
There are different ways of thinking about how triangles are related in dynamic geometry.

In dynamic geometry, a generic or "scalene" triangle with no special constraints on its sides or angles may be dragged into special cases, like a right triangle or an equilateral triangle. However, it does not have the constraints of a right angle vertex or equal sides built into it by its construction, so it will not necessarily retain the special-case characteristics when it is dragged again.

You can think of a hierarchy of kinds of triangles: an equilateral triangle can be viewed as a special case of an isosceles acute triangle, which can be viewed as a special case of an acute triangle, which can be viewed as a special case of a scalene triangle.

Can your team list all the distinct kinds of triangles?

Can your team connect them in a hierarchy diagram? Create a hierarchy diagram like the one shown in the tab. Add more kinds of triangles to it. You may want to reorganize the structure of the diagram.



What we noticed:

What we wondered:

Exploring Quadrilaterals

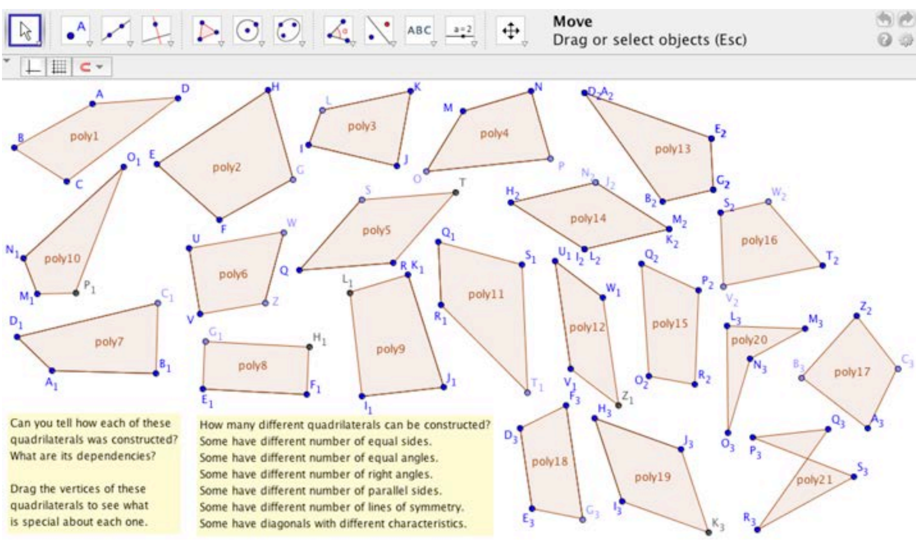
Before working on this topic with your team, you may want to look at ***Tour 6: "GeoGebra Videos and Resources"*** *near the end of this booklet. It lists some YouTube videos that you may enjoy watching.*

The term "quadrilateral" means "four-sided" in Latin. Just join four segments at their endpoints. There are many different kinds of quadrilaterals and complex relationships among their related constructions.

Dragging Different Quadrilaterals

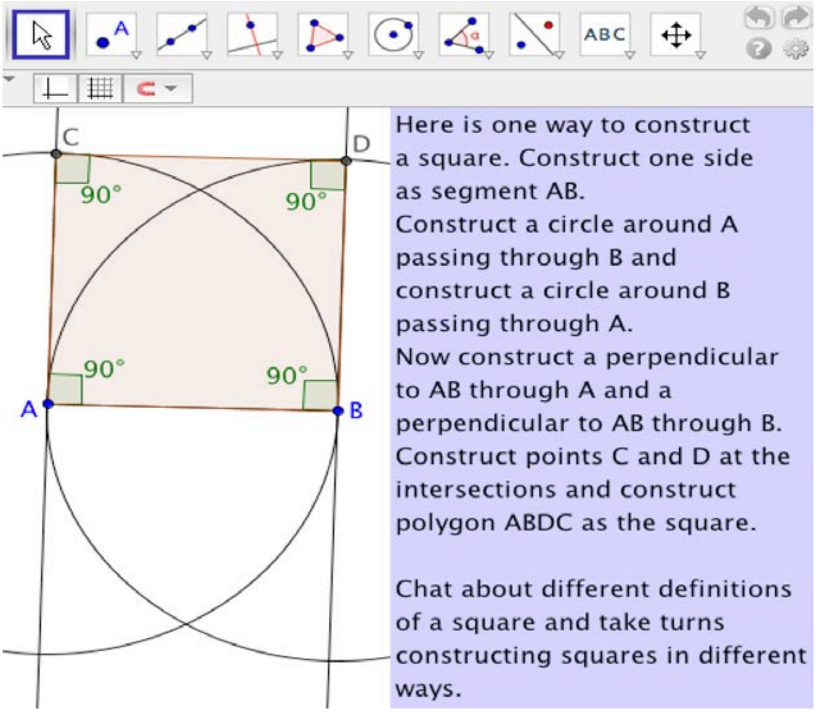
There are many different quadrilaterals with specific construction characteristics and dynamic behaviors.

Identify the constraints on each of the quadrilaterals in this tab. Can you think of any possible quadrilaterals that are not there? Does everyone on your team agree?



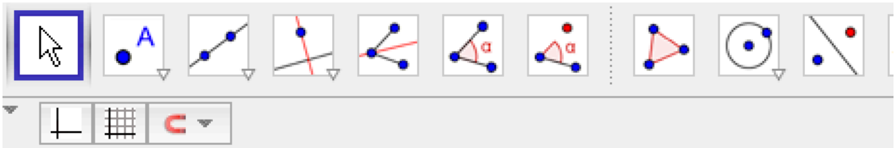
Constructing a Square

There may be many ways to construct a quadrilateral with specific constraints. How many ways can you list to construct a square? Would you say that the construction in this tab defines the constraints as three equal sides and two right angles?



Constructing Different Quadrilaterals

Can you construct the different kinds of quadrilaterals pictured here?



Take turns constructing each of the different kinds of quadrilaterals with different constraints (some equal sides; some equal angles, some right angles, some parallel sides).

As you construct, chat about how you are doing it and why. (Notice that there are tools for constructing perpendiculars, parallels, right angles.)

Be sure to use the drag test to make sure the constraints are working. Make sure that everyone in the team understands how to do the constructions.

Are there more than one way to construct certain kinds of quadrilaterals?

Which quadrilaterals can be dragged to match which other ones?

What we noticed:

What we wondered:

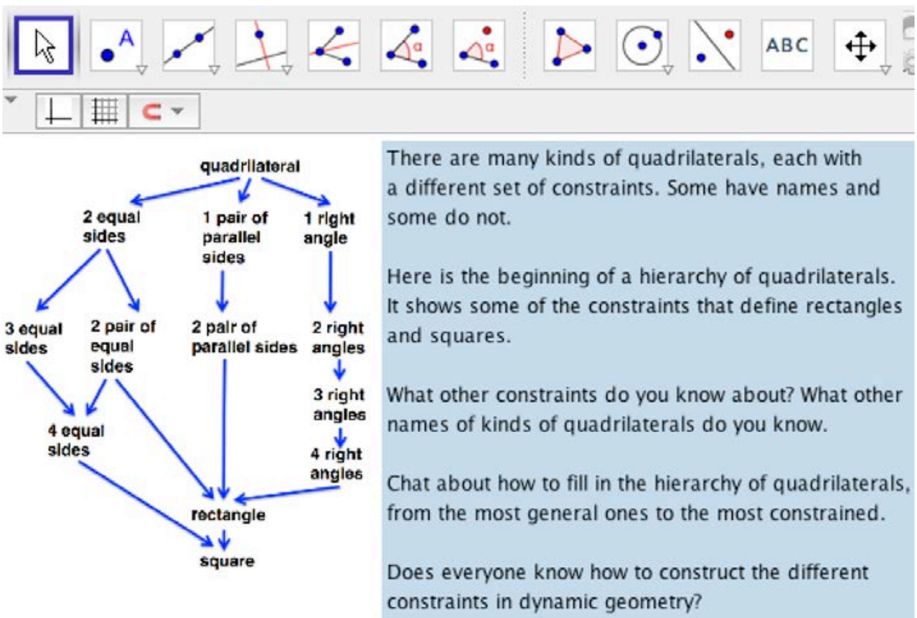
Building a Hierarchy of Quadrilaterals

(**Note**: *This is a long topic. Try to explore some of the tabs on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session.*)

Quadrilaterals have many special properties or relationships among their parts.

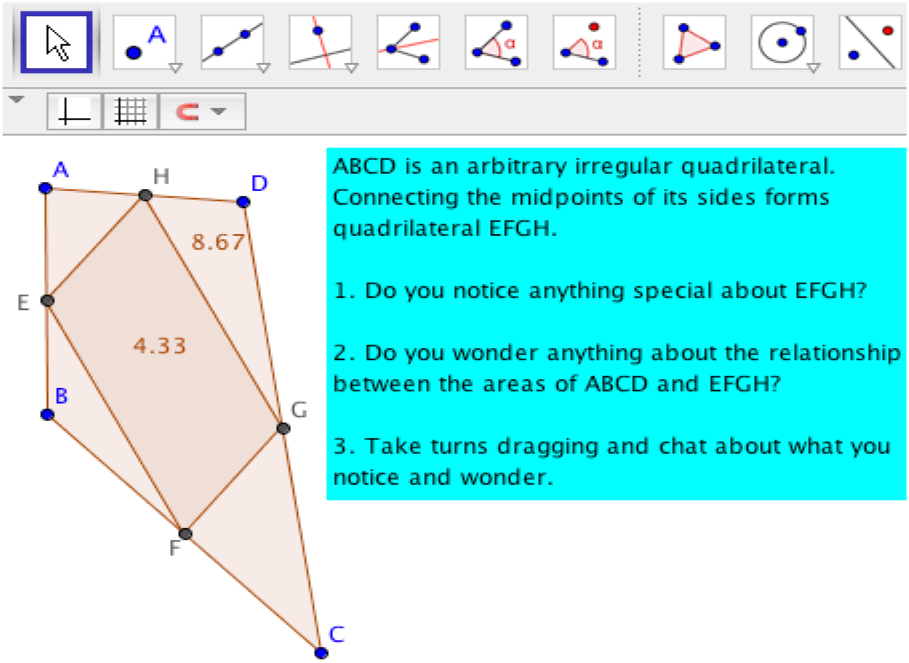
The Hierarchy of Quadrilaterals

Can you construct a hierarchy of types of quadrilaterals? How many types can you include? Do you have square, rectangle, rhombus, trapezoid, parallelogram, kite and many types that do not have common names? Can your team connect the different types of quadrilaterals in a hierarchy diagram? Create a hierarchy diagram like the one shown in the tab. Add more kinds of triangles to it. You may want to reorganize the structure of the diagram. You can use the descriptions of constraints, use the common names like rhombus and kite or use a combination of these, like in the tab.



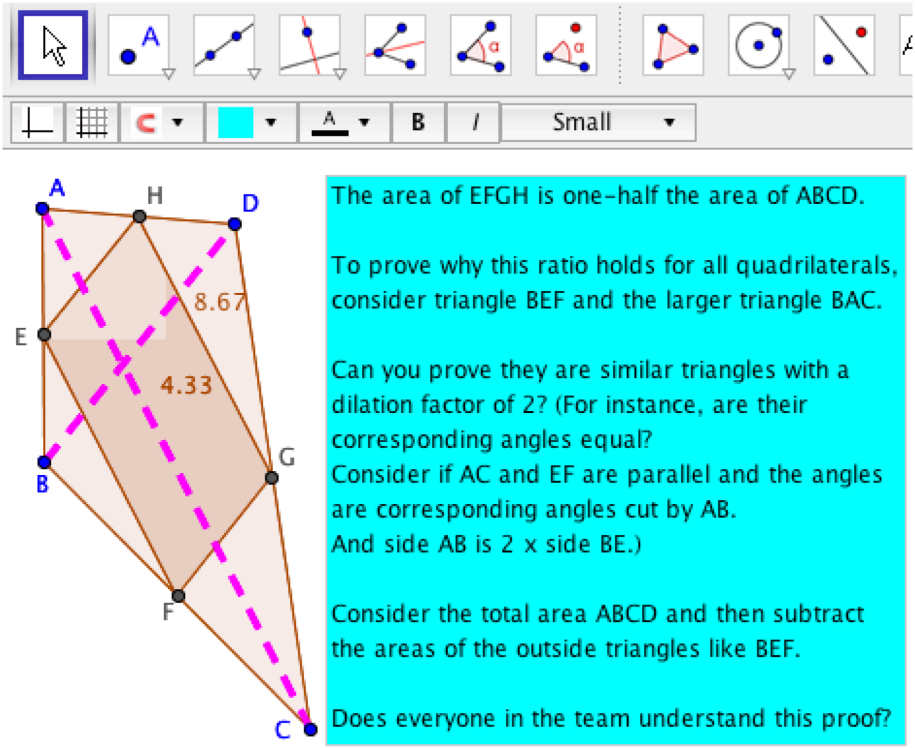
Connecting the Midpoints of a Quadrilateral's Sides

You may be surprised by the quadrilateral formed by connecting an irregular quadrilateral's sides. Drag ABCD and see how EFGH changes. Why do you think it is like this?



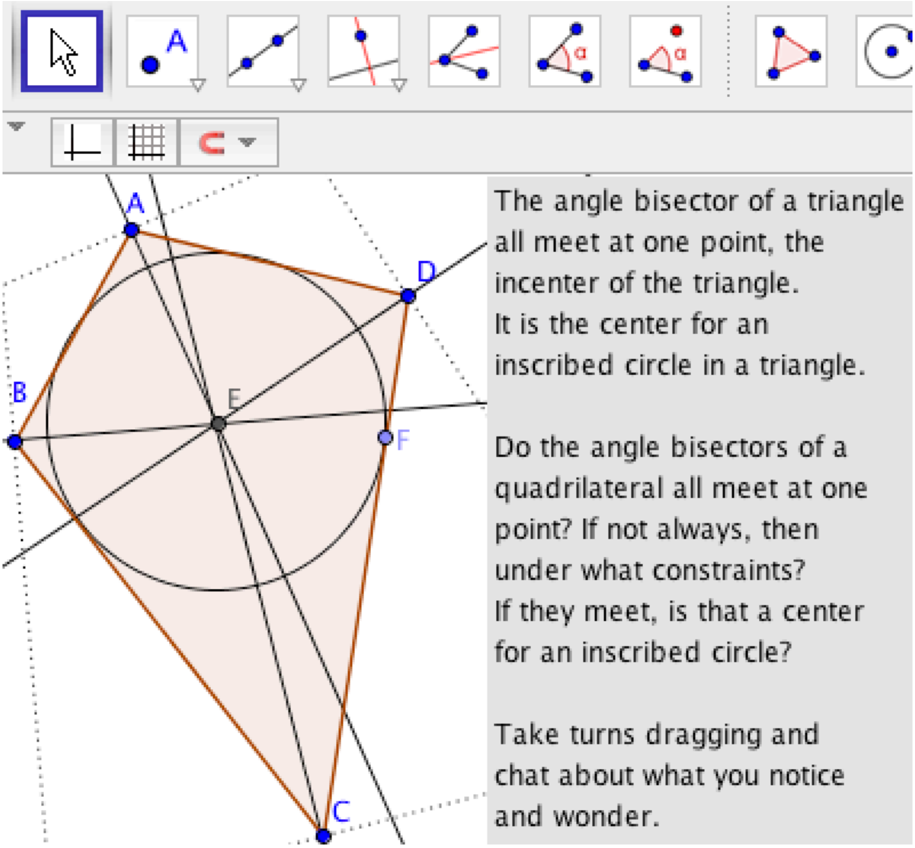
A Proof about Areas of Quadrilaterals

Can you prove that the quadrilateral formed by connecting the midpoints is a parallelogram with half the area of the original quadrilateral?



Angle Bisectors of Quadrilaterals

In certain cases – but not in all cases – the angle bisectors of a quadrilateral all meet in one point, which can serve as the center of an inscribed circle.



What we noticed:

What we wondered:

Individual Transition Activity

Before working on this topic with your team, you should read ***Tour 7: "Creating VMT Chat Rooms"*** *near the end of this booklet. It will show you how to create your own VMT chat rooms so you can invite friends to work on dynamic-geometry topics you define.*

(**Note**: *This is a topic for you to do on your own. It will help you to continue to use GeoGebra in your future mathematics studies.*)

In this topic, you will use the Theorem of Thales (see 0) to construct a tangent to a circle using the geometry tools of GeoGebra that you already know. Then, you will do the same thing in a very different way, using the algebra tools of GeoGebra. This will introduce you to GeoGebra's tools that go beyond geometry to algebra and other area of dynamic mathematics. GeoGebra integrates the different areas in interesting ways. You will see how geometry and algebra are integrated in this topic. As Descartes (the medieval French philosopher, scientist, and mathematician) did in his analytic geometry and calculus, the location of a point is now given (x, y) coordinates on a grid.

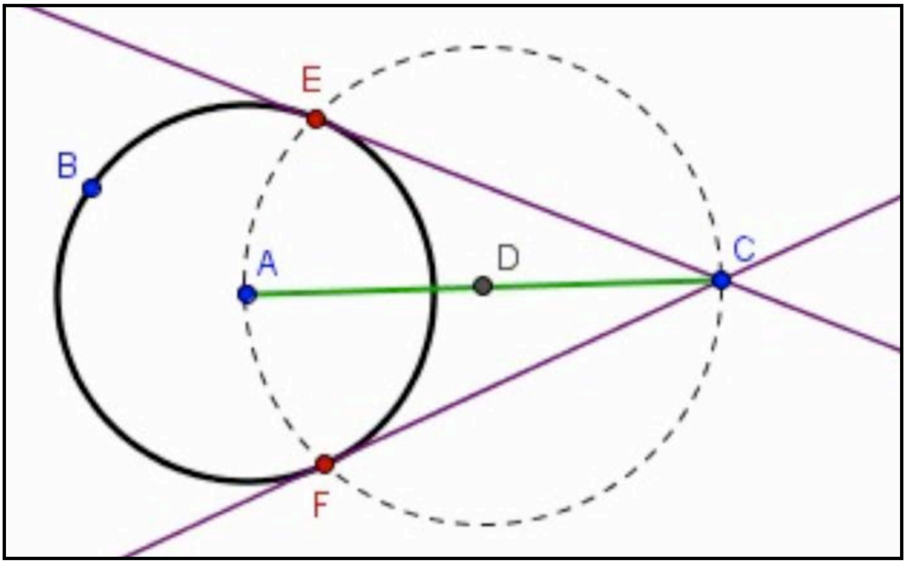
Construct Tangents to a Circle Geometrically

Task: Given a circle and an arbitrary point outside the circle, construct the tangents to the circle going through the point.

A "tangent" to a circle touches the circle at one and only one point. The tangent is perpendicular to a radius from the center of the circle to the point of tangency (see 0).

You can use Thales Theorem to construct the tangent through a point C to a circle with center A if you construct another circle whose diameter is segment AC. According to Thales Theorem, the angle formed between line CE and a line from A to point E (at the intersection of the two circles) will be a right angle, making line CE a tangent to the circle centered at A.

Discuss in chat what tools to use and how to do the construction. Take turns doing the construction and checking the dependencies.



A geometric construction of tangents.

Construct a circle with center at point A, going through a point B. Also, construct a point C outside the circle.

Then, construct the tangents to the circle, going through point C, as indicated in the image above. Note that there are two tangents and that the diagram is symmetric along AC.

Hint: construct segment AC and its midpoint D. Then construct a circle centered on D and passing through C and A. Now the tangent goes from point C to point E or F.

Construct a supplementary segment AE and the angle AEC to check if the tangent is perpendicular to the radius. Drag point C to see if the relationships hold dynamically.

Explain in your summary what you observed in this activity. What is the Theorem of Thales and how did it help you to construct the tangent to the circle? State this in your own words and post it in the chat.

Construct Tangents with the Algebra Interface

In this activity, you will use the Algebra interface of GeoGebra to do the same construction you did in the last activity with geometry tools. This will introduce you to the multiple representations of GeoGebra. (GeoGebra also supports statistics, spreadsheets, algebra, solid geometry, 3D geometry, trigonometry, calculus and other areas of mathematics—but that is all for you to explore on your own or with your friends later.)

GeoGebra has the ability to deal with algebra variables and equations as well as geometry points and lines. These two views are coordinated in GeoGebra: an expression in the algebra window corresponds to an object in the geometry window and vice versa.

GeoGebra's user interface consists of a graphics window and an algebra window. On the one hand, you can operate the provided geometry tools with the mouse in order to create geometric constructions in the graphics window. On the other hand, you can directly enter algebraic input, commands and functions into the input field (at the bottom of the tab) by using the keyboard. While the graphical representation of all objects is displayed in the graphics window, their algebraic numeric representation is shown in the algebra window.

GeoGebra offers algebraic input and commands in addition to the geometry tools. Every geometry tool has a matching algebra command. In fact, GeoGebra offers more algebra commands than geometry tools.

Tips and Tricks:

- Name a new object by typing in `name =` in front of its algebraic representation in the Input Field. Example: `P = (3, 2)` creates point P.
- Multiplication needs to be entered using an asterisk or space between the factors. Example: `a*x` or `a x`
- Raising to a power is entered using ^. Example: `f(x) = x^2 + 2*x + 1`
- GeoGebra is case sensitive! Thus, upper and lower case letters must not be mixed up. Note: Points are always named with upper case letters. Example: `A = (1, 2)`
- Segments, lines, circles, functions… are always named with lower case letters. Example: circle `c: (x – 2)^2 + (y – 1)^2 = 16`
- The variable x within a function and the variables x and y in the equation of a conic section always need to be lower case. Example: `f(x) = 3*x + 2`
- If you want to use an object within an algebraic expression or command, you need to create the object before using its name in the input field. Examples: `y = m x + b` creates a line whose parameters are already existing values

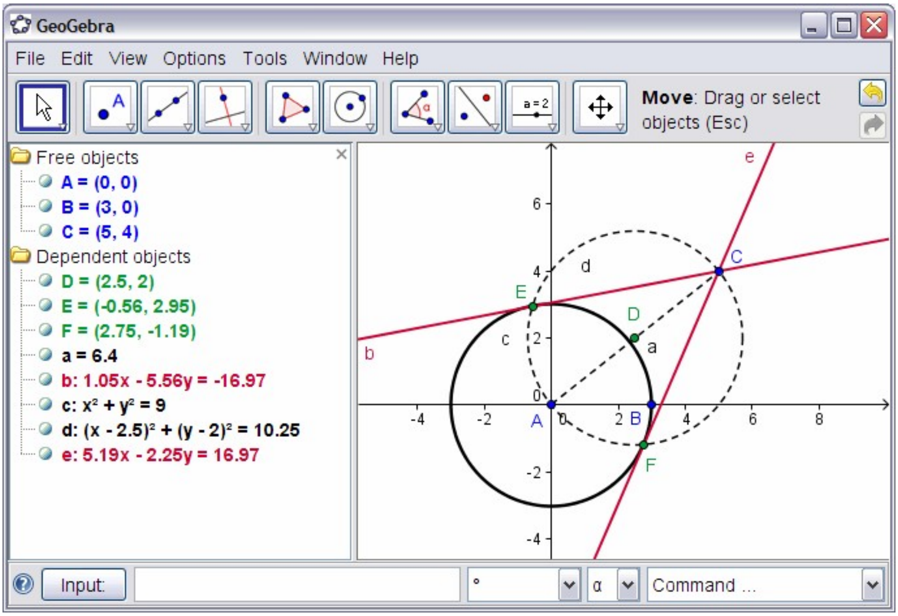
m and b (e.g. numbers / sliders). `Line[A, B]` creates a line through existing points A and B.

- Confirm an expression you entered into the input field by pressing the Enter key.
- Open the "Input Help" panel for help using the input field and commands by clicking the "?" button next to the input field.
- Error messages: Always read the messages – they could possibly help to fix the problem!
- Commands can be typed in or selected from the list next to the input field. Hint: If you do not know which parameters are required within the brackets of a certain command, type in the full command name and press key F1. A pop-up window appears explaining the syntax and necessary parameters of the command.
- Automatic completion of commands: After typing the first two letters of a command into the input field, GeoGebra tries to complete the command. If GeoGebra suggests the desired command, hit the Enter key in order to place the cursor within the brackets. If the suggested command is not the one you wanted to enter, just keep typing until the suggestion matches.

Check out the list of textual algebraic commands next to the Input Help and look for commands corresponding to the geometry tools you have learned to use.

Preparation

Select the "Algebra View" and "Graphics View" from the View menu. Use the View menu and the Graphics View tool bar to make sure the Input Bar, the Algebra window and the Coordinate Axes are all displayed.



An algebraic construction of tangents.

Construction Process

Enter the following entries into the Algebra input field:

Step	Input Field entry	Object created
1	A = (0, 0)	Point A
Hint: Make sure to close the parenthesis.		
2	(3, 0)	Point B
Hint: If you do not specify a name objects are named in alphabetical order.		
3	c = Circle[A, B]	Circle with center A through point B
Hint: Circle is a dependent object		

Note: GeoGebra distinguishes between free and dependent objects. While free objects can be directly modified either using the mouse or the keyboard, dependent objects adapt to changes of their parent objects. It does not matter how an object was initially created (by mouse or keyboard)!

Hint 1: Activate Move mode and double click an object in the algebra window in order to change its algebraic representation using the keyboard. Hit the Enter key once you are done.

Hint 2: You can use the arrow keys to move free objects in a more controlled way. Activate move mode and select the object (e.g., a free point) in either window. Press the up / down or left / right arrow keys in order to move the object in the desired direction.

4	C = (5, 4)	Point C
5	s = Segment[A, C]	Segment AC
7	D = Midpoint[s]	Midpoint D of segment AC
8	d = Circle[D, C]	Circle with center D through point C
9	Intersect[c, d]	Intersection points E and F of the two circles
10	Line[C, E]	Tangent through points C and E
11	Line[C, F]	Tangent through points C and F

Checking and Enhancing the Construction

Perform the drag-test in order to check if the construction is correct.

Change properties of objects in order to improve the construction's appearance (e.g., colors, line thickness, auxiliary objects dashed, etc.).

Discussion

Did any problems or difficulties occur during the construction process?

Which version of the construction (mouse or keyboard) do you prefer and why?

Why should we use keyboard input if we could also do it using tools?

Hint: There are algebra commands available that have no equivalent geometric tool.

Does it matter in which way an object was created? Can it be changed in the algebra window (using the keyboard) as well as in the graphics window (using the mouse)?

Congratulations

Your team has completed the core topics of "*Topics in Dynamic Geometry for Virtual Math Teams*"! You can now explore lots of mathematics—either in small groups or on your own—using GeoGebra. You can create your own VMT chat rooms and invite people to collaborate on your own topics. In addition, you can download GeoGebra from www.geogebra.org to use on your own. Enjoy, explore, create!

Following are some open-ended topics that invite you and your team to create new forms of mathematics using the skills you have just learned.

What we noticed:

What we wondered:

Proving with Dependencies

(**Note**: *This is a long topic. Explore this topic on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the topic after your team session.*)

This topic explores how identifying dependencies in a dynamic-geometry construction can help you prove a conjecture about that construction.

In Euclid's construction of an equilateral triangle, he made the lengths of the three sides of the triangle dependent on each other by constructing each of them as radii of congruent circles. Then to prove that the triangle was equilateral, all he had to do was to point out that the lengths of the three sides of the triangle were all radii of congruent circles and therefore they were all equal.

In this topic, you will look at a more complicated conjecture about triangles, namely relationships having to do with the incenter of a triangle. Remember that the "incenter" of a triangle is located at the intersection of the bisectors of the three vertex angles of the triangle (see 0). The conjecture has a number of parts:

1. The three bisectors of the vertex angles all meet at a single point. (It is unusual for three lines to meet at one point. For instance, do the angle bisectors of a quadrilateral always intersect at one point?)
2. The incenter of any triangle is located inside of the triangle. (Other kinds of centers of triangles are sometimes located outside of the triangle. For instance, can the circumcenter of a triangle be outside the triangle?)
3. Line segments that are perpendiculars to the three sides passing through the incenter are all of equal length.
4. A circle centered on the incenter is inscribed in the triangle if it passes through a point where a perpendicular from the incenter to a side intersects that side.
5. The inscribed circle is tangent to the three sides of the triangle.

These may seem to be surprising conjectures for a simple triangle. After all, a generic triangle just consists of three segments joined together at their endpoints. Why should a triangle always have these rather complicated relationships?

Construct the incenter of a general dynamic triangle and observe how the dependencies of the construction suggest a proof for these five parts of the conjecture about a triangle's incenter.

Construct an incenter with a custom incenter tool

In a previous topic, 0, you programmed your own custom incenter tool. Open the .ggt file for it with the menu "File" | "Open." Then select your custom incenter tool or use the custom "my_incenter" tool that is already on the tool bar in this tab. Click on three points A, B and C to define the vertices of a triangle. The tool will automatically construct the triangle as a polygon ABC and a point D at the incenter of triangle ABC. You can then use a perpendicular tool to construct a line through point D and perpendicular to side AB of the triangle at point E. Next construct a circle centered on D and passing through E. That is the state shown the figure.

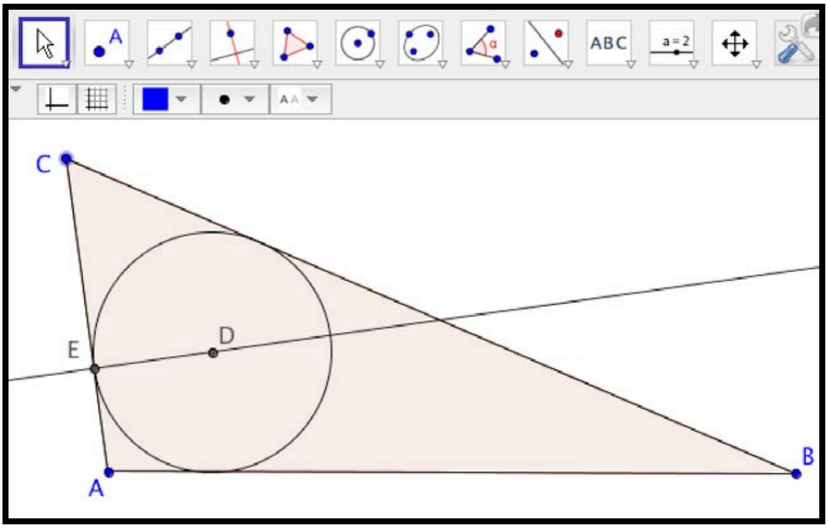


Figure 0-1: Given triangle ABC, its incenter D has been constructed with a custom tool.

Drag this figure around. Can you see why the five parts of the conjecture should always be true?

Add in the three angle bisectors and the other two perpendiculars through point D. You can change the properties of the perpendicular segments to show the value of their lengths. Drag the figure now. Do the three angle bisectors all meet at the same point? Is that point always inside the triangle? Are the three perpendicular segments between D and the triangle sides all equal? Is the circle through D always inscribed in the triangle? Is it always tangent to the three sides? Can you explain why these relationships are always true? Can you identify dependencies built into the construction that constrain the circle to move so it is always tangent to all three sides?

Construct the incenter with standard GeoGebra tools

This time, construct the incenter without the custom tool, simply using the standard GeoGebra tools. Construct a simple triangle ABC. Use the angle-bisector tool (pull down from the perpendicular-line tool) to construct the three angle bisectors. They all meet at point D, which is always inside the circle. Now construct perpendiculars from D to the three sides, defining points E, F and G at the intersections with the sides. Segments DE, DF and DG are all the same length. Construct a circle centered on D and passing through E. The circle is tangent at E, F and G. That is the state shown in Figure 0-2.

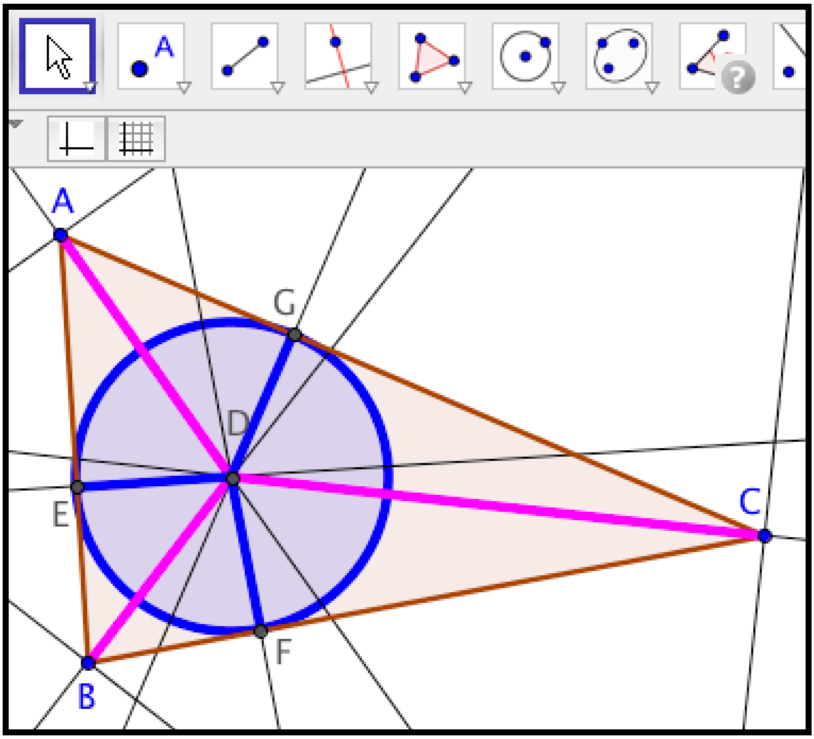


Figure 0-2: Given a triangle ABC, its incenter has been constructed with the GeoGebra angle-bisector tool.

Drag this figure around. Can you see why the five parts of the conjecture should always be true? Can you identify dependencies built into the construction that constrain the incenter to move in response to movements of A, B or C so that the five parts of the conjecture are always true?

Construct the incenter with elementary line and circle tools

A formal deductive proof of the conjecture would normally start from a completed diagram like Figure 0-2. Rather than starting from this completed figure, instead proceed through the construction step by step using just elemental straightedge (line) and compass (circle) tools. Avoid using the angle-bisector tool, which hides the dependencies that make the produced line a bisector.

As a first step, construct the angle bisectors of vertex A of a general triangle ABC (see Figure 0-3). Construct the angle bisector by constructing a ray AF that goes from point A through some point F that lies between sides AB and AC and is equidistant from both these sides. This is the dependency that defines an angle bisector: that it is the locus of points equidistant from the two sides of the angle. The constraint that F is the same distance from sides AB and AC is constructed as follows: First construct a circle centered on A and intersecting AB and AC—call the points of intersection D and E. Construct perpendiculars to the sides at these points. The perpendiculars necessarily meet between the sides—call the point of intersection F. Construct ray AF.

AF bisects the angle at vertex A, as can be shown by congruent right triangles ADF and AEF. (Right triangles are congruent if any two sides are congruent because of the Pythagorean relationship, which guarantees that the third sides are also congruent.) This shows that angle BAF equals angle CAF, so that ray AF bisects the vertex angle CAC into two equal angles. By constructing perpendiculars from the angle sides to any point on ray AF, one can show by the corresponding congruent triangles that every point on AF is equidistant from the sides of the triangle.

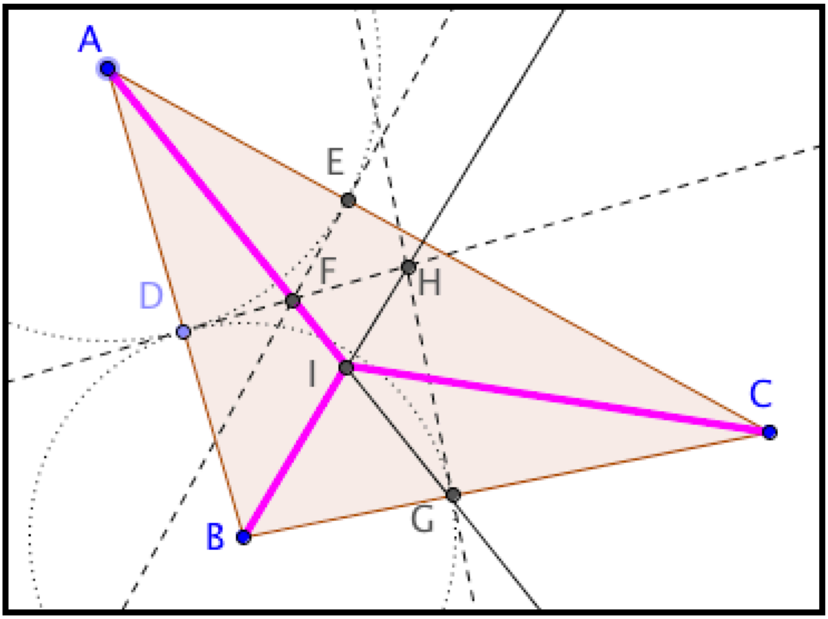


Figure 0-3: Given a triangle ABC, its incenter has been constructed with basic tools.

As the second step, construct the bisector of the angle at vertex B. First construct a circle centered on B and intersecting side AB at point D—call the circle's point of intersection with side BC point G. Construct perpendiculars to the sides at these points. The perpendiculars necessarily meet between the sides AB and BC—call the point of intersection H. H has been constructed to lie between AB and BC. Construct ray BH. BH bisects the angle at vertex B, as can be shown by congruent right triangles BDH and BGH, as before.

For the third step, mark the intersection of the two angle-bisector rays AF and BH as point I, the incenter of triangle ABC. Construct segment CI. You can see that CI is the angle bisector of the angle at the third vertex, C in Figure 0-4 as follows. Construct perpendiculars IJ, IK, IL from the incenter to the three sides. We know that I is on the bisector of angles A and B, so IJ=IK and IJ=IL. Therefore, IK=IL, which means that I is also on the bisector of angle C. This implies that triangles CKI and CLI are congruent, so that their angles at vertex C are equal and CI bisects angle ACB. You have now shown that point I is common to the three angle bisectors of an arbitrary triangle ABC. In other words, the three angle bisectors meet at one point. The fact that the bisectors of the three angles of a triangle are all concurrent is a direct consequence of the dependencies you imposed when constructing the bisectors.

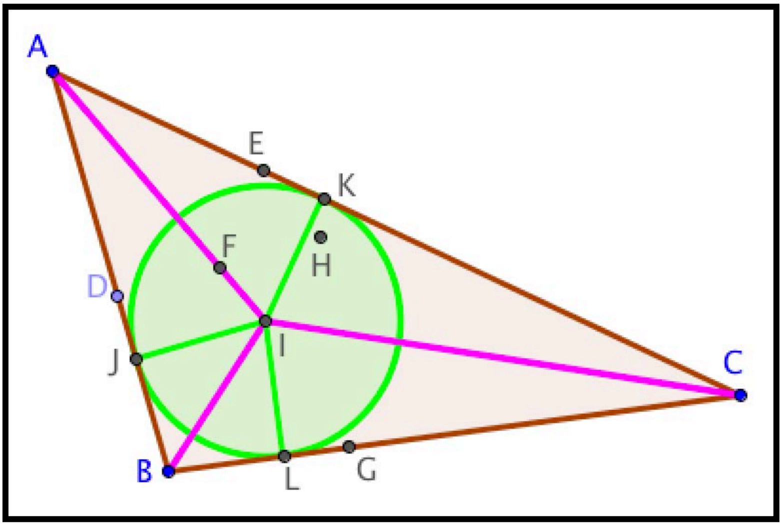


Figure 0-4: The incenter, I, of triangle ABC, with equal perpendiculars IJ, IK, and IL, which are radii of the inscribed circle.

Now construct a circle centered on the incenter, with radii IJ, IK, and IL. You have already shown that the lengths of IJ, IK and IL are all equal and you constructed them to be perpendicular to the triangle sides. The circle is inscribed in the triangle because it is tangent to each of the sides. (Remember, a circle is tangent to a line if its radius to the intersection point is perpendicular to the line.)

Drag the vertex points of the triangle to show that all the discussed relationships are retained dynamically.

Review the description of the construction. Can you see why all of the parts of the conjecture have been built into the dependencies of the figure? None of the parts seem surprising now. They were all built into the figure by the various detailed steps in the construction of the incenter.

When you used the custom incenter tool or even the GeoGebra angle-bisector tool, you could not notice that you were thereby imposing the constraint that DF=EF, etc. It was only by going step-by-step that you could see all the dependencies that were being designed into the figure by construction. The packaging of the detailed construction process in special tools obscured the imposition of dependencies. This is the useful process of "abstraction" in mathematics: While it allows you to build quickly upon past accomplishments, it has the unfortunate unintended consequence of hiding what is taking place in terms of imposing dependencies.

In Figure 0-5, only the elementary "straightedge and compass" tools of the point, line and circle have been used. The perpendiculars have been constructed without even using the perpendicular tool. All of the geometric relationships, constraints and dependencies that are at work in Figure 0-1 or Figure 0-4 are visible in Figure 0-5. This construction involved the creation of 63 objects (points, lines and circles). It is becoming visually confusing. That is why it is often useful to package all of this in a special tool, which hides the underlying complexity. It is wonderful to use these powerful tools, as long as you understand what dependencies are still active behind the visible drawing.

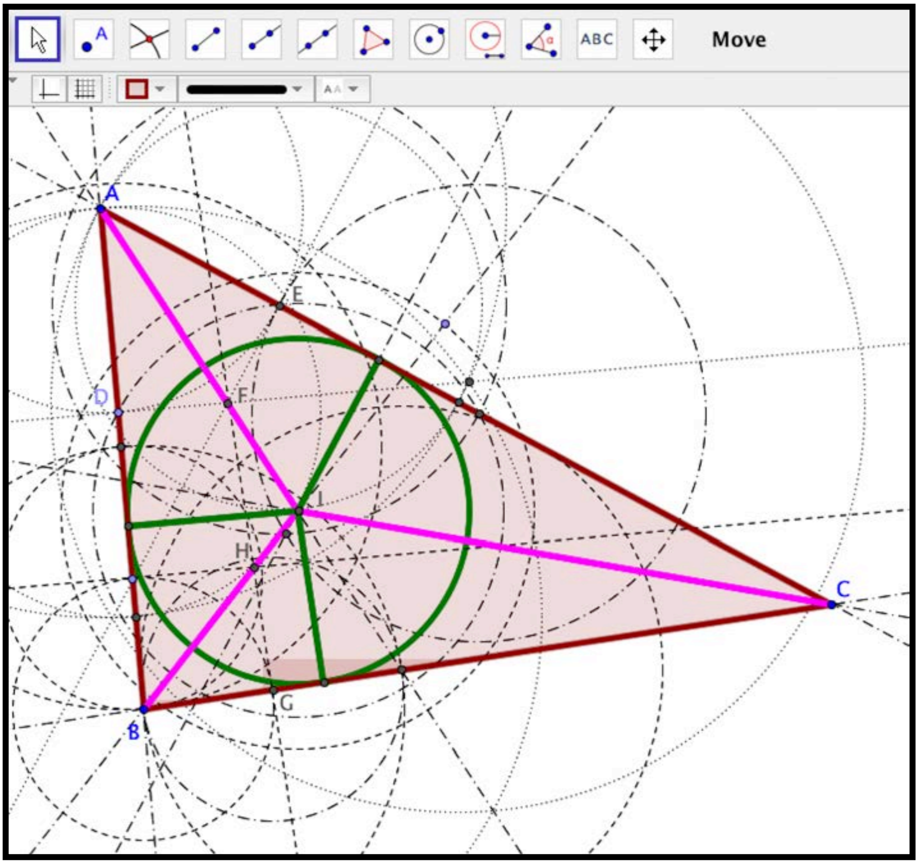


Figure 0-5: Given a triangle ABC, its incenter has been constructed with only elementary point, line and circle tools.

Construct Euler's Segment and the Nine Point Circle

In 0, you explored Euler's Nine-Point Circle. The construction of this circle involves the orthocenter, the centroid and the circumcenter in addition to the incenter.

Construct the four different centers and the nine-point circle shown in Figure 0-6 using just elementary point, line and circle tools. You can start with the construction shown in Figure 0-5. When you are finished, drag the triangle to see that the relationships shown in Figure 0-6 remain dynamically. Using the dependencies that you have constructed, explain why the nine-point circle passes through those nine points.

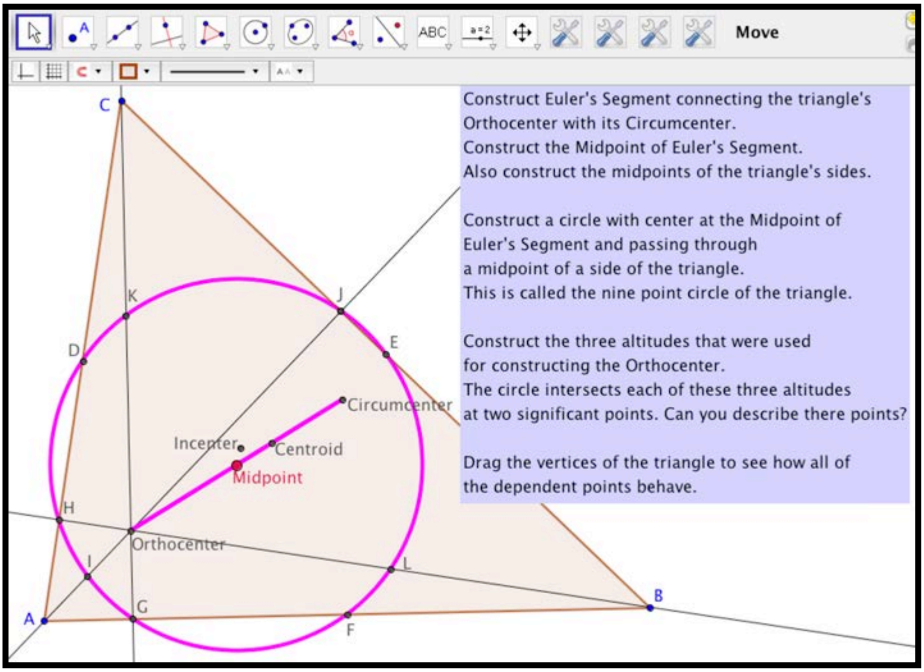


Figure 0-6: Euler's Segment and the Nine Point Circle of a triangle.

This construction is quite complicated. You will soon see why abstracting sequences of constructions into specialized tools is handy. Just as construction of dependencies is closely related to proving conjectures, the packaging of construction sequences in custom tools can be closely related to establishing theorems or propositions. In similar ways, the tools and the theorems help you to build up systems of complex mathematics.

While dragging figures that have already been constructed and even constructing with a large palette of construction tools can be extremely helpful for exploring geometric relationships and coming up with conjectures to investigate, such an approach can give the misimpression that the relationships are abstract truths to be accepted on authority and validated through routinized deduction. It is also important—at least when you want a deeper understanding of what is going on—to be able to construct figures for yourself, using the basic tools of line and circle (analogous to the classic tools of straightedge and compass). You should then understand how other tools are built up from the elementary construction methods and should know how to create your own custom tools, for which you understand the incorporated procedures.

Prove Euler's Segment using its Construction Dependencies

Most discussions of Euler's segment and the nine-point circle talk about them as wonderful mysteries: how can a common, simple triangle contain such marvelous relationships and surprising but elegant complexities? By now, you should suspect that these relationships are not inherent in the essence of the plain triangle, but are built into the Euler segment by the construction of the various centers. We have just seen that there are not just the three segments that form the triangle and one additional segment in the middle, but a couple hundred inter-dependent points, lines and circles.

In fig, we see two diagrams highlighting a small number of the lines used in the construction of the centers and the segment that joins them. Accompanying the diagrams is a textual sketch of a proof of the **Euler Line Theorem**:

The orthocenter, the circumcenter and the centroid of a triangle are co-linear and the centroid is a third of the way between the orthocenter and the circumcenter.

The informal proof relies upon the dependencies built into the construction of the orthocenter, circumcenter and centroid by altitudes, perpendicular bisectors and medians of the triangle.

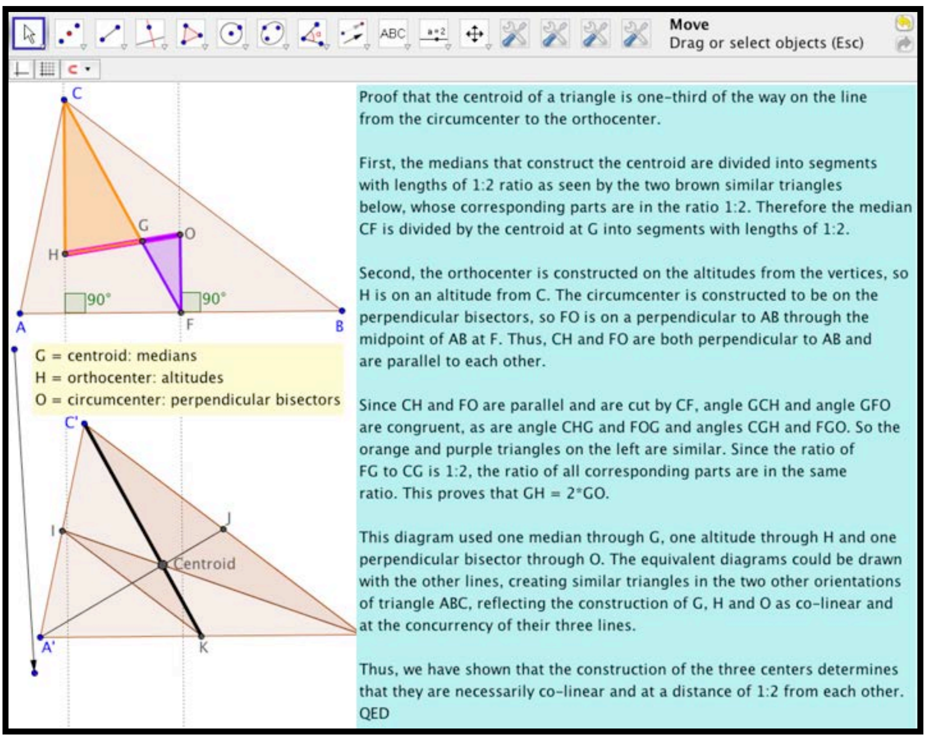


Figure 0-7: Sketch of a proof of Euler's segment.

Euler's theorems sparked renewed interest in geometry research, involving more centers of triangles, further axiomatization and the new field of topology. Hofstadter (1997)—whose popular books about math are always entertaining and thought provoking—extended the exploration of the Euler segment and circle in an interesting way using dynamic geometry. Venema (2013) provides a more systematic introduction to exploring advanced, post-Euclidean geometry using GeoGebra (see the "Further Readings" Tour).

What we noticed:

What we wondered:

Transforming a Factory

(**Note**: *This is a long topic. Explore this topic on your own before your team meets together. Do not spend too much time on any one section. Continue to work on the topic after your team session.*)

In this topic, you will conduct mathematical studies to help design a widget factory. The movement of polygon-shaped widgets, which the factory processes, can be modeled in terms of rigid transformations of polygons. You will explore physical models and GeoGebra simulations of different kinds of transformations of widgets. You will also compose multiple simple transformations to create transformations that are more complex, but might be more efficient. You will apply what you learned to the purchase of widget-moving machines in a factory.

Designing a Factory

Suppose you are the mathematician on a team of people designing a new factory to process widgets. In the factory, special machines will be used to move heavy widgets from location to location and to align them properly. There are different machines available for moving the widgets. One machine can flip a widget over; one can slide a widget in a straight line, one can rotate a widget. As the mathematician on the team, you are supposed to figure out the most efficient way to move the widgets from location to location and to align them properly. You are also supposed to figure out the least expensive set of machines to do the moving.

The factory will be built on one floor and the widgets that have to be moved are shaped like flat polygons, which can be laid on their top or bottom. Therefore, you can model the movement of widgets as rigid transformations of polygons on a two-dimensional surface. See what you can learn about such transformations.

Experiment with Physical Transformations

Before you get together with your team online, take a piece of cardboard and cut out an irregular polygon. This polygon represents a widget being processed at the factory. Imagine it is moved through the factory by a series of machines that flip it, slide it and rotate it to move it from one position to another on the factory floor.

Place the polygon on a piece of graph paper and trace its outline. Mark that as the "start state" of the polygon. Move the cardboard polygon around. Flip it over a number of times. What do you notice? Rotate it around its center or around another point. Slide it along the graph paper. Finally, trace its outline again and mark that as the "end state" of the transformation.

Place the polygon at its start state position. What is the simplest way to move it into its finish state position? What do you notice about different ways of doing this?

Now cut an equilateral triangle out of the cardboard and do the same thing. Is it easier to transform the equilateral triangle from its start state to its finish state than it was for the irregular polygon? What do you notice about flipping the triangle? What do you notice about rotating the triangle? What do you notice about sliding the triangle?

The other people in your group cannot see your cardboard polygon moving. Explain to them in the chat, what you did and what you noticed. Share what you are wondering about transformations of polygons and discuss these questions.

Transformational Geometry

In a previous activity with triangles, you saw that there were several kinds of rigid transformations of triangles that preserved the measures of the sides and the angles of the triangles. You also learned about GeoGebra tools that could transform objects in those ways, such as:

- Reflect Object about Line
- Rotate Object around Point by Angle
- Translate Object by Vector

These tools can transform any polygon in these ways and preserve the measures of their sides and angles. In other words, these geometric transformations can model the movement of widgets around the factory.

Composing Multiple Transformations

In addition to these three kinds of simple transformations, you can "compose" two or more of these to create a more complicated movement. For instance, a "glide reflection" consists of reflecting an object about a line and then translating the reflected object by a vector. Composing three transformations means taking an object in its start state, transforming it by the first transformation into a second state, then transforming it with the second transformation from its second state into a third state, and finally transforming it with the third transformation from its third state into its end state. You can conceive of this as a single complex transformation from the object's start state to its end state.

The study of these transformations is called "transformational geometry." There are some important theorems in transformational geometry. Maybe you can discover some of them and even find some of your own. These theorems can tell you what is possible or optimal in the widget factory's operation.

An Example of Transformations in GeoGebra

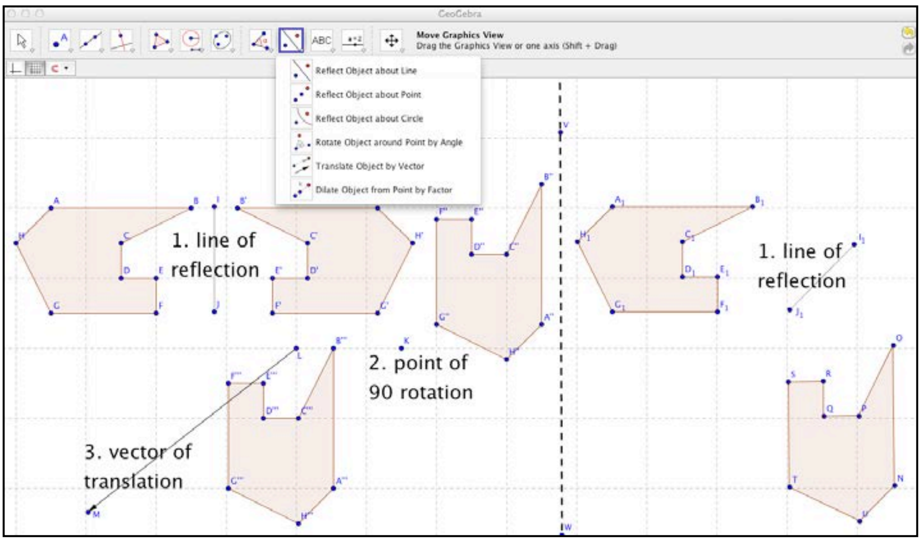


Figure 0-8. Transformations of a polygon.

In Figure 0-8, an irregular polygon ABCDEFGH has gone through 3 transformations: a reflection (about line IJ), a rotation (about point K), and a translation (by vector LM). A copy of the polygon has gone through just 1 transformation (a reflection about line I_1J_1) and ended in the same relative position and orientation. There are many sequences of different transformations to transform a polygon from a particular starting state (position and orientation) to an end state (position and orientation). Some possible alternative sequences are simpler than others.

Discuss with your group how you want to proceed with each of the following explorations. Do each one together with your group, sharing GeoGebra constructions. Save a construction view for each exploration to include in your summary. Discuss what you are doing, what you notice, what you wonder, how you are constructing and transforming polygons, and what conjectures you are considering.

Exploration 1

Consider the transformations in Figure 0-8. Drag the line of reflection (line IJ), the point of rotation (point K), the translation vector (vector LM) and the alternative line of reflection (line NO). How does this affect your ability to substitute the one reflection for the sequence on three transformations? What ideas does this give you for the lay-out of work-flow in a factory?

Exploration 2

Consider just simple rotations of an irregular polygon. Suppose you perform a sequence of several rotations of the polygon widget around different points. Would it be possible to get from the start state to the end state in a fewer number of rotations? In other words, can the factory be made more efficient?

Consider the same question for translations of widgets.

Consider the same question for reflections of widgets.

Exploration 3

Perhaps instead of having a machine in the factory to flip widgets and a different machine to move the widgets, there should be a machine that does both at the same time. Consider a composite transformation, like a glide reflection composed of a reflection followed by a translation. Suppose you perform a sequence of glide reflections on an irregular polygon. Does it matter what order you perform the glide reflections? Would it be possible to get from the start state to the end state in a fewer number of glide reflections?

Does it matter if a glide reflection does the translation before or after the reflection?

Consider the same questions for glide rotations.

Exploration 4

Factory managers always want to accomplish tasks as efficiently as possible. What is the minimum number of simple transformation actions needed to get from any start state of the irregular polygon in the figure to any end state? For instance, can you accomplish any transformation with three (or fewer) simple actions: one reflection, one rotation and one translation (as in the left side of the Figure 0-8)? Is it always possible to achieve the transformation with fewer than three simple actions (as in the right side of the figure)?

Exploration 5

Factory managers always want to save costs. If they can just buy one kind of machine instead of three kinds, that could save money. Is it always possible to transform a given polygon from a given start state to a specified end state with just one *kind* of simple transformation – e.g., just reflections, just rotations or just translations? How about with a certain composition of two simple kinds, such as a rotation composed with a translation or a reflection composed with a rotation?

Exploration 6

Help the factory planners to find the most direct way to transform their widgets. Connect the corresponding vertices of the start state and the end state of a transformed polygon. Find the midpoints of the connecting segments. Do the midpoints line up in a straight line? Under what conditions (what kinds of simple transformations) do the midpoints line up in a straight line? Can you prove why the midpoints line up for some of these conditions?

If you are given the start state and the end state of a transformed polygon, can you calculate a transformation (or a set of transforms) that will achieve this transformation? This is called "reverse engineering" the transformation. *Hint*: constructing the perpendicular bisectors of the connecting segments between corresponding vertices may help in some conditions (with some kinds of simple transformations).

Exploration 7

Different factories process differently shaped widgets. How would the findings or conjectures from Explorations 1 to 5 be different for a widget which is an equilateral triangle than they were for an irregular polygon? How about for a square? How about for a hexagon? How about for other regular polygons?

Exploration 8

So far, you have only explored rigid transformations – which keep the corresponding angles and sides congruent from the start state to the end state. What if you now add dilation transformations, which keep corresponding angles congruent but change corresponding sides proportionately? Use the Dilate-Object-from-Point-by-Factor tool and compose it with other transformations. How does this affect your findings or conjectures from Explorations 1 to 5? Does it affect your factory design if the widgets produced in the factory can be uniformly stretched or shrunk?

Factory Design

Consider the factory equipment now. Suppose the factory needs machines for three different complicated transformations and the machines have the following costs: a reflector machine $20,000; a rotator machine $10,000; a translator machine $5,000. How many of each machine would you recommend buying for the factory?

What if instead they each cost $10,000?

Summarize

Summarize your trials with the cardboard polygons and your work on each of the explorations in your chat discussion. What did you notice that was interesting or surprising? State your conjectures or findings. Can you make some recommendations for the design of the factory? If you did not reach a conclusion, what do you think you would have to do to reach one? Do you think you could develop a formal proof for any of your conjectures in the explorations?

What we noticed:

What we wondered:

Navigating Taxicab Geometry

(**Note**: *This is a long topic. Explore this topic on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the topic after your team session.*)

In this topic, you will explore an invented transformational geometry that has probably never been analyzed before (except by other teams who did this topic). Taxicab geometry is considered a "non-Euclidean" form of geometry, because in taxicab geometry the shortest distance between two points is not necessarily a straight line. Although it was originally considered by the mathematician Minkowski (who helped Einstein figure out the non-Euclidean geometry of the universe), taxicab geometry can be fun for amateurs to explore. Krause (1986) wrote a nice introductory book on it that uses an inquiry approach, mainly posing thought-provoking problems for the reader. Gardner devoted his column on mathematical games in *Scientific American* to clever extensions of it in November 1980.

An Invented Taxicab Geometry

There is an intriguing form of geometry that is called "taxicab geometry" because all lines, objects and movements are confined to a grid. It is like a grid of streets in a city where all the streets either run north and south or they run east and west. For a taxicab to go from one point to another in the city, the shortest route involves movements along the grid. Taxicab geometry provides a model of urban life and navigation.

In taxicab geometry as we will define it for this topic, all points are at grid intersections, all segments are confined to the grid lines and their lengths are confined to integer multiples of the grid spacing. The only angles that exist are multiples of 90° — like 0°, 90°, 180°, 270° and 360°. Polygons consist of segments connected at right angles to each other.

How would you define the rigid transformations of a polygon in taxicab geometry? Discuss this with your team and decide on definitions of rotation, translation and reflection for this geometry. (See 0 for an example.)

Use GeoGebra with the grid showing. Use the grid icon on the lower toolbar to display the grid; the pull-down menu from the little triangle on the right lets you activate "Snap to Grid" or "Fixed to Grid. The menu "Options" | "Advance" | "Graphics" | "Grid" lets you modify the grid spacing when you have Control. Only place points on the grid intersections.

Construct several taxicab polygons. Can you use GeoGebra's transformation tools (rotation, translation and reflection)? Or do you need to define custom transformation tools for taxicab geometry? Or do you have to manually construct the results of taxicab transformations? Rotate (by 90° or 180°), translate (along grid lines to new grid intersections) and reflect (across segments on grid lines) your polygons.

Explore Taxicab Transformational Geometry

Now consider the question that you explored for classical transformational geometry in 0 Can all complex transformations be accomplished by just one kind of transformation, such as reflection on the grid? What is the minimum number of simple transformations required to accomplish any change that can be accomplished by a series of legal taxicab transformations?

In Euclidean geometry, if a right triangle has sides of length 3 and 4, the hypotenuse is 5, forming a right triangle with integer lengths. In taxicab geometry, it seems to have a hypotenuse of 7, which can be drawn along several different paths. In the grid shown (Figure 0-9), a 3-4-7 right triangle ABC (green) has been reflected about segment IJ (blue), then translated by vector KL (blue), and then rotated 180° clockwise about point C'' (brown). Equivalently, ABC (green) has been reflected about segment BC (red), then reflected about the segment going down from C'_1 (red), and then reflected about segment A'''M''' (brown). Thus, in this case, the composition of a reflection, a translation and a rotation can be replicated by the composition of just reflections, three of them.

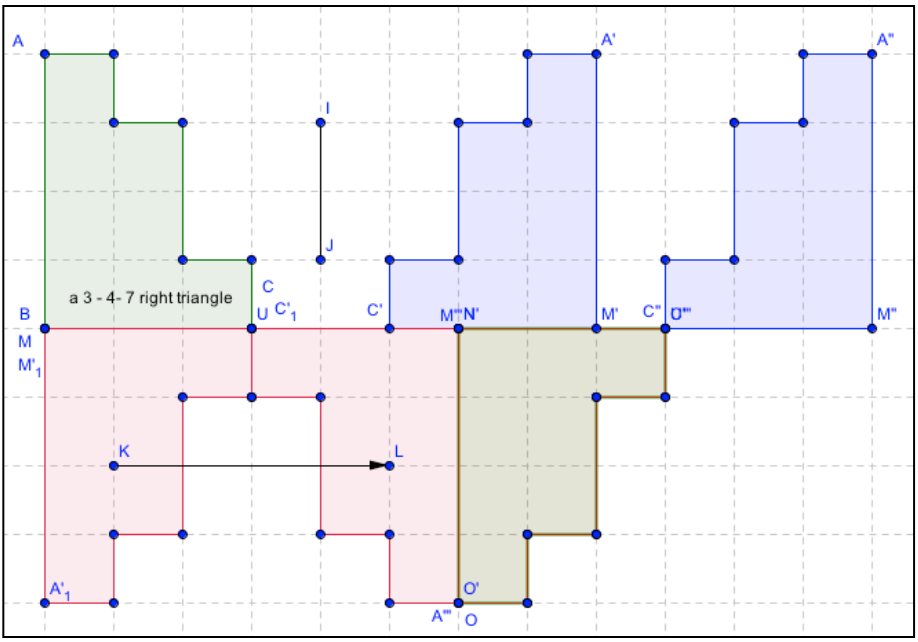


Figure 0-9. Transformations in taxicab geometry.

Explore Kinds of Polygons and their Symmetries

What distinct kinds of "polygons" are possible in taxicab geometry? Can you work out the hierarchy of different kinds of "taxicab polygons" with each number of sides? E.g., are there right or equilateral taxicab triangles? Are there square or parallelogram taxicab quadrilaterals?

Discuss and Summarize

What has your group noticed about taxicab transformational geometry? What have you wondered about and investigated? Do you have conjectures? Did you prove any theorems in this new geometry? What questions do you still have?

Be sure to list your findings in the chat, as well as wonderings that you would like to investigate in the future.

Congratulations!

You have now completed the topics in this series. You are ready to explore dynamic geometry and GeoGebra on your own or to propose further investigations for your team. You can also create new VMTwG chat rooms with your own topics and invite people to work together in them. If you are a teacher, you can set up rooms with topics from this series for groups of your students.

What we noticed and what we wondered:

4. Dynamic-Geometry Activities with GeoGebra for Virtual Math Teams

Introduction

Dynamic-Geometry Activities with GeoGebra for Virtual Math Teams introduces you to dynamic mathematics using collaboration software. This booklet consists of activities for individuals, small groups and classes to get started with dynamic-math geometry discussions.

The **VMT online environment** is designed for people to discuss mathematical topics of interest in small online collaborative groups, known as "virtual math teams." The VMT environment provides a lobby for selecting mathematical activities, chat rooms for exploring math, and a wiki for sharing ideas with other groups. One of the kinds of tabs available in VMT chat rooms lets people share a multi-user version of GeoGebra.

GeoGebra is an interactive environment for visualizing and exploring geometry and algebra, as well as other areas of mathematics. GeoGebra lets you construct dynamic-mathematics figures and investigate them interactively. VMT-with-GeoGebra (VMTwG) lets you share this exploration in a VMT chat room. A group can observe dynamic-math figures, notice characteristics, wonder about their relationships and discuss the mathematics.

The set of activities in this booklet is designed to encourage people to use VMTwG to visualize and explore dynamic constructions of geometry, with their dependencies, relationships and proofs. It encourages collaborative learning through textual chat about stimulating and challenging dynamic-geometry activities. It provides opportunities to learn how to discuss mathematics in small groups.

The activities start with explorations of triangles. These activities cover many of the classical theorems in Book I of Euclid's *Elements*. They cover much of the basic geometry content in the new *Common Core* standards. They can be used to supplement most high school geometry books with visualizations and explorations of the central concepts and theorems. The activities encourage significant mathematical discourse on these topics within small collaborative groups of peer learners.

The tours and early activities introduce the use of the most important tools in the VMTwG environment. This prepares students to conduct their own explorations with these flexible and powerful tools for investigating and discussing mathematics.

The activities with triangles conclude with investigations of symmetry and rigid transformations. These activities are more open-ended and challenging, allowing groups of students to explore in different directions, following their own interests. These activities then segue into applying the construction techniques and the concepts of congruence, symmetry and transformation to quadrilaterals and many-sided polygons.

The activities end by introducing GeoGebra's integration of geometry with algebra, and providing a sample of challenge problems and open-ended topics for further exploration. At this point, students should be sufficiently proficient at online collaboration and discourse in VMT and at construction and dynamic exploration in GeoGebra to continue to take advantage of the VMTwG environment as a powerful tool for supplementing their future mathematical studies.

Tour 1: Joining a Virtual Math Team

In this tour, you will explore the VMT-with-GeoGebra environment and learn how to use it. You will learn about many special features of the VMT system, which you will need to use in the following activities.

The Virtual Math Teams (VMT) environment

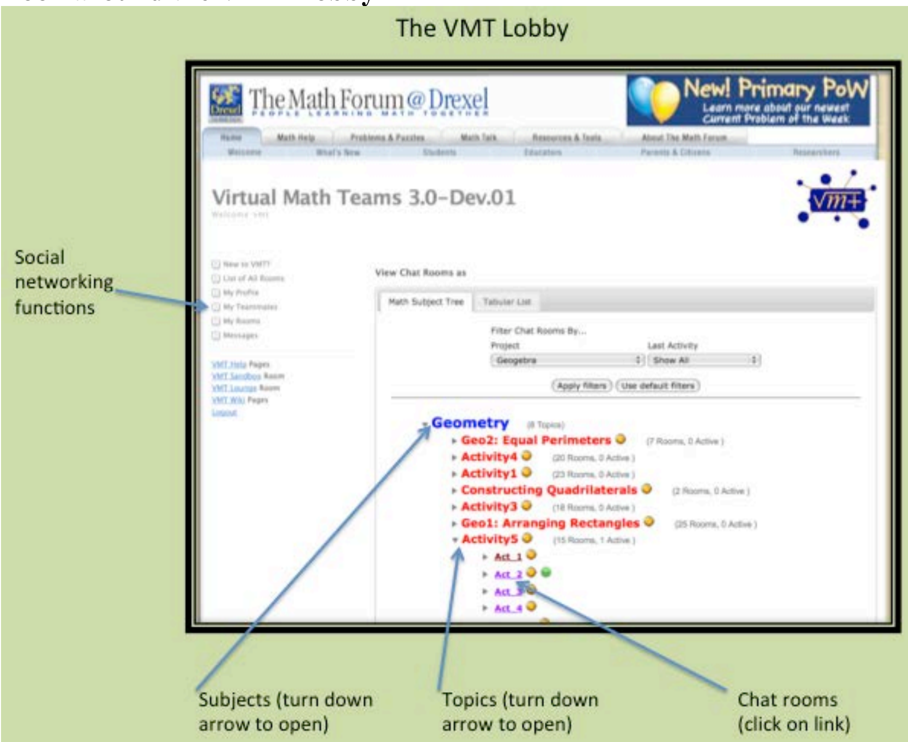
The VMT system has been developed to support small groups of people to discuss mathematics online. It has tabs and tools to help individuals, small groups (about 2-6 people) and larger groups (like classes) to explore math collaboratively.

Register and log in to VMT

Go to the VMT Lobby at http://vmt.mathforum.org/VMTLobby.

Log in (if you do not have a VMT login, then first register). If you are using VMT in a class, your instructor may have already registered you and assigned your username and password. If not, then choose a username that you want to be known by online in VMT. Choose the project that is defined for your class or group.

Look around the VMT Lobby



Interface of the VMT Lobby.

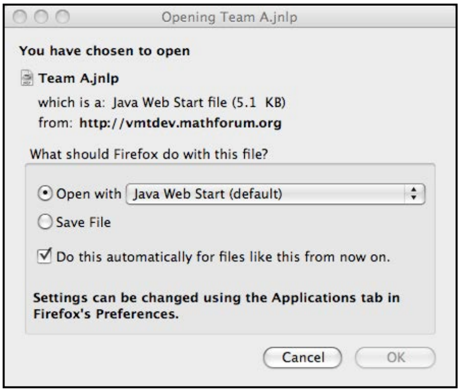
In the center of the Lobby is a list of math subjects. For each subject, you can view activity topics related to that subject. For each topic, there are links to chat rooms for discussing that topic. Find the room where you are supposed to meet with your group. Click on the link for that room to open a window with the chat room.

On the left of the Lobby is a list of links to other functions. The link, "List of All Rooms", displays the list of math subjects. The link, "My Profile", allows you to change your login name, password or information about you. The link, "My Rooms", lets you see links to chat rooms that you have been invited to by your teacher or a friend, as well as rooms that you have been in before.

You can use the "VMT Sandbox" link to open a practice chat room. However, it is better to meet with the members of your group in a chat room that has been created for your group to do an activity. You should be able to find it in the "List of all Rooms" under your project, the subject "Geometry", the topic, and the name of your group. It may also be listed under "My Rooms" or you might have been given a direct link to the room.

Enter a VMT chat room

When you click on a chat room link to open it, your computer will download VMT files. This may take a couple minutes, especially the first time it is done on your computer. You will see a dialog box window asking if you want to open the file with Java Web Start. Just select "Open with Java Web Start" and press the OK button. (See "Appendix: Fix a Technical Problem" at the end of this document if you have problems at this point.)



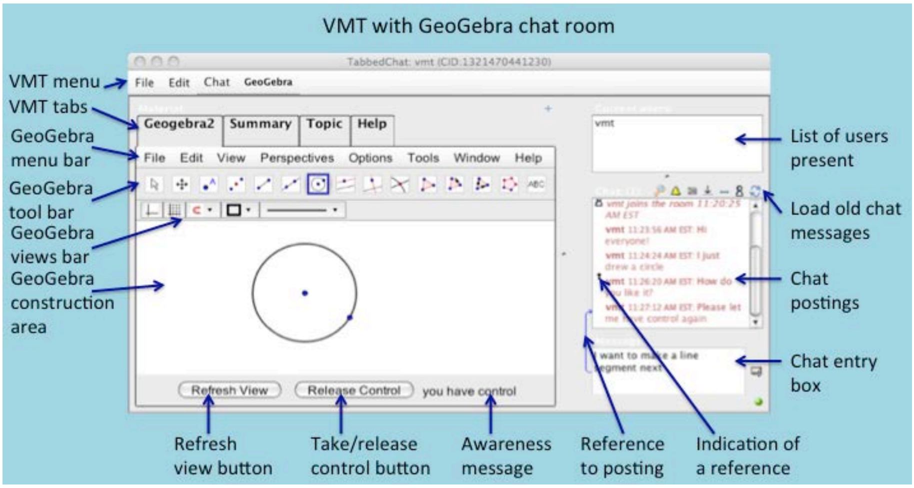
Dialog box for Java Web Start.

You will learn more about how to use the VMT tools in future tours. For now, just click on the tab for GeoGebra and proceed with the next activity.

Tour 2: GeoGebra for Dynamic Math

Go to the VMT chat room and open the GeoGebra tab

Open the GeoGebra tab in your chat room and identify the parts listed in the figure below. You will be using this GeoGebra tab most of the time in the following activities.



The GeoGebra tab interface in VMT.

Take turns

This is a multi-user version of GeoGebra. What you see in the team's GeoGebra tab is the same as what everyone in the VMT chat room with you also sees in their GeoGebra tab (except that they may have their view options set differently, like having the tab opened wider or smaller than you do).

Two people cannot be creating and manipulating objects at the same time in GeoGebra, so you have to take turns. While someone else is constructing or dragging, you can be watching and chatting.

Use the chat to let people know when you want to "take control" of the GeoGebra construction. Use the chat to tell people what you notice and what you are wondering about the construction.

Decide in the chat who will go first. That person should press the "Take Control" button and do some drawing. Then release control and let the others draw.

Before you start to draw, say in the chat what you plan to do. After you release control, say in the chat what you discovered if anything surprised you. You can also ask other people in your group questions about what they drew and how they did it.

There is a history slider on the left side of the GeoGebra tab. You can only use the history slider in the GeoGebra tab when you are *not* "in control". Sliding the history slider shows you previous versions of constructions in the GeoGebra tab, so you can review how your group did its work.

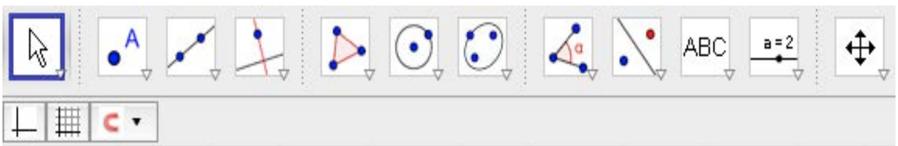
Create a practice tab

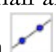
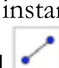
To create a new GeoGebra tab for yourself, use the "+" button in the upper-right corner above the tabs.

This way, you can create your own GeoGebra tab, where you can practice doing things in GeoGebra before you get together with your team in the team's GeoGebra tab. You can use your own tab to try out the drawing tools described below. At the beginning of each activity, there may be tasks for you to try yourself in your own tab; then you will discuss them and share your findings in the team GeoGebra tab. Anyone can view any tab, so you can post a chat invitation to other people to go to your GeoGebra tab and see what you have done. You can even let someone else "take control" in your tab to help you construct something or to explore your construction. After your group constructs something in the group GeoGebra tab, you should make sure that you can do it yourself by doing the construction in your own tab.

Some drawing tools in GeoGebra

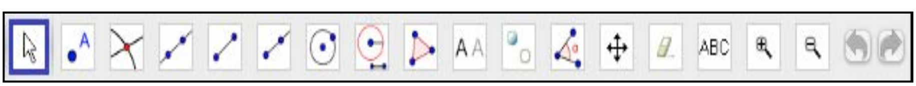
When you open a GeoGebra tab, the tool bar may look something like this:



Notice that you can "pull down" many different tools by clicking on the small arrow at the bottom of each icon in the tool bar. For instance, from the third icon , you can select the Line tool , the Segment tool  and the Ray tool . You can change the menu and other settings by clicking on the small arrow in the middle along the right side of the tab. You can select the "Basic Geometry" or the "Geometry" perspective. If there are grid lines, you can remove them with the Grid button below the tool bar. If there are coordinate axes, you can remove them with the coordinates button below the tool bar. You can change the color or thickness of a selected line with the other buttons there.

Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked so that new points you create will have their names showing.

Here are some of the first tools you will be using in GeoGebra:



These tools correspond to the traditional Euclidean geometry construction tools of straightedge and compass. *The first several tools let you construct dynamic points and lines (including lines, segments, rays and circles), much as you would with a pencil and paper using a straightedge for the lines, segments and rays or a compass for the circles.*

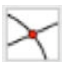
Check out this video for an overview and some tips on the use of these tools: http://www.youtube.com/watch?v=2NqblDIPl38

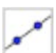
Here is how to use these tool buttons. Try each one out in the construction area of your own GeoGebra tab. First click on the button for the tool in the tool bar, then click in the construction area to use the tool. The tool will remain selected in the tool bar until you select another one:

Use the Move tool  to select a point that already exists (or segment or circle) and drag it to a new position. Everyone will see the object being dragged.

Use the Point tool 

to create some points. Each place you click with the Point tool will leave a point. These points will appear in the GeoGebra tab of everyone in your chat room. By convention, points are named with capital letters – and lines (as well as segments, rays, circles and polygons) are named with lowercase letters.

Use the Intersection tool  to mark the intersection of two objects—like a line and a circle—with a new point. When you click on the intersection of two objects, both objects should get thicker to show they have been selected. You can also select the two objects separately, one after another and the new point will be on their intersection. If you click at a location where three objects meet, you will get a pull-down menu to select the two objects that you want.

Use the Line tool  to create a line with no endpoints. A line has to pass through two points. You can either select two existing points or click with the Line tool to create the points while you are constructing the line.

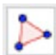
Use the Segment tool  to connect two points with a line segment. You can also create points as you click for the ends of the segment. See what happens when two segments use the same point for one of their endpoints.

Use the Ray tool  to connect two points with a ray. First click for the starting point of the ray and then click for a point along the ray. You can also select existing points for the endpoint and the other point.

Use the Circle tool  to draw a circle. You must click to place a point where you want the center to be and then click again for a point on the circumference of the circle. You can also use existing points for the center and the other point.

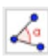
Use the Compass  to draw a circle whose radius is equal to the distance between two points and whose center is at a third point. First click on two points to define the length of the radius. Then without releasing the cursor, drag the circle to the point where you want its center to be. This tool is like a mechanical compass, where you first set the size of the opening and then fix one end at a center and draw a circle around it. The Compass tool is very handy for copying a length from one part of a construction to another in a way that will be preserved through any dragging; if you change the original length, the copied length will change automatically to still be equal to the original one.

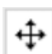
The following tools can be used for modifying the display of a construction to make it easier to see what is going on with the dependencies of the construction.

The Polygon tool  is used to display a two-dimensional polygon. For instance, if three segments connecting three points form a triangle, then you can use the Polygon tool to display a filled-in triangle. Click on the vertex points in order around the polygon and then complete the figure by clicking on the first point again.

Show/Hide Label . Select this tool. Then click on an object to hide its label (or display it if it was hidden).

Show/Hide Object . Select this tool. Then click on an object to hide it (or to display it if it was hidden).

Use the Angle tool  to display an angle. Click on the three points that form an angle *in clockwise order*—if you do it in counterclockwise order it will display the exterior angle, which you probably do not want. You can also click on the two lines that form the angle in clockwise order.

Use the Move Graphic tool  to shift the whole construction area.

To delete a point, either use the Delete tool  or select the object and press the "delete" button on your keyboard. You can also use the Undo button  at the far right of the tool bar to remove the last item or action. Before you delete something that someone else created, be sure to ask in the chat if everyone agrees that it should be deleted.

Insert Text . This tool can be used to place text on the drawing surface. You can add a title, a comment, etc.

You can use the Zoom in  and Zoom out  tools to change the scale of your view of the construction area. On a Mac computer, you can also use two-finger gestures for zooming; on a Windows computer, you can use a mouse scroll wheel or right button. Changing your view will not affect what others see in their views.

Use the Un-do tool to return to the state before the last construction action. Use the Re-do tool to restore an action that was un-done. Remember, do not un-do someone else's action without their agreement in the chat. In fact, these buttons may be disabled to avoid conflicts.

The Algebra view

A good way to view the locations, lengths, areas or other values of all the GeoGebra objects is to open the Algebra View from the GeoGebra "View" | "Algebra" menu. This opens a window listing all the free and dependent objects that you have constructed. You can un-attach this window with the little window icon that is above the Algebra View:

Top of the Algebra View and the Graphics View in GeoGebra.

The "drag test"

This is where dynamic geometry gets especially interesting. Select an object in the construction area with the Move tool. Drag the object by holding down the Move tool on the object and moving it. Observe how other parts move with the selected object. That is because the other parts are "dependent" on the part you are dragging. For instance, a segment depends on its end-points; when the points move, the segment must also move. If two segments both depend on the same point, then they will always move together; if you drag one of the two segments, it will drag the common end-point, which will drag the other segment. Dragging is an important way to check that parts have the correct connections or "dependencies" on other parts. GeoGebra lets you construct objects that have the dependencies that are important in geometry and in other branches of mathematics.

A thorough explanation of a simple construction with a dependency is given in a YouTube video using GeoGebra tools that are equivalent to straightedge and compass: http://www.youtube.com/watch?v=AdBNfEOEVco

Explore!

Construct some lines that share the same points. Think about how the figures are connected. State what you think will happen if certain objects are dragged. Then try it out. Take control and drag part of a figure. Discuss the dependencies in chat.

Hint

If two elements share a point – for instance, if a line segment starts at a point on a circle, then we say there is a "dependency" between the segment and the circle. That is, the position of the segment depends on the position of the circle, and when you move one, the other also moves. *Geometry is all about such dependencies*. A dynamic-math environment lets you see how the dependencies work and lets you explore them. Check out these videos of complicated dependencies:

http://www.youtube.com/watch?v=Oyj64QnZIe4&NR=1

http://www.youtube.com/watch?v=-GgOn66knqA&NR=1

Activity: Constructing Dynamic-Geometry Objects

Goal of this activity

In this activity, you will see how the computer representations of points, lines and circles in GeoGebra are dynamic and how they reveal relationships that can only be imagined otherwise.

When geometry was invented about 2,450 years ago in ancient Greece, geometric diagrams were constructed using just a *straightedge* (like a ruler without measurements on it for drawing straight line segments) and a *compass* (a hinged device also without measurements for drawing circles or arcs). The Greek geometers developed a graphical system of constructing two-dimensional diagrams with well-defined relationships and a deductive language for proving dependencies among the graphical objects. We call their system "Euclidean plane geometry".

A painting of Euclid constructing with straightedge and compass on a clay tablet.

The computer tools you will use in this activity allow you to construct dynamic-geometry diagrams that are equivalent to paper-and-pencil drawings with straightedge-and-compass tools—although of course your constructions will be dynamic. You will be able to drag them around changing their measurements, but maintaining the dependencies that you design into your construction.

Prepare for the activity

In a web browser on your computer, login to VMT-with-GeoGebra.[5] Find your chat room for this activity.

When the chat room is finished loading, click on the "+" button in the upper-right corner. Define a new GeoGebra tab with your login name as the name of the tab. Now click on the tab with your name on it. Use the construction area in this tab to explore some things in your own GeoGebra tab before you join your group to discuss your findings and questions.

This activity will only use GeoGebra tools that are equivalent to compass-and-straightedge tools for paper-and-pencil geometry constructions.

[5] See the "Tour: Joining a Virtual Math Team" for details on logging in. You may receive special instructions from your instructor.

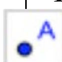
Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked so your points will have their names showing.

Try it on your own

Here are some things you should try to do before you work with the rest of your group. Do this in your own tab. If you have any problems or questions, communicate with the people in your group through the chat; they can probably help you because they are doing the same activities.

Create points

Create two points in your tab's construction area:

1. Press the "Take Control" button on the bottom of the tab to activate the tool bar. If there is already something on the construction area, clear it with the menu "File" | "New" | "Don't Save".
2. Select the Point tool  in the tool bar. Click in the construction area of your tab to create a point A somewhere.
3. Create another point B anywhere else with the Point tool .
4. Select the Move tool  and use it to move point A exactly where point B is. Look at the Algebra window to see the coordinates of A and B (use the menu "View" | "Algebra"). Show the grid to help locate the points exactly (use the grid button below the tool bar).

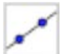
When a geometry problem says, "Imagine an arbitrary point," or a theorem says, "Given a point," then what follows is meant to be true for any point at any location. But when you represent a point by drawing one on paper, that point will always only be at the specific location where you drew it. Notice how this is different in GeoGebra—point B can be at any location, including at point A. So if you are trying to prove something about an arbitrary point B, then you might want to move it around to see if the same thing is true of point B when it is in some other locations, including special locations like the location of another point.

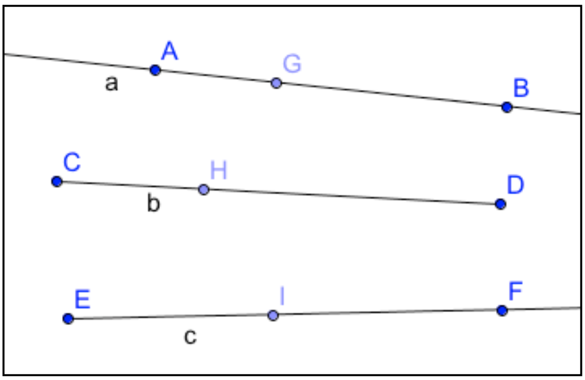
Notice that points A and B are listed in the Algebra window as "free" objects. If you move point A, does that change point B in any way? If you move point B, does that change point A in any way?

Create lines, segments and rays

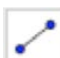
The dynamic nature of GeoGebra becomes more important when we move from points to lines.

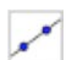
5. Clear anything on the drawing area with the menu “File” | “New” | “Don’t Save”.

6. Select the Line tool  from the tool bar. Create a line AB anywhere, by clicking to create the points A and B that line AB passes through. Note that a line is defined by two points; it passes through the points and continues on in both directions forever. Can you define a line with one point? With two points? With three points?



A line, segment and ray with an extra point on each.

7. Select the Segment tool  by pulling it down from the Line tool . Create a segment CD anywhere, by clicking to create the points C and D that define the endpoints of segment CD.

8. Select the Ray tool  by pulling it down from the Line tool . Create a ray EF anywhere, by clicking to create the points E and F that ray EF passes through. What is the difference between a line, a segment and a ray? Can you create a short line? A short segment? A short ray?

9. Now select the Point tool  and create a point G somewhere on line AB. Create a point H somewhere on segment CD. Create a point I somewhere on ray EF.

10. Select the Move tool  and try to drag the lines (line AB, segment CD, ray EF). Try to drag each of the points on them. Notice that in GeoGebra,

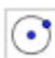
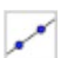
some objects are colored blue and some are black. Do you wonder what the difference is?

11. After you construct something in GeoGebra, you can use the Move tool  to conduct a "drag test" to see if the dependencies you wanted are in effect. Notice how you can drag different parts of an object and get different results. For instance, drag ray EF by point E, by point F or by dragging the whole ray. Does point I stay on it? What happens when you drag point I on ray EF? How far can you drag point I?

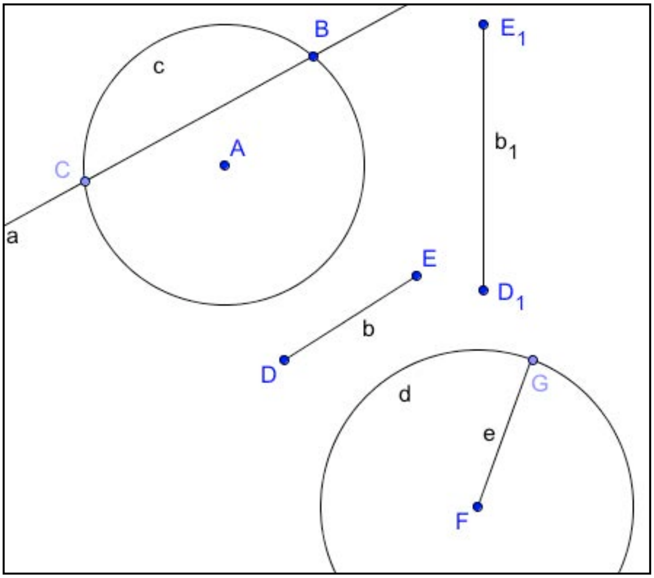
Notice the different ways the lines and points move as other objects are dragged. Also, notice that some objects are constrained to only move in certain ways because of their dependence on other objects. Which objects are dependent on which other objects? Does that explain how everything moves? How were the dependencies defined in your construction?

Dependencies of circles

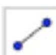
Circles or a compass are very useful for defining dependencies in geometry. Let us see how a circle can define a useful constraint. Remember that the circumference of a circle is defined as the points that are all a given distance (the length of a radius) from the center point.

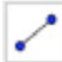
12. Clear anything on the drawing area from the menu "File" | "New" | "Don't Save".
13. Select the Circle tool . Create a circle c by clicking for its center A and for a point B on its circumference. Now select the Move tool  and try to drag point A, point B or circle c. Notice how things move differently in each case. Do you understand why they move this way? Do you think that is the way they should move to maintain the dependencies you just defined: that A is always at the center of the circle and B is always on the circumference?
14. Select the Point tool  and create another point, C, on the circumference of the circle.
15. Select the Line tool  and create a line through points B and C.

Select the Move tool  and try to drag each of the objects (the points, the line and the circle). Notice the variety of constraints to movement caused by the dependencies on the radius of the circle (which is an implicit segment AB, not drawn). The dynamic geometry decides where objects can move. When you cannot move a point, think about how that is related to constraints that you just constructed.



Construction of free and dependent circles.

16. Now let's see how we can use this dependency on the radius of a circle. Select the Segment tool  and create a new segment DE.
17. Select the Move tool  and click on segment DE. Use the menu "Edit" | "Copy" and then "Edit" | "Paste" to make a copy of segment DE, named segment D_1E_1. Drag the points D, E, D_1, E_1. Notice that even though D_1E_1 is a "copy" of DE, it is not constrained by DE in any way. It was copied from segment DE, but it was not *constructed* to be *dependent* on segment DE.
18. Select the Compass tool  by pulling down from the Circle tool . Click on points D and E. Then click somewhere else to create point F as the center of a circle d. Is the radius of circle d constrained to always be as long as the length of segment DE? Can you confirm this with the drag test? You can check the length of segments in the Algebra window (use the menu "View" | "Algebra").
19. Select the Point tool  and place a point G on the circumference of the new circle d.

20. Select the Segment tool  and create a segment FG, which is a radius of circle d.

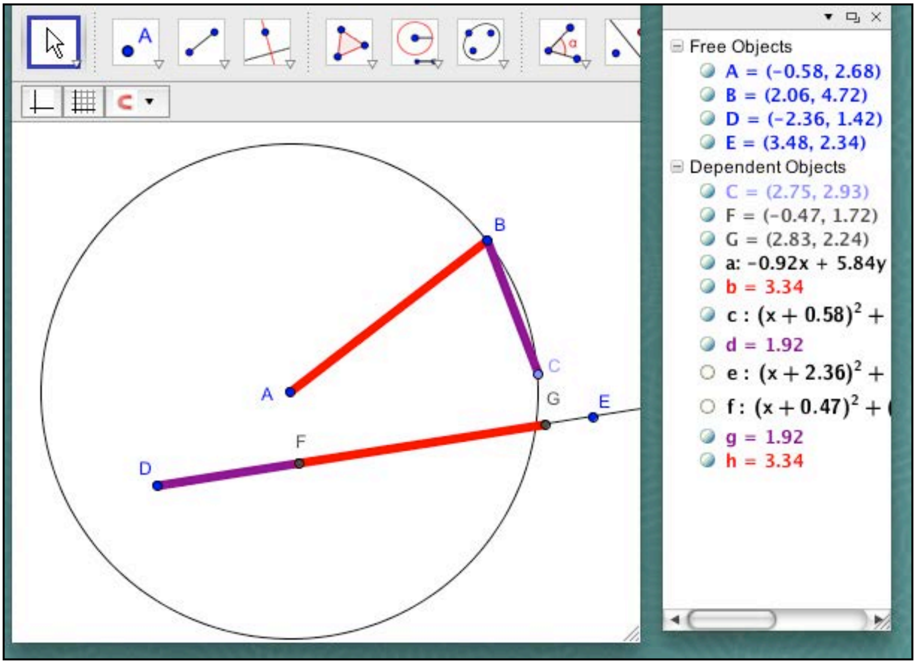
Select the Move tool  and explore the dependence of the length of FG on the length of DE. Is it true that these two segments can be anywhere, in any direction, as long as they maintain the same length? Is this a two-way dependency? Do you understand how point G can move?

Work together

When you have finished working on this activity by yourself, announce in the chat that you are ready to work together with your group. Use the team "GeoGebra" tab for the following work. Take turns taking control and releasing control in the team tab. Use the chat to ask for control, to say what you want to do, and to describe your work:

First, answer questions that anyone on the team has about this activity. If someone could not do one of the constructions or is not sure they did it correctly, go through the steps as a group in the group tab. Say what you are doing in the chat. Makes sure that everyone in the group understands and agrees with each step.

If your whole team thinks that they understood all of the previous individual activities described above, then as a group do this construction. Do *not* just copy the drawing in the figure – construct it by following the steps below:



Construction of DG = AB + BC.

Challenge: To construct a segment DG along ray DE, whose length is equal to the sum of the length of a radius AB of a circle plus the length of a segment BC connecting two points on the circumference of the circle.

Clear anything on the drawing area using the menu “File” | “New” | “Don’t Save” and do the construction.

Use the compass tool to copy the lengths of AB and BC onto ray DE.

You can color segments BC and DF one color and segments AB and FG another color with the color and line-thickness tools under the tool bar. You can see the lengths of these segments in the Algebra window.

Check your construction with the drag test to see how your segments change as you drag points A, B, C, D or E.

Take turns being in control of the construction. Say what you are doing in the chat. Make sure that everyone in the group understands and agrees with each step.

Does DG = AB + BC no matter how you drag any of the objects?

You can hide some of the construction objects like the compass circles by double clicking on them and changing their “Object Properties” by un-checking “Show Object”. Then your construction should look similar to the figure.

- When the group is finished doing this task in the shared GeoGebra tab, try to do it yourself in your own tab. This is an important skill; be sure that you completely understand it and can do it yourself. Make sure that everyone in your group can do it themselves.

Discuss it

In the chat, state in your own words:

- The difference between lines, segments and rays in GeoGebra.
- The meaning of "constraint" and "dependency" in dynamic geometry.

What were the most interesting movements for you?

- The way a line pivots around one point when the other defining point is dragged.
- The way a line through two points on a circle is constrained.
- The way a third point on a line maintains its proportionate spacing between the two points that define the line, as one of those points is dragged.
- The way that a point on a circle is constrained by the center and radius.
- Some other movement.

Can you think of any ways you could use the dependency created with the compass tool or circle tool to construct other geometric figures or relationships?

Think about different kinds of points in different kinds of geometries:

- In the *real world*, points have some size and other characteristics. For instance, your town is at some point in your country. If you look closely enough, that point is quite complex.
- In the *mathematical world* of Euclidean plane geometry, a point is defined as having no size, no color, no thickness and no other characteristics except for its precise location. Every point is identical, except for its location. If you move point B to the location of point A, then they become the same point in mathematical geometry.
- In *paper-and-pencil representations* of mathematical geometry in school on a chalk blackboard or on a piece of paper, a point has size, shape and color, depending on how you draw it.
- In *GeoGebra*, points have size, shape and color—so that you can see them and move them around. You can change their size, shape and color to help you see what is happening in a complicated construction. If you move point B to the location of point A, do they become the same point in GeoGebra?

Challenge: Compare the answers to the following questions (a) in the real world, (b) in Euclidean geometry, (c) in pencil-and-paper drawings and (d) in GeoGebra:

- Can two points, A and B, be at the same exact location?
- Can you use the same two points to define a line AB, a segment AB and a ray AB all in the same construction?
- If you have defined a line, a segment and a ray all with the same two points and then you define a third point, C, on the segment, is it also on the line and the ray—and vice versa?
- Can you move the third point along the whole line?
- What happens to the line, ray, segment and third point if the two defining points, A and B, are moved to exactly the same location?
- If two lines intersect, how many points can there be at their intersection?
- Can three lines intersect at the same point?

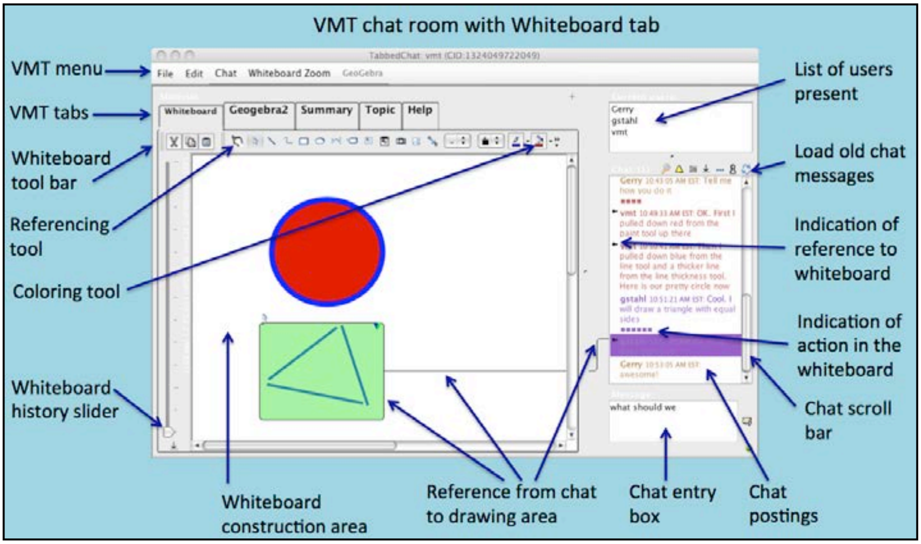
Hint: If you select the menu "View" | "Algebra" you will display a list of all the points, lines and polygons, including their coordinates, lengths or areas and which are "free objects" and which are "dependent objects".

Tour 3: VMT to Learn Together Online

In this tour, you will explore the VMT-with-GeoGebra environment and learn how to use it to collaborate. You will learn about many special features of the VMT system, which you will need to use in the following activities.

Enter a VMT chat room

When the VMT chat room is open, it will look something like this:



Interface of a VMT chat room with a whiteboard tab.

See what is going on

See the list of users present in the upper right. It shows all the people who are currently logged into this chat room

Awareness messages near the bottom of the window state who is currently typing a chat message or drawing in the shared whiteboard. You should see all the messages that anyone posted in the chat room and all the drawings that anyone did in the whiteboard as soon as they finish typing (after they post the message by pressing the Return key on their keyboard) or drawing (and after they click on the whiteboard background).

Post a greeting message

Type in the chat input box. Press the Return or Enter key on your keyboard to post your message for others to read. Your message should appear above in the chat messages area with your login name and the current time. Other people in the same chat room will also see your message.

Look back in chat and whiteboard history

Load old messages if you are joining a room where people have already been chatting.

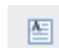
Use the reload icon  (two curved arrows). You can scroll back in the chat if there are too many messages to be displayed at once.

The whiteboard also has a history slider so you can see how the images in the whiteboard tab changed over time.

Reference a previous chat message
Point to a previous chat message by double clicking on the previous message while you are typing a new chat message. This will create an arrow from your new chat message to the previous chat message. Everyone will see this arrow when your message is posted or if they click on your message later.

If a reference arrow exists and you want to delete it, then press the ESC key on your keyboard before you post the message.

Leave a message on a shared whiteboard
Click on the different tabs to see the different work areas. The Summary Tab is just like the Whiteboard Tab. Your group can use the Summary Tab to summarize your work on an activity.

Go back to the Whiteboard Tab. Open a textbox (the icon for this  is in the middle of the Whiteboard tool menu; it has an "A" in it; if you roll your cursor over it, it says "Add a textbox".) Type a message in the textbox. Double-click on someone else's textbox to edit and add to what they wrote.

You can draw a square or circle and change its color, outline, etc.

Reference an object in the whiteboard
You can also create an arrow from your new chat message to an object or an area on the whiteboard, just like you did from a new chat message to a previous one.

Point to the square or circle with the reference tool. First click on the referencing tool (the pointing hand  in the whiteboard tool bar — see screenshot of "VMT chat room with Whiteboard tab" above). Then select the square or circle — or else drag the cursor to select a rectangular area around the square or circle. Finally, type a chat message and post it. You should see a line connecting your chat posting to the object in the whiteboard. This is handy to use when you want to make a comment or ask a question about an object in the Whiteboard or in GeoGebra.

Draw two triangles
Draw an equilateral triangle (where all three sides are of equal length) on the shared whiteboard.

Or draw a right triangle (where one angle is a 90-degree right angle) on the shared whiteboard.

Try to move these triangles around.

What do you notice about them? Is it hard to rotate or move the triangle around? What would you like to be able to do?

If you drag one end of a line to change the lengths of the sides, are the triangles still equilateral or right triangles?

GeoGebra has other ways of drawing triangles. You will be doing a lot of that in the following activities.

Open extra tabs
Use the "+" button (above the upper right corner of the tab) to create a new tab. This is handy if a whiteboard tab or a GeoGebra tab gets filled up and you want to open a new one without erasing everything.

Get some help on math notation
Go to the Help tab to learn more about VMT. For instance, look up how to enter Mathematical Equations/Expressions in the chat, in Whiteboard textboxes and in the VMT wiki. They use the $ to indicate math notation. You can cut and paste these expressions between the chat, Whiteboard textboxes and the VMT wiki.

Review your team's work
Use the history sliders on the left side of the whiteboard and on the left side of the chat to get an overview of what your group has done and discussed if you come in late or return on another day. Discuss a summary of your work with your group. You can put a textbox with this summary in your Summary Tab. You can even start with an outline or a first draft of a summary in the Summary Tab and then have everyone discuss it in the chat and edit it in the Summary Tab.

Try to create a reference from a chat posting about an idea in the summary to the sentence in the Summary tab.

What have you learned in this activity? What do you wonder about it? What did you not understand or what do you want to know about? Ask the other people in your group—they may have some answers for you or be able to help you find the answers.

Activity: Exploring Triangles

Goal of the activity

In this activity, you will drag four different dynamic-math triangles to explore their built-in dependencies. Then your group will try to create an isosceles triangle with its dependencies.

Prepare for the activity

In a web browser on your computer, login to VMT-with-GeoGebra. Find your chat room for this activity.

Open the tab with your name on it. If there is not a tab for you, use the "+" icon in the upper-right corner to create a GeoGebra tab with your name.

When you open your tab, you should see four colored triangles. If you do not see this, then download the file http://vmt.mathforum,org/activities/four_triangles.ggb and use the menu "File" | "Open" to load the triangles in your construction area. Press the "Refresh View" button.

Try it on your own

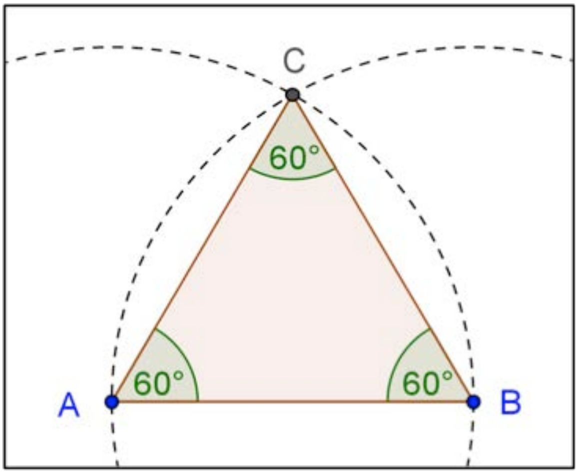
Here are some things you should try to do before you meet with your group for this activity. You should do this in the tab with your name. If you have any problems or questions, communicate with the people in your group through the chat; they can probably help you because they are doing the same activities.

Constructing an equilateral triangle

Now you will see how one of the triangles was created. You will see how to use the compass tool to make sure that the three sides of a triangle will always be of equal length. This was the first construction in Euclid's geometry book.

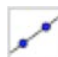
Here is a video showing one way to construct an equilateral triangle in GeoGebra: http://www.youtube.com/watch?v=ORIaWNQSM_E

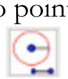
Think about the properties of an equilateral triangle before you start the construction. You should end up with a construction that looks like the following one. (Your labels and other details may be different.)

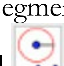


An equilateral triangle.

1. Take control. Clear anything on the drawing area with the menu "File" | "New" | "Don't Save". Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked.

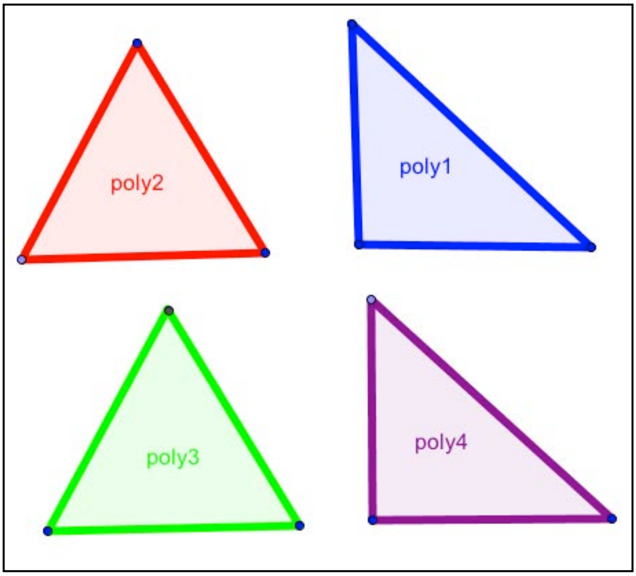
2. Use the Segment tool  (pull down from the Line tool ) to construct segment AB. This will be the base (one side) of your triangle. The endpoints might have different labels for you. Labels of points, lines, angles and other objects are important in geometry; they help you to refer to the objects when you discuss them. GeoGebra automatically labels points alphabetically when they are created. Your points may not have the same labels as the diagrams here and you may have to "translate" between the two sets of labels. You can try to change your labels by control-clicking (on a Mac) or right-clicking (in Windows) on a label and renaming it. Points have names with capital letters and you cannot have two points with the same name.

3. Use the Compass tool  (pull down from the Circle tool ) to construct a circle with center at A (one endpoint of the segment constructed above) and passing through B (the other endpoint of the segment), so that the radius of the circle is equal to the length of segment AB. Any segment from point A to a point on the circumference of the circle will always be the same length as segment AB. When we create point C on this circle, which will make the length of side AC *dependent* on the length of the base side AB – even if the length of segment AB or the size of the circle are changed.

4. Use the Compass tool  to construct a circle with center at B and passing through A. When we create point C on this circle, that will make the length of side BC *dependent* on the length of the base side AB.

5. Use the Point tool or the Intersect tool (pulled down from the Point tool) to construct point C at an intersection of the two circles. Now select the Move tool and try to drag point C. If you put it at the intersection of the two circles, you should not be able to drag it because it is dependent on staying on both circles. If point C is not dependent on both circles, then create a new point that is at the intersection. Point C is *constrained* to be the same distance from point A and point B. Do you see why this is? Explain it in your textbox.
6. Use the Polygon tool to construct polygon ABC (click on points A, B, C, A).
7. Use the angle tool to construct angle ACB, angle CBA and angle BAC (you must click on the three vertices in clockwise order to define each angle).
8. Check your construction: Select the move tool . Do a drag test of your dependencies. Can you drag point C? Can you drag point A, point B, segment AB, segment AC, segment BC, polygon ABC?

Exploring different triangles



Four triangles.

In your GeoGebra tab, press the "Take Control" button and use the Move tool to drag the vertices of each of the triangles. Explore the different triangles, and notice as much as you can about their shapes, their sides, their angles, and any relationships between these shapes, sides and angles.

Then go to the Summary tab and create a textbox. Type your name in the top of the textbox and record what you notice about the shapes, sides and angles of each of the triangles. Be as thorough as you can.

You can move some of the triangles so that their vertices match up and the two triangles lie on top of each other (try it!). Try it with the other pairs of triangles, and decide which pairs can and cannot match up:

List the pairs that you can match up in your Summary textbox.

Notice & wonder

Are the results of your constructions what you would expect? Can you explain why they are that way?

Did you notice anything that you were not expecting? Can you explain why it is that way?

Enhance your construction: Select the Show/Hide Label tool (pull down from the Move Graphics tool). Click on the three angles to make sure their values are showing or open the Algebra view. You can also select segment AB, segment AC and segment BC to display their lengths. You can display the area of polygon ABC. Do the sides of the triangle stay the same as each other when you drag the triangle around and change its size and area? Do the angles stay the same? Change the length of segment AB and observe what happens to the lengths of segments AC and BC. Is that what you expected?

Select the Show/Hide Object tool (pull down from the Move Graphics tool). Click on each of the circles to hide them. They are not needed as part of the final triangle. However, their dependencies remain even when the circles are not visible. What would happen if you deleted a circle instead of just hiding it? (You can select the circle , delete it , drag objects around, and then un-do the deletion with the undo button.)

What did you learn? Answer these questions in your Summary tab text box:

- What did you learn about equilateral triangles? How did you define your dynamic equilateral triangle ABC to be equilateral by your construction of it?

- What did you learn about constructing geometric figures? What dependency did you impose by locating point C at the intersection of the two circles?
- What did you learn about dynamic math? Do the three sides of the triangle stay congruent (the same length as each other) when you drag the triangle in various ways? Do the three angles stay congruent (the same size as each other) when you drag the triangle in various ways? Can you explain clearly why this is?

Work together

When you have finished working on this activity by yourself, announce in the chat that you are ready to work together with your group in the team "GeoGebra" tab.

Discuss the red, green, blue and purple triangles. Do you all agree on what type of triangle each one is? An *equilateral* triangle has all sides of equal length. An *isosceles* triangle has two sides of equal length. A *right* triangle has one 90° angle. A *scalene* triangle has no equal sides and no right angles.

Do you all agree on which triangle can exactly cover which other triangle? Make sure that everyone agrees to the answers, and anyone could demonstrate why any specified pairing does or does not work. Look at everyone's answers in the Summary tab. Create a textbox in the Summary tab that states the answer that the whole group agrees on.

Now work as a team to do the following challenge in the group GeoGebra tab. Use the chat to decide who does what. Take turns with the "Take/Release Control Button." Describe what you are doing and why you are doing it in the chat.

> *Challenge*: Try to construct an isosceles triangle. *Only use tools that are equivalent to straightedge and compass*: Point, Line, Segment, Ray, Circle, Compass. Think about how to build in the dependencies for their sides, angles and shapes. You have already constructed a triangle that is constrained to have all three sides of equal length (an equilateral triangle). How would you construct a triangle that is constrained to have just two sides of equal length?

Do you agree on how to construct an isosceles triangle?

Make sure that anyone in the group could construct a scalene, isosceles or equilateral triangle from scratch on demand, and those constructions would be as general as possible (not over-constrained, such as an isosceles triangle that has to be equilateral).

Discuss it

In the Summary tab, describe the key constraints that have to be built into the different kinds of triangles. Summarize the approach to constructing triangles with these dependencies.

Here are some final things to write about on your group's Summary tab for this activity:

- Discuss what you understand the terms "constraint" and "dependency" to mean. Give some examples from the constructions in this activity.
- What is the difference between drawing a triangle on paper – or in the Summary whiteboard tab – and constructing it with dependencies in a GeoGebra tab?
- Triangles are often used in building bridges because a triangle is a very stable shape that constrains movement and distortion. Can you think of other places that triangles are used in the real world?

Tour 4: The VMT Wiki for Sharing

There is a special wiki page for your chat room. You can find it from the VMT Lobby.

Click on the wiki icon  that is displayed after the link to your chat room in the VMT Lobby. A wiki page is a page on the Internet that anyone can easily add to or edit.

Go to the wiki page for your chat group and copy the summary of your group's work from the Summary tab of your chat room to this wiki page. Copy and paste text from your Summary tab into the "edit" tab of the wiki page and then format it for the wiki. Instructions for editing are available from the "Help" link in the "navigation" panel on the left side of the wiki page. (The VMT wiki is edited the same way as Wikipedia).

A wiki page for chat room "Trial33_1" on topic "Activity1", subject "Geometry", community "Geogebra".

The wiki page for your chat room is automatically linked to all the wiki pages for its activity and also to all the wiki pages for its subject. Finally, there is a wiki page that links to all the groups or teams in your whole project. This way, you can compare your group's findings with everyone else's.

For instance, if your chat room has the topic "Activity1", then go to the wiki page for Activity1 by clicking on the Category link for "Activity1" at the bottom of your group's wiki page and browse to see summaries of other groups. Now return to your group's wiki page and comment on how your work compares to that of other groups.

Activity: Creating Construction Tools

Goal of the activity

In this activity, you will use the equivalent of straightedge-and-compass tools to construct perpendicular lines, parallel lines and a midpoint. Then you will construct a right triangle. These are basic constructions and relationships, which are used repeatedly in geometry. To make it easier to do these frequent constructions, you can program your own custom tools in GeoGebra. In this activity, you will program a new custom tool for constructing a dynamic-geometry perpendicular.

Prepare for the activity

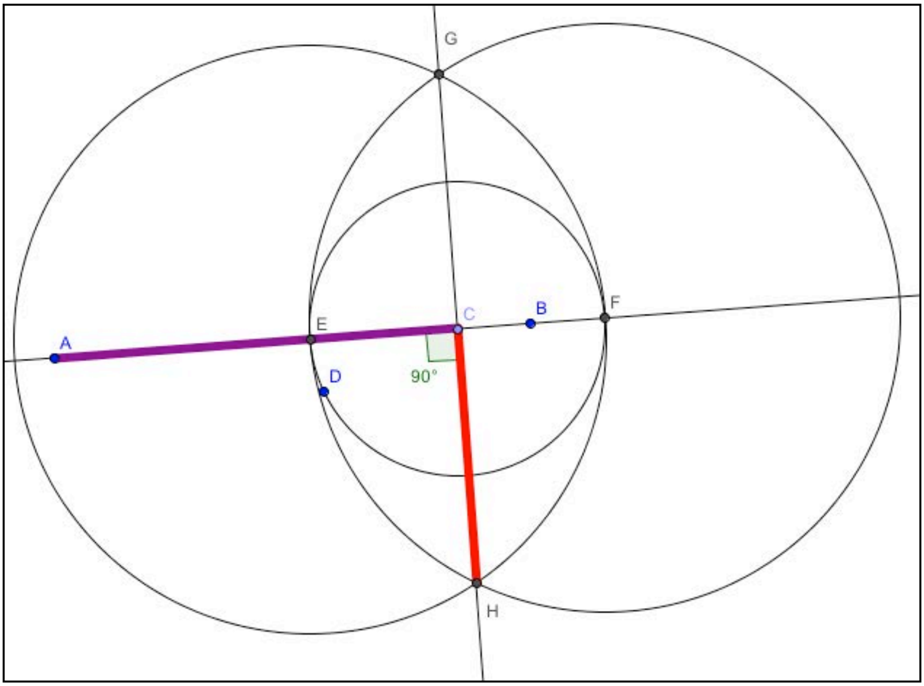
Think about how you would use the straightedge-and-compass GeoGebra tools you already know to construct a right angle (a segment perpendicular to a given segment) and a right triangle (a triangle with one of its angles a right angle).

Try it on your own

Here are some things you should try to do before you meet with your group for this activity. You can do this in your own GeoGebra tab. If you have not already created this tab, use the GeoGebra "+" icon. Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked so your points will have their names showing.

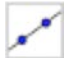
Construction of a perpendicular at a point

We want to construct a line GH perpendicular to line AB and passing through point C to intersect line AB at point C.



Construction of a perpendicular.

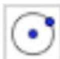
1. Clear anything on the drawing area with the menu "File" | "New" | "Don't Save".

2. Construct line AB with the Line tool . Construct an arbitrary point C with the Point tool  somewhere on line AB. Now you want to construct a perpendicular to line AB, which intersects line AB at point C.

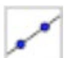
3. Construct a circle with center at C using the Circle tool  (passing through any point D not on AB).

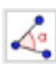
4. Use the intersect tool  to construct points E and F at the two intersections of the circle with line AB. Notice that points E and F are equidistant from point C.

5. Construct a second circle  with center at E passing through F.

6. Construct a third circle  with center at F passing through E (and therefore having the same radius as the previous circle).

7. Use the intersect tool  to construct points G and H at the two intersections of the circles (with centers at E and F) with each other.

8. Construct line GH .

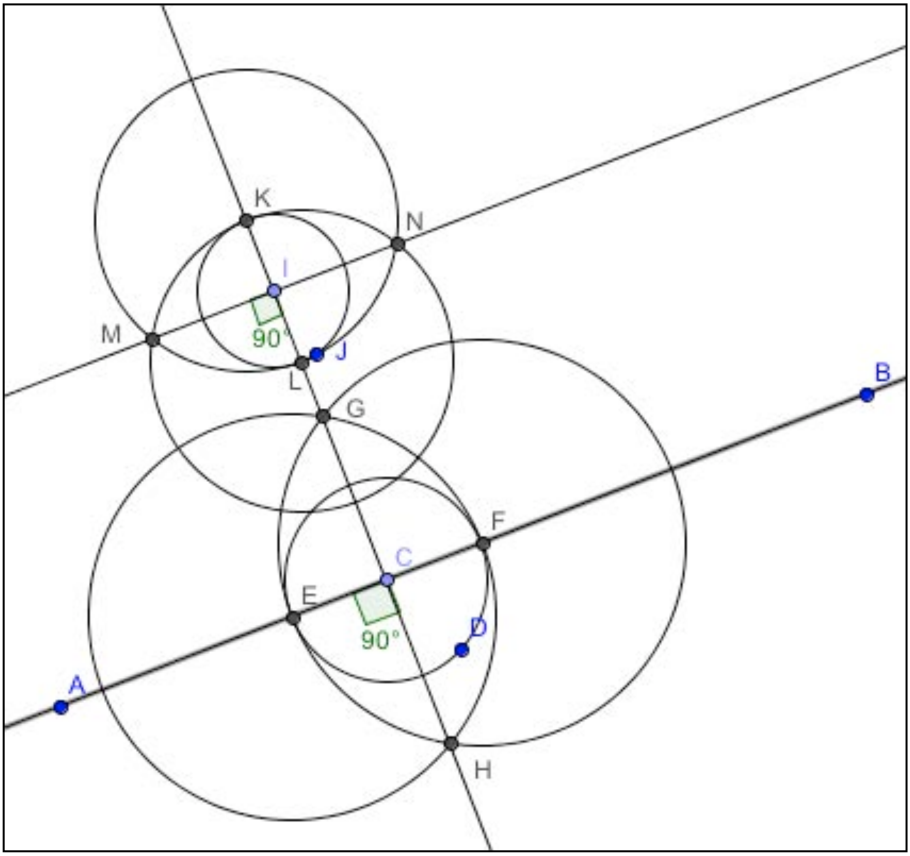
Use the angle tool  for angle ACH to see if line GH is perpendicular (90°) to line AB at point C.

Use the drag test to see if line GH stays perpendicular to line AB at point C.

Think about why GH is perpendicular to AB at point C. Was every step necessary? Can you simplify the construction?

Construct a parallel line

You can construct a line parallel to line AB by constructing a perpendicular to a perpendicular. Look at the construction. Line AB is parallel to line MN because line GH is perpendicular to AB and MN is perpendicular to GH.



Construction of parallel lines

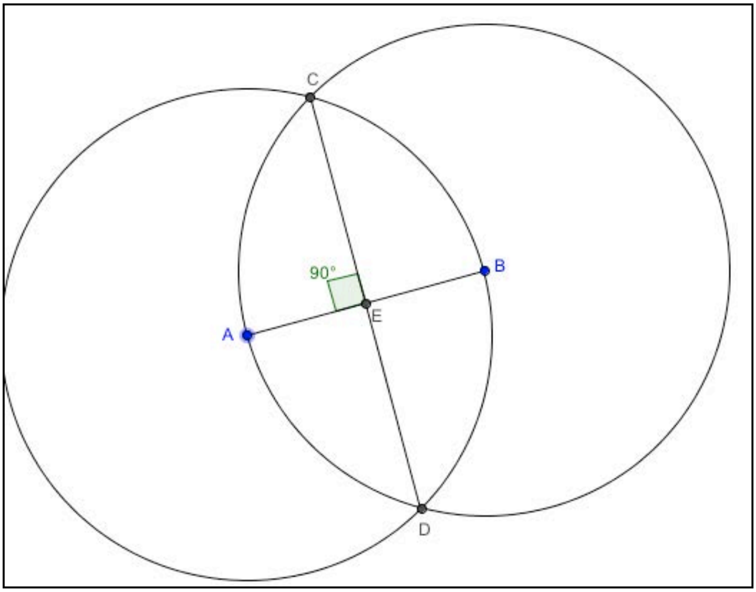
9. Construct an arbitrary point  I on line GH.
10. Construct a new line perpendicular to line GH at point I, using the steps you followed above.

Use the angle tool and the drag test to see if the new line stays parallel to line GH.

Think about why the new line is constrained to be parallel to line GH by the dependencies of the construction.

Construct a midpoint of a segment

Here is a variation of the previous construction of a perpendicular. It constructs the perpendicular that passes through the midpoint of a segment and thereby constructs the segment's midpoint.



Construction of a midpoint.

1. Clear anything on the drawing area with the menu "File" | "New" | "Don't Save".
2. Construct segment AB.
3. Construct a circle with center at A passing through B.
4. Construct a circle with center at B passing through A.
5. Construct a point C at one intersection of the two circles. (What are the dependencies of this point?)
6. Construct a point D at the other intersection of the two circles.
7. Construct segment CD connecting points C and D.
8. Construct point E at the intersection of segment AB and segment CD. Point E is the bisector of segment AB.

Think about why point E is the *bisector* of segment AB. How do you know that it divides segment AB into two segments whose lengths must be equal? Drag points A and B toward and away from each other; points C and D move up and down the perpendicular bisector. Points C and D stay equidistant from points A and B as the construction is dynamically dragged.

Discuss why segment CD is perpendicular to segment AB. What does it mean to say that two segments are perpendicular to each other?

Segment CD is called the "perpendicular bisector" of segment AB because CD is perpendicular to AB and it bisects AB into two equal halves through the midpoint of AB. Finding the perpendicular bisector is very useful in practical tasks as well as in solving many geometry problems. You will use this skill in several of the following activities.

In this activity, you used only dynamic-geometry tools equivalent to the classic-geometry tools of compass and straightedge to do the following:

- Construct a perpendicular to a given line through a given point.
- Construct a line parallel to a given line.
- Construct a midpoint of a given segment.

Now you can use these skills to construct objects with right angles. Also, you can make a custom tool to automate these constructions.

Notice & wonder

When you have finished working on this activity by yourself, announce in the chat that you are ready to work together with your group in the team GeoGebra tab.

Here are some things to think about and to discuss with your team in the chat:

- Does it matter if AD is a segment, a ray or a line in your custom tool creation?
- What if you also saved point D in your custom tool creation and used it for creating a right triangle from the base segment?
- Can you make a custom tool that outputs a right-triangle polygon directly from a segment? How general is it? Can it be isosceles?
- What new GeoGebra tools would you like to have?
- Can the way custom tools are defined in GeoGebra be called "programming"?
- Do you think that any of the tools in GeoGebra's menu were programmed this way?

Work together

Do the following together with your group:

- Discuss why GH was perpendicular to line AB through point C in the first construction of this activity. Can you prove it? Write an answer that your whole group agrees with on the wiki page for your group's chat room.
- Discuss the difference in constructing a perpendicular to AB through C if C is (i) on AB or (ii) off AB.
- Can you design another perpendicular tool that works differently but would be useful for some constructions?

Post your group conclusions on the wiki page.

Make sure that you can do all the constructions that the group discussed. Try them yourself in your own GeoGebra tab.

Check with the other people in your team to make sure that they all understand how to do these constructions and how these constructions impose the dependencies that are needed.

Don't forget

Write on your group's wiki page your group's responses to these:

- Discuss what you learned in this activity?
- Discuss what you learned about right triangles.
- Discuss what you learned about constructing geometric figures.
- Discuss what you learned about dynamic math.
- Discuss how you think programming new tools can be useful.

Tour 5: VMT Logs & Replayer for Reflection

Get a log of your group's work

You can get a spreadsheet containing a log of all the chat postings in a VMT chat room. You can use this log for documentation of your group's work by pasting excerpts from the log into a report. This can also be useful for reflection on the work of your group or for analysis of the interaction and knowledge building that took place.

To view the log, go to the VMT Lobby and find the chat room. In front of the link to the room is an arrow, which you can click on to turn down. You will see a button that says "View Chat Log". This will display the spreadsheet. You can cut and paste from the display window to your report document. There are also three "Get Log" options for downloading the chat log as a spreadsheet file: with each participant's posting in a different column, with all postings in one chronological column or in a special format for automated analysis. The spreadsheets can be filtered by event type to display a subset of events.

Geometry (10 Topics)

Activity1 (25 Rooms, 0 Active)

Act_1

Username	# of Messages	Last Active
abp	2	Jan 11, 2012 10:24
andicat	2	Dec 1, 2011 13:25
jason	4	Nov 17, 2011 13:33
loretta	8	Jan 3, 2012 08:31
roberthg	2	Nov 17, 2011 11:29

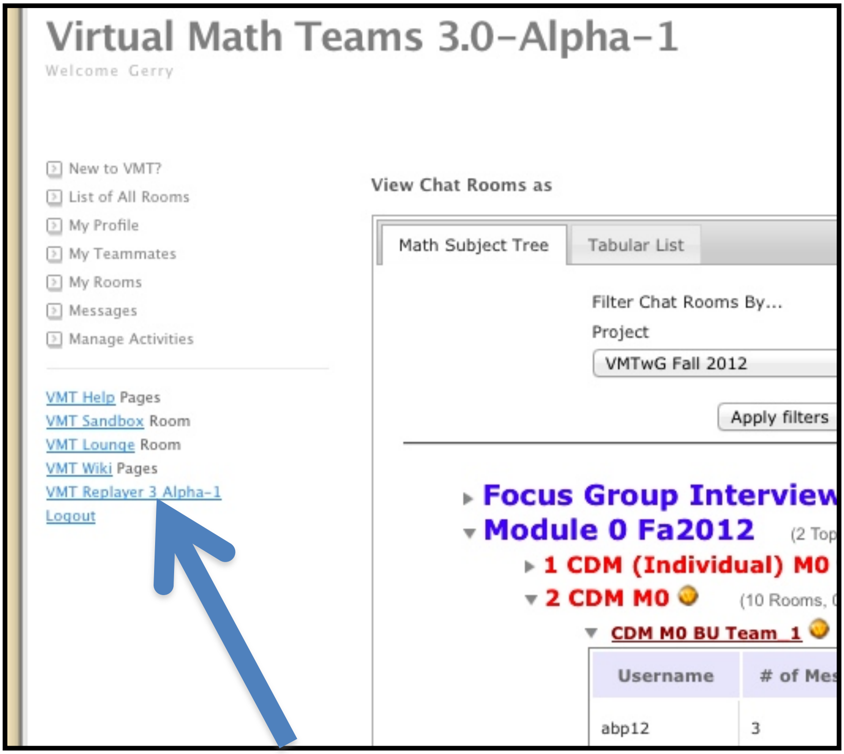
Add to Favorites | Save as JNO | View Chat Log

Get Log: columns for each user

Get Log: one column for all users | Get Log: Informatics

Replay your group's work

You can replay all the chat and constructions in a VMT chat room with the VMT Replayer. Then you can save a screenshot of any stage of your session to include in a report. This can also be useful for reflection on the work of your group or for analysis of the interaction and knowledge building that took place. Save a complete history of a VMT chat room with the "Save as JNO" button; this will download your room's .JNO file to your desktop.



Click on the "VMT Replayer 3 Alpha-1" link in the Lobby to download "vmtPlayer.jnlp". Start the Replayer. Select menu item "File" | "Open Session" and browse to your room's .JNO file. It may take a few minutes for the Replayer to open with the chat room history, depending upon how much activity took place in the room. When the room is opened in the Replayer, it will look just like the original VMT room, except that at the bottom it will have a history slider and some buttons to replay the entire session at a selected speed or to step through the interaction one action at a time with your keyboard's arrow keys. Scroll the timeline back to the start of the session.

Try it on your own

When you are finished working with your group on the next activity, download the Replayer and the JNO file for your room. Compose a brief report on your group experience and include excerpts from the chat and screen shots from the Replayer on your group's wiki page. You may want to download a free screen-capture application like Grab to make images of the Replayer on your computer screen.

Activity: Constructing Triangles

Goal of the activity

In this activity, you will use the equivalent of straightedge-and-compass tools to construct perpendicular lines, parallel lines and a midpoint. Then you will construct a right triangle. These are basic constructions and relationships, which are used repeatedly in geometry. To make it easier to do these frequent constructions, you can program your own custom tools in GeoGebra. In this activity, you will program a new custom tool for constructing a dynamic-geometry perpendicular.

Warning: This activity has many steps. Give yourself plenty of time to work on this before your group meeting.

Prepare for the activity

Think about how you would use the straightedge-and-compass GeoGebra tools you already know to construct a right angle (a segment perpendicular to a given segment) and a right triangle (a triangle with one of its angles a right angle).

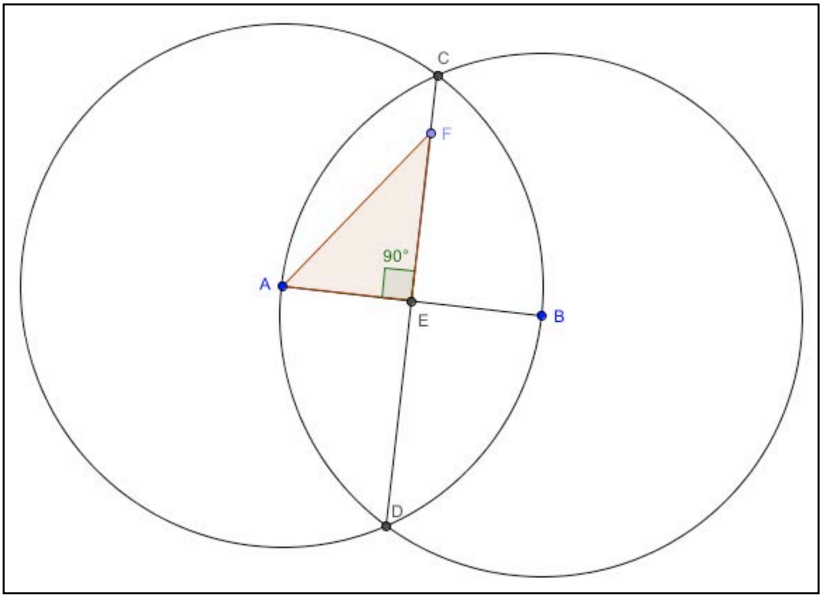
Try it on your own

Here are some things you should try to do before you meet with your group for this activity. You can do this in your own GeoGebra tab. If you have not already created this tab, use the GeoGebra "+" icon. Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked so your points will have their names showing.

Construct a right triangle

Now you can construct a triangle that is constrained to always have a right angle. Right angles are very important in all forms of practical construction of shaped objects, such as in carpentry, bridge design, architecture, computer graphics, etc.

To construct a right triangle, create a segment for the base of the triangle and then construct a perpendicular bisector to this segment. Finally, connect a point on the base to a point on the perpendicular.



Construction of a right triangle.

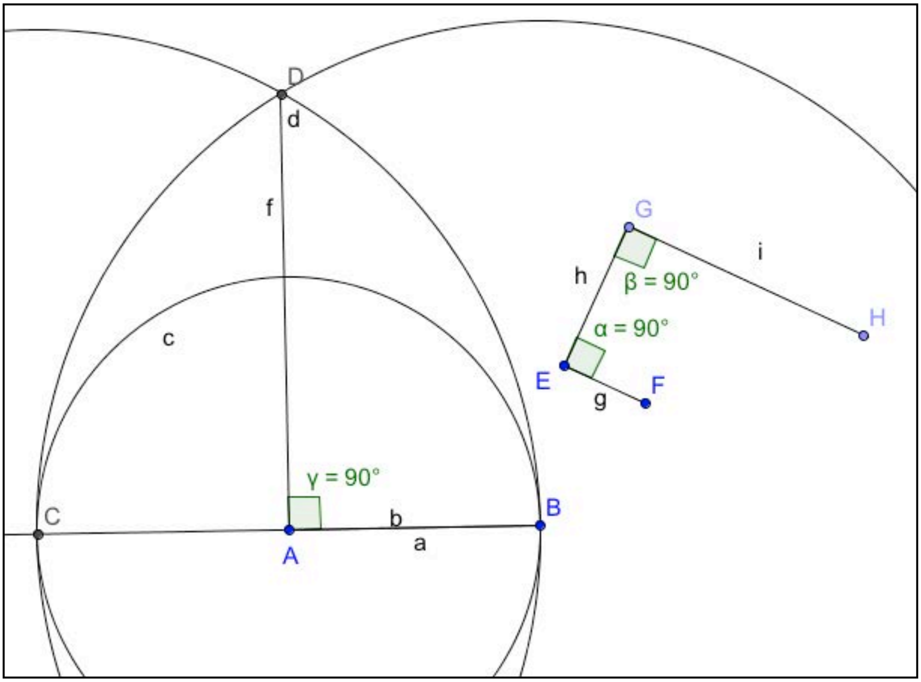
1. Clear anything on the drawing area with the menu “File” | “New” | “Don’t Save”.
2. Construct segment AB.
3. Construct a circle with center at A passing through B.
4. Construct a circle with center at B passing through A.
5. Construct a point C at one intersection of the two circles.
6. Construct a point D at the other intersection of the two circles.
7. Construct segment CD passing through points C and D.
8. Construct point E at the intersection of segment AB and segment CD. Point E is the midpoint or bisector of segment AB.
9. Construct point F on segment CE. How is point F constrained?
10. Construct a polygon AFEA .
11. Show angle FEA .
12. Hide all the objects except polygon AFE, its vertex points A, C and E, and its angle CEA.
13. Check your construction with the move tool to do a drag test of your dependencies.

Can you drag point F? Can you drag point A, point F, segment AE, segment EF, segment AF, polygon AEF? Are these results what you expected? Did you notice anything that you were not expecting?

Build your own new tool: a custom perpendicular tool

Now that you know how to construct perpendicular, parallel and bisecting lines, you can add a custom tool to the tool bar to create a perpendicular segment without going through all the steps in future activities. In this way, geometry builds on discoveries in order to help discover more complicated constructions and discoveries.

GeoGebra allows you to define a new tool to construct perpendicular bisectors of segments with two clicks. You can define a button that automatically imposes the dependencies of the construction that you learned in this activity. Here is how you define the new tool: You create a construction and then you program a custom tool by defining input objects and output objects. For instance, you will program a custom tool called “My Perpendicular Tool”. When this tool is selected, if you click on two endpoints of a segment, the tool automatically creates a segment that is perpendicular to the segment through the first endpoint of the segment. To program this tool, first construct a perpendicular to a segment. Then open GeoGebra’s “Create New Tool” window. Select the two endpoints of the segment as the input objects and select the perpendicular segment and its endpoint as the output. Then just give your tool its name and you are finished. Here are the steps to do this:

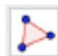


Creating and using a custom tool.

1. Clear anything on the drawing area with the menu "File" | "New" | "Don't Save".
2. Construct segment AB. Points A and B will be the inputs to your new tool.
3. Construct a ray BA so that you can locate a point C on the other side of A from B.
4. Construct a circle with center at A passing through B.
5. Mark the intersection of this circle with the ray as point C.
6. Construct a circle with center at C passing through B.
7. Construct a circle with center at B passing through C.
8. Mark an intersection of the two circles as point D.
9. Construct segment, ray or line AD. It will be the output from your new tool.
10. Go to the GeoGebra menu "Tools" | "Create New Tool". This will open a dialog box to define your new tool.
11. Select the tab "Input Objects" on the dialog box and pull down the list to select objects. Select "Point A" and "Point B". They should appear in the new list.
12. Select the tab "Output Objects" on the dialog box and pull down the list to select objects. Select "Segment [A, D]". It should appear in the new list.
13. Select the tab "Name & Icon" on the dialog box. For "Tool name", enter your name for your new tool, such as "My Perpendicular Tool". The

"Command name" can be the same. For "Tool help" you can say something like "point A and B of the segment". Make sure the checkbox for "Show in Toolbar" is checked.

14. Press the "Finish" button. You should see an Info box saying "New tool created successfully". You should also see a custom tool icon  at the right end of the tool bar.
15. Try out your new tool. First create a segment EF. Then select the custom tool icon  on the tool bar – or from the "Tools" | "Custom Tools" menu. Click on point E and then on point F. A perpendicular segment EG should appear. With the custom tool icon still selected, now click on point G and then point E. A perpendicular segment GH should appear. GH is perpendicular to EG and parallel to EF. (Wasn't that easier than drawing all those circles?) Drag this construction to see that it maintains the dependencies among its points, segments and angles.

Use your custom tool  and the polygon  tool to quickly create several right triangles in your own GeoGebra tab.

Note: This custom tool will only be available in your current construction. To save your custom tool, you must save the construction as a ".ggb" file and then load the construction later when you want to re-use your custom tool. Use the VMT GeoGebra "File" menu to do this.

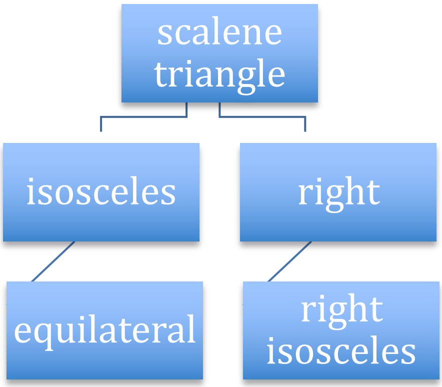
A hierarchy of triangle types

You have constructed different types of triangles with different combinations of dependencies:

- A scalene (generic) triangle consists of three segments whose endpoints meet at three vertices.
- An isosceles triangle is constrained to have two segments of equal length.
- A right triangle is constrained to have one segment perpendicular to another one.
- A right isosceles triangle is constrained to have two segments of equal length and to have one segment perpendicular to another. It is both isosceles and right. Which segments can be equal?
- An equilateral triangle is constrained to have all three segments of equal length. It is a special case of an isosceles triangle, in which the third side is also of equal length to the other two.

Can you think of any other types of triangles? Where would they go on the hierarchy? What about a triangle with all acute angles (smaller than a right angle)? What about a triangle with an obtuse angle (larger than a right angle)? What about a triangle with two or three equal angles?

We can represent this hierarchy of triangles as follows:



Notice & wonder

When you have finished working on this activity by yourself, announce in the chat that you are ready to work together with your group in the team GeoGebra tab.

Here are some things to think about and to discuss with your team in the chat:

Can you make a custom tool that outputs a right-triangle polygon directly from a segment? How general is it? Can it be isosceles?
What new GeoGebra tools would you like to have?
Can the way custom tools are defined in GeoGebra be called "programming"?
Do you think that any of the tools in GeoGebra's menu were programmed this way?

Work together

Discuss how to use your custom tool to build several right triangles:

Challenge: Construct an isosceles right triangle.
Super-Challenge: Create a custom tool to output an isosceles right triangle directly given a base segment.
Can you use this tool to build an equilateral right triangle or a triangle with two right angles?

Post your group conclusions on the wiki page.

Make sure that you can do all the constructions that the group discussed. Try them yourself in your own GeoGebra tab.

Check with the other people in your team to make sure that they all understand how to do these constructions and how these constructions impose the dependencies that are needed.

Don't forget

Write on your group's wiki page your group's responses to these:

- Discuss what you learned in this activity?
- Discuss what you learned about right triangles.
- Discuss what you learned about constructing geometric figures.
- Discuss what you learned about dynamic math.
- Discuss how you think programming new tools can be useful.

Tour 6: GeoGebra Videos & Resources

GeoGebra was created to harness the power of personal computers to help people learn about how exciting geometry can be as an interactive and creative world of exploration and expression. The original developer of GeoGebra discusses his vision and the worldwide response to it in this YouTube video:

http://www.youtube.com/watch?v=w7lgMx8-1c0

Another video shows students engrossed in artistic, evolving and three-dimensional images of mathematical phenomena constructed in the GeoGebra environment:

http://www.youtube.com/watch?v=9IrZAYHpGfk

A third video provides a sampling of advanced GeoGebra constructions, showing the boundless possibilities of the system for representing mathematical objects:

www.youtube.com/watch?v=rZnKMwicW_M

Check out this video for an overview and some tips on the use of the GeoGebra tools that are equivalent to traditional straightedge and compass:

http://www.youtube.com/watch?v=2NqblDIPl38

A thorough explanation of a simple construction with a dependency is given in a YouTube video using GeoGebra tools that are equivalent to straightedge and compass:

http://www.youtube.com/watch?v=AdBNfEOEVco

Here is a video showing how to construct an equilateral triangle with those tools in GeoGebra: http://www.youtube.com/watch?v=ORIaWNQSM_E

Check out these videos of complicated dependencies:

http://www.youtube.com/watch?v=Oyj64QnZIe4&NR=1

http://www.youtube.com/watch?v=-GgOn66knqA&NR=1

There are a large number of YouTube tutorials for GeoGebra. Some of them are collected on the GeoGebra channel:

http://www.youtube.com/geogebrachannel

A good place to begin these videos is:

www.youtube.com/watch?v=2NqblDIPl38

There is a GeoGebra wiki site with resources for students and teachers:

www.geogebra.org

http://www.geogebratube.org

GeoGebra becomes even more powerful in its multi-user version, as part of the VMT (Virtual Math Teams) software environment. Here are some YouTube demos of important aspects of the VMT-with-GeoGebra system:

The multi-user version of GeoGebra—each person sees the actions of the others as they happen:

http://youtube.googleapis.com/v/4oBBynYVrY0

GeoGebra's history slider—you can go back and forth to see how a diagram evolved step-by-step in a GeoGebra or WhiteBoard tab of VMT:

http://youtube.googleapis.com/v/DRlDnadcfRE

The VMT Replayer—you can replay an entire session, including all the tabs. The chat is coordinated with the drawings as you scroll or replay. You can speed up the replaying at multiple speeds. You can stop and step through, action-by-action, forward or backward to analyze the group interaction in detail:

http://youtube.googleapis.com/v/3IzkcVSyYjM

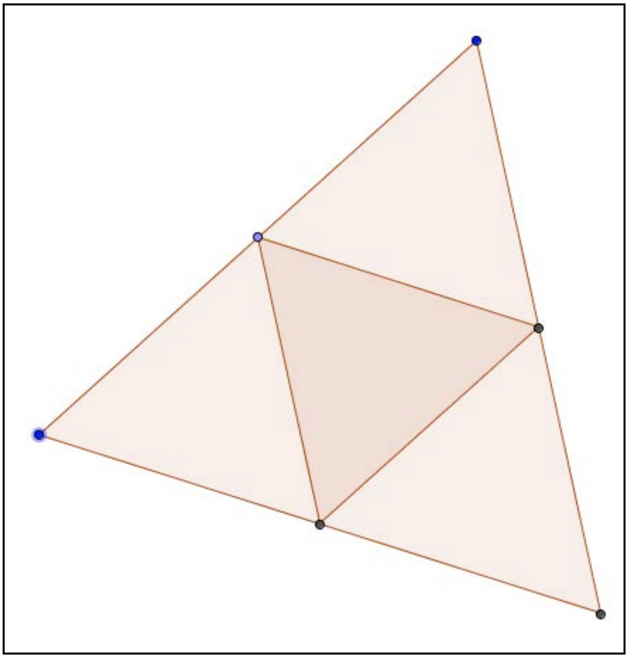
The three videos on VMTwG are integrated in a PowerPoint slide show introduction to VMTwG, available at: http://GerryStahl.net/pub/vmtdemo.pptx

Activity: Inscribing Polygons

This activity provides a challenging construction and an associated proof.

Equilateral triangles

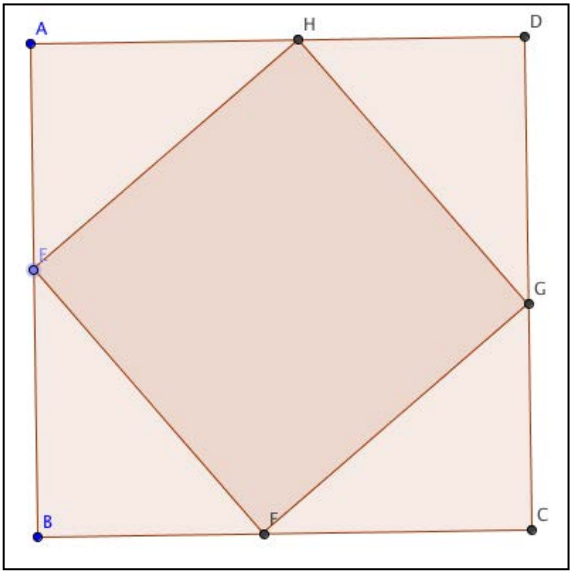
The challenge is to construct an equilateral triangle inscribed in another equilateral triangle. Then drag each of the triangles to explore the constraints on this construction. Here is a screenshot:



Inscribed equilateral triangles.

A square inside a square

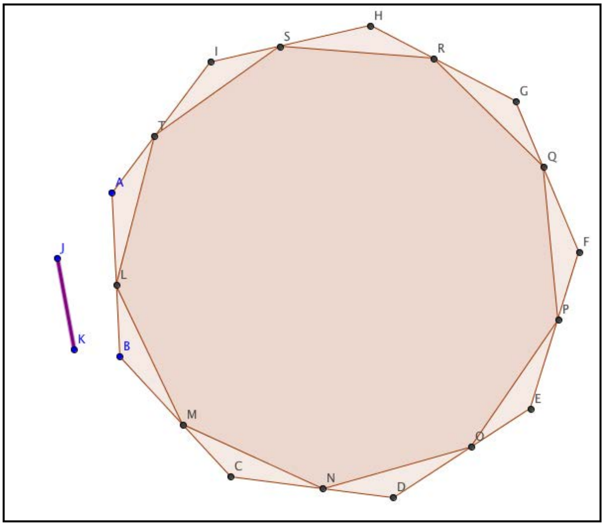
Now try to construct a square inside a square, such that every vertex of the smaller square is on a side of the larger square, as seen in this screenshot:



Inscribed squares.

An n-sided regular polygon inscribed in an n-sided regular polygon

Do you have a conjecture about regular inscribed polygons for any n≥3? Try the construction for a 9-sided regular polygon:



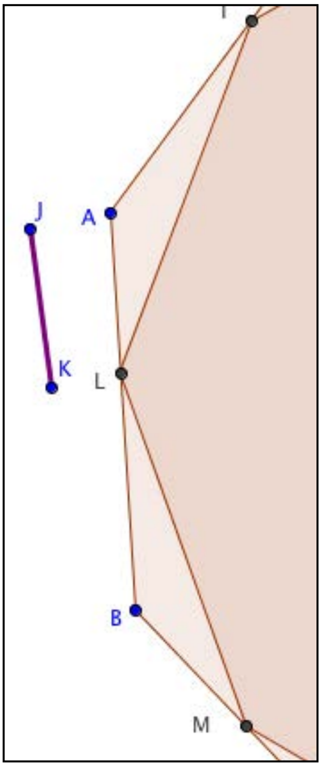
Inscribed regular polygons.

Proof of the construction

If you got this far, you may have a conjecture similar to the following:

> If an n-sided regular polygon (Polygon1) is inscribed inside another, larger n-sided regular polygon (Polygon2) such that every vertex of Polygon1 is located along a side of Polygon2, then the distance from each vertex of Polygon1 to the closest vertex of Polygon2 clockwise is the same distance.

You may find the following detail from the n=9 case to be a useful reference:



Detail on inscribed regular polygons.

Proof:

Given an n-sided regular polygon (Polygon1) inscribed inside another, larger n-sided regular polygon (Polygon2) such that every vertex of Polygon1 is located along a side of Polygon2. *Prove* that the distance from each vertex of Polygon1 to the closest vertex of Polygon2 clockwise must be the same distance.

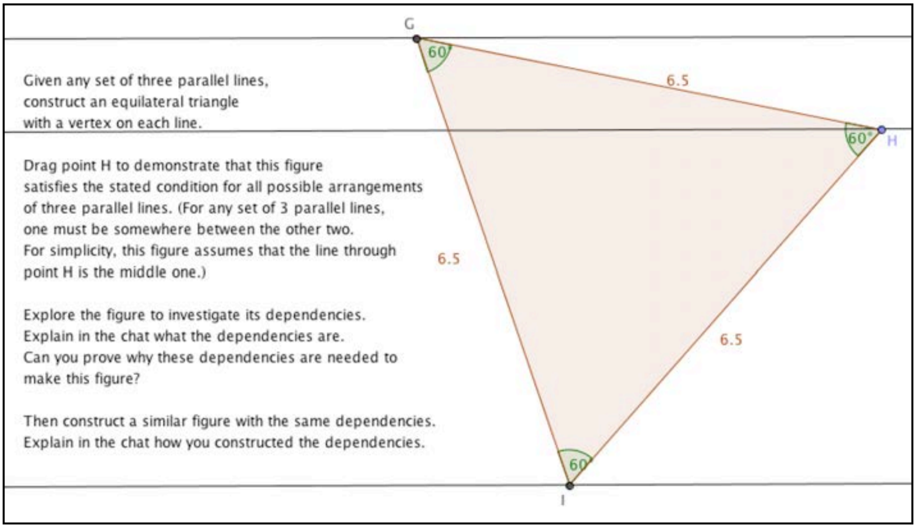
Approach: We will see that the n small triangles (like BLM or ATL) formed between the two polygons are all congruent, so that their corresponding sides (MB and LA) are equal. We will apply the ASA theorem for triangle congruence.

- The sides LM and LT are equal because they are sides of a regular polygon.
- We know that the internal angles at the vertices of both polygons (angles MBL, LAT and MLT) are each y=180*(n-2)/n degrees. E.g., for the 9-sided polygons, these angles are all 140°.
- First consider the three angles in triangle BLM, which add to 180°. Assume that angle BLM is x degrees. Then angle BML is 180-x-y or 180(1-(n-2)/n)-x.
- Now consider the three angles BLM, MLT and ALT, which add up to a 180° straight line AB. We assumed that angle BLM is x degrees and we know that angle MLT is y=180*(n-2)/n degrees. That leaves angle ALT as 180-x-y or 180(1-(n-2)/n)-x, exactly the same as angle BML.
- Finally, consider the three angles in triangle ATL, which add to 180°. We showed that angle LAT is y=180*(n-2)/n degrees and that angle ALT is 180-x-y. Therefore angle ATL is 180-y-(180-x-y)=x, the same as we assumed for angle BLM.
- We have shown that all the corresponding angles of triangles BLM or ATL are congruent and that the corresponding sides ML and LT are congruent. Therefore, the triangles are congruent, so that the corresponding sides AL and MB are necessarily of equal length.

These equal distances are the distances from vertices of Polygon1 to the closest vertex of Polygon2 clockwise. We did this for an arbitrary vertex, so the same holds for all the vertices of Polygon1.

Equilateral triangles on parallel lines

The challenge is to construct an equilateral triangle whose vertices are on three parallel lines, given any three parallel lines. Then drag each of the triangles to explore the constraints on this construction. Here is a screenshot:



An equilateral triangle on three parallel lines.

Activity: The Many Centers of Triangles

Goal of the activity

In this activity you will circumscribe a circle around a triangle. By doing this, you start to explore the relationships between triangles and circles.

Then, you will construct four different "center points" of a triangle and explore their uses and relationships. This builds on the previous activity, exploring some advanced relationships. You will build your own custom tools, which can be used for solving practical problems like finding points on a map that have the shortest total paths to other points.

Try it on your own

Here are some things you should do before you meet with your group for this activity:

Construction process

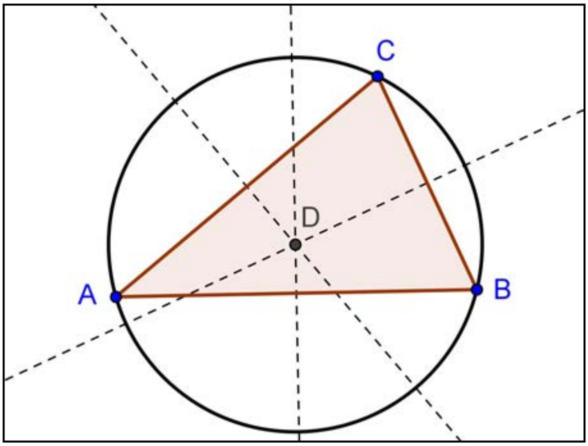
1. Use the polygon tool to create an arbitrary triangle ABC.

2. Construct the perpendicular bisector for each side of the triangle by pulling down the Perpendicular Bisector tool and using it to construct a perpendicular bisector to each side through its midpoint.

3. Use the intersection tool to construct point D at the intersection of two of the perpendicular bisectors. (Notice that the three bisectors all meet at the same point! However, the intersection tool cannot be applied to the intersection of three lines. Either select two of the three line bisectors, or click on the intersection point and if a list of the lines opens up then select one line at a time from the pull-down list.)

4. Construct a circle with center D through one of the vertices of triangle ABC. (Notice that the circle actually goes through all three vertices!) Point D is known as the "**circumcenter**" of the triangle because it is the center of the circle that circumscribes the triangle.

5. Perform the drag test to check if your construction is correct.



The circumcenter of a triangle.

Challenge

Modify your construction to answer the following questions:

1. Can the circumcenter of a triangle lie outside the triangle? If yes, for which types of triangles is this true?

2. Try to find an explanation for using perpendicular bisectors in order to create the circumcenter of a triangle.

3. Compare the area of the triangle to the area of the circle.

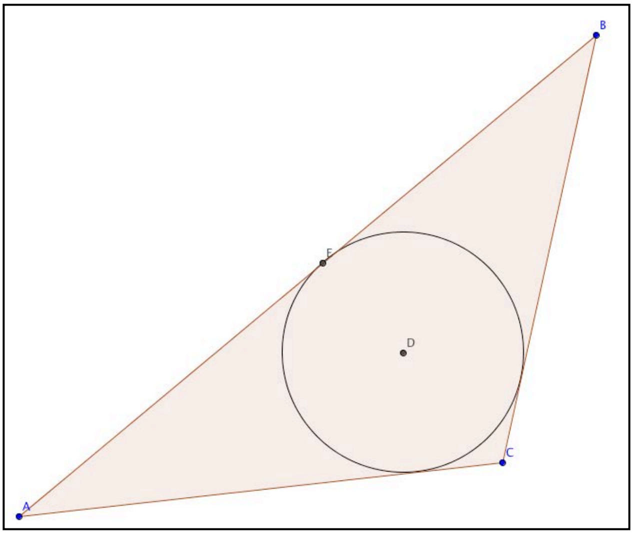
4. Explain why the three bisectors all meet at the same point.

5. Explain why the circle actually goes through all three vertices.

6. Explain why the ratio of the area of the triangle to the area of its circumscribing circle is always the same.

Inscribe a circle

How would you inscribe a circle inside a triangle? Where would you locate the center of the circle? Are the sides of the triangle all tangent to the circle?



A circle inscribed in a triangle.

Define your own custom tools

In this activity, you will be using four different kinds of center points of a triangle repeatedly. It will be convenient to define a custom tool to create each of these kinds of center points. This is easy in GeoGebra. Just clear the drawing area and construct a triangle. Then construct the center point. D, where the three perpendicular bisectors meet. Go to the GeoGebra menu item "Tools" | "Create New Tool…". This will open a "Create New Tool" dialog box. Select the tab in the dialog for "Input Objects". Then use the pull-down list to select the three points forming the triangle's vertices, e.g., "Point A", "Point B", "Point C" (they may have already be listed in the window for Input Objects). Now select the tab in the dialog for "Output Objects". Then use the pull-down list to select the center point, e.g., "Point D: intersection point of d, e".

Next select the tab in the dialog for "Name & Icon". Enter a Tool name for your new tool, such as "my circumcenter". Make sure the check box for "Show in Toolbar" is checked so that your new tool will be displayed in the GeoGebra tool bar as well as be listed in the GeoGebra menu under "Tools" | "Custom Tools". You can use this procedure to make four new custom tools for the four kinds of points of concurrency in a triangle.

Points of concurrency in a triangle

There are four sets of lines that cross in a triangle, forming points with these names:

Centroid: *The point where the medians cross*. To construct in GeoGebra, mark the midpoints of each side and connect it with a segment to the opposite vertex. Use the Intersect-two-Objects tool to mark the intersection of the three medians. Now make a custom tool to create a centroid given the three points that form a triangle.

Circumcenter: *The point where the perpendicular bisectors cross*. To construct in GeoGebra, construct a perpendicular bisector of each side. Mark the intersection of the three bisectors. Now make a custom tool to create a circumcenter given the three points that form a triangle.

Orthocenter: *The point where the altitudes cross*. To construct in GeoGebra, construct a perpendicular of each side from the opposite vertex. Mark the intersection of the three altitudes. Now make a custom tool to create an orthocenter given the three points that form a triangle.

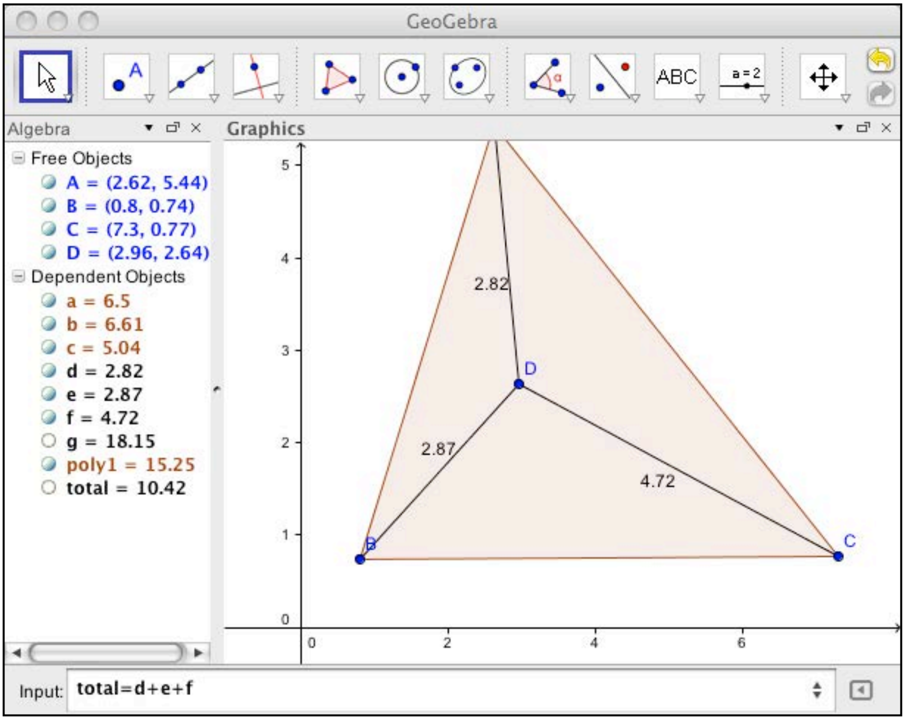
Incenter: *The point where the angle bisectors cross*. To construct in GeoGebra, use the Angle-Bisector tool to construct the bisector of each vertex. Mark the intersection of the three bisectors. Now make a custom tool to create an incenter given the three points that form a triangle.

Work together

Here are some things for you to do while you are online together with your group:

Hierarchy of types of triangles

In an earlier activity, you developed a hierarchy of types of triangles based on how many sides were equal (e.g., an isosceles triangle) or the measure of the largest angle (e.g., a right triangle). You have now explored other relationships, such as rotational symmetry, reflective symmetry and the relationships among the circumcenter, centroid, orthocenter and incenter. Reconstruct your hierarchy of types of triangles using your understanding of these relationships. How many distinct types of triangles can you define? What is the simplest definition in terms of dependencies or constraints? Can you use GeoGebra to construct each type of triangle with the dependencies it needs to retain its type when dragged?



Shortest paths in a triangle.

Shortest paths

People often want to know the shortest paths from one point to another. What point in a triangle has the shortest total paths to the three vertices?

Clear the workspace with “File” | “New”. Select “Perspectives” | “Algebra and Graphics” to display the measures of objects. Construct triangle ABC with a point D inside. Add the lengths of AD, BD and CD. You can do this by typing in the “Input” box at the bottom an equation like: “ total=d+e+f ”. Assuming that d, e and f are the segments connecting point D to the vertices, this equation defines a variable “total” that displays the sum of these three lengths in the algebra pane of the GeoGebra tab. Drag D around to get the smallest possible value for this total path. Do you have a conjecture about point D? Discuss your conjecture in the chat.

What point, E, has the shortest total paths to the three sides? Construct triangle ABC with point E inside. Construct a segment from E to each side, perpendicular to the side. Add the lengths of these three segments. Drag E around to get the smallest possible value for this total path. Drag the triangle to check special types of triangles. Do you have a conjecture about this point? Discuss your conjecture in the chat.

What point, F, has the same distance to the three vertices or the three sides? If you can construct a circle with center at F that is tangent to each side (that is inscribed in the triangle), then F is equidistant from the threes sides. If you can construct a circle with center at F that crosses the three vertices (that circumscribes the triangle), then F is equidistant from the three vertices. Why is this true? How would you construct these circles – where would you locate point F? What kinds of triangles can be inscribed in a circle? Why? What kinds of triangles can inscribe a circle? Why? Discuss this in chat and summarize your group's conclusions in your group's wiki page.

Activity: More Centers of Triangles

Goal of the activity

In this activity you will circumscribe a circle around a triangle. By doing this, you start to explore the relationships between triangles and circles.

Then, you will construct four different "center points" of a triangle and explore their uses and relationships. This builds on the previous activity, exploring some advanced relationships. You will build your own custom tools, which can be used for solving practical problems like finding points on a map that have the shortest total paths to other points.

Try it on your own

Here are some things you should do before you meet with your group for this activity:

Points of concurrency in a triangle

There are four sets of lines that cross in a triangle, forming points with these names:

Centroid: *The point where the medians cross*. To construct in GeoGebra, mark the midpoints of each side and connect it with a segment to the opposite vertex. Use the

Intersect-two-Objects tool to mark the intersection of the three medians. Now make a custom tool to create a centroid given the three points that form a triangle.

Circumcenter: *The point where the perpendicular bisectors cross.* To construct in GeoGebra, construct a perpendicular bisector of each side. Mark the intersection of the three bisectors. Now make a custom tool to create a circumcenter given the three points that form a triangle.

Orthocenter: *The point where the altitudes cross.* To construct in GeoGebra, construct a perpendicular of each side from the opposite vertex. Mark the intersection of the three altitudes. Now make a custom tool to create an orthocenter given the three points that form a triangle.

Incenter: *The point where the angle bisectors cross.* To construct in GeoGebra, use the Angle-Bisector tool to construct the bisector of each vertex. Mark the intersection of the three bisectors. Now make a custom tool to create an incenter given the three points that form a triangle.

Explore the points of concurrency

Clear the workspace with "File" | "New". Construct a generic scalene triangle ABC with the polygon tool. Construct its centroid with your custom centroid tool. Drag the triangle. Discuss how its centroid moves. Where is it for an equilateral triangle, an isosceles triangle, an acute triangle, an obtuse triangle, or a right triangle? Is it ever outside the triangle?

Do the same for the cirumcenter of a triangle. (Create it with your custom circumcenter tool in the same triangle as the centroid. Use the text tool to mark the different center points.)

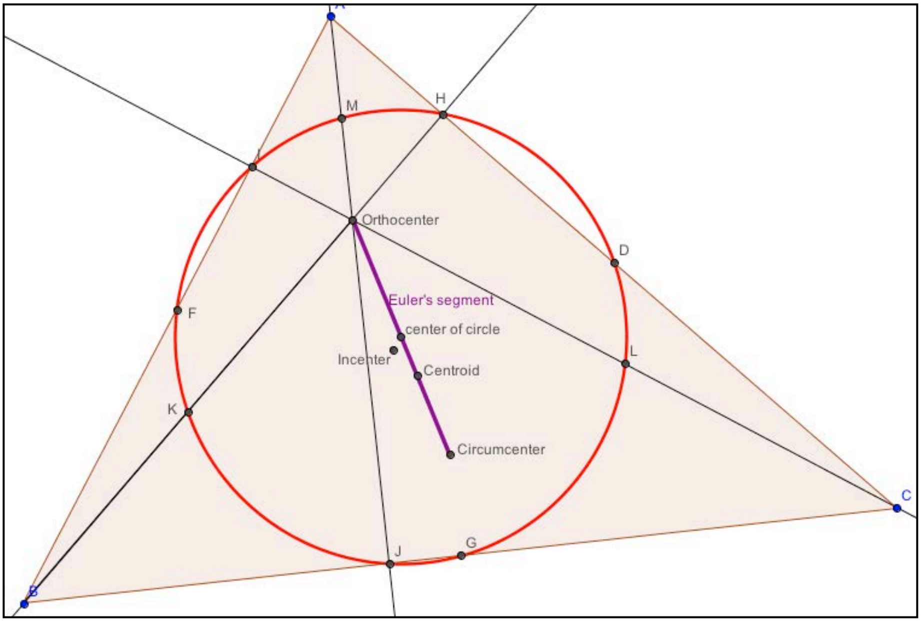
Add the orthocenter of the triangle and explore its behavior.

Add the incenter of the triangle and explore its behavior.

Watch all four of these points as you drag the triangle. What do you notice? Do you have a conjecture?

Euler's segment

The famous mathematician, Euler (pronounced "oiler"), constructed a segment connecting three of these points. It is called Euler's segment. Which points do you think he connected? Connect two points with a segment that also passes through a third point. Why do you think Euler found this segment interesting? Drag the triangle and consider the special cases of different types of triangles: How does Euler's segment behave?



Euler's segment.

Challenge: The nine-point circle

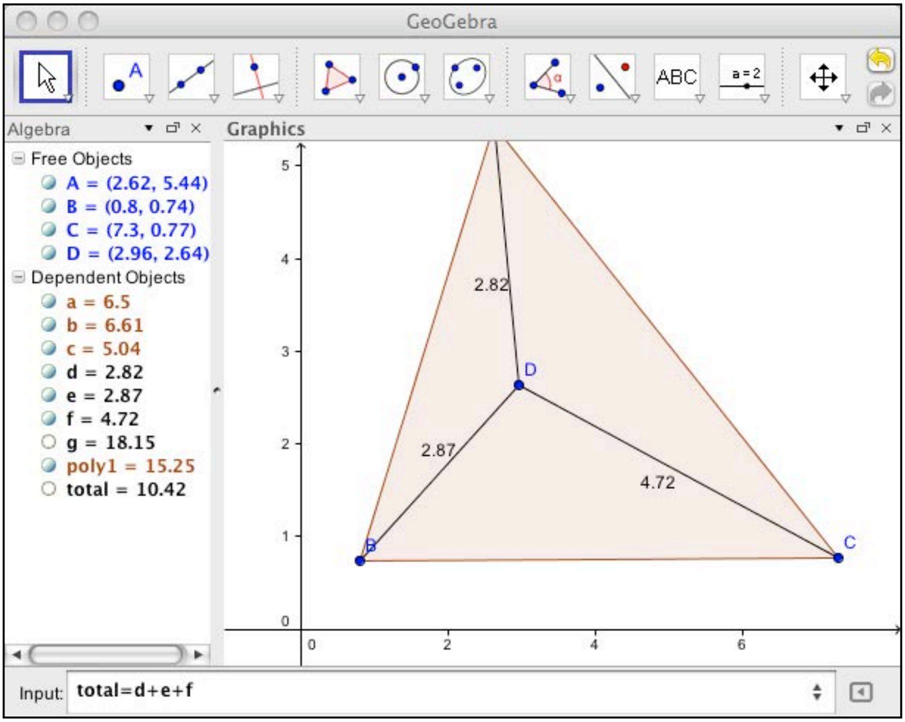
Construct a triangle with its Euler segment. Mark the midpoint of Euler's segment and the midpoints of the triangle's sides. Construct a circle with its center at the midpoint of the Euler segment and passing through the midpoint of a side of the triangle. This circle passes through nine special points on the triangle, mostly related to the construction of the orthocenter. Can you identify all nine points? What happens to the circle and these points as you drag the triangle to change its shape?

Work together

Here are some things for you to do while you are online together with your group:

Hierarchy of types of triangles

In an earlier activity, you developed a hierarchy of types of triangles based on how many sides were equal (e.g., an isosceles triangle) or the measure of the largest angle (e.g., a right triangle). You have now explored other relationships, such as rotational symmetry, reflective symmetry and the relationships among the circumcenter, centroid, orthocenter and incenter. Reconstruct your hierarchy of types of triangles using your understanding of these relationships. How many distinct types of triangles can you define? What is the simplest definition in terms of dependencies or constraints? Can you use GeoGebra to construct each type of triangle with the dependencies it needs to retain its type when dragged?



Shortest paths in a triangle.

Shortest paths

People often want to know the shortest paths from one point to another. What point in a triangle has the shortest total paths to the three vertices?

Clear the workspace with "File" | "New". Select "Perspectives" | "Algebra and Graphics" to display the measures of objects. Construct triangle ABC with a point D inside. Add the lengths of AD, BD and CD. You can do this by typing in the "Input" box at the bottom an equation like: " total=d+e+f ". Assuming that d, e and f are the segments connecting point D to the vertices, this equation defines a variable "total" that displays the sum of these three lengths in the algebra pane of the GeoGebra tab. Drag D around to get the smallest possible value for this total path. Do you have a conjecture about point D? Discuss your conjecture in the chat.

What point, E, has the shortest total paths to the three sides? Construct triangle ABC with point E inside. Construct a segment from E to each side, perpendicular to the side. Add the lengths of these three segments. Drag E around to get the smallest possible value for this total path. Drag the triangle to check special types of triangles. Do you have a conjecture about this point? Discuss your conjecture in the chat.

What point, F, has the same distance to the three vertices or the three sides? If you can construct a circle with center at F that is tangent to each side (that is inscribed in the triangle), then F is equidistant from the threes sides. If you can construct a circle with center at F that crosses the three vertices (that circumscribes the triangle), then F is equidistant from the three vertices. Why is this true? How would you construct these circles – where would you locate point F? What kinds of triangles can be inscribed in a circle? Why? What kinds of triangles can inscribe a circle? Why? Discuss this in chat and summarize your group's conclusions in your group's wiki page.

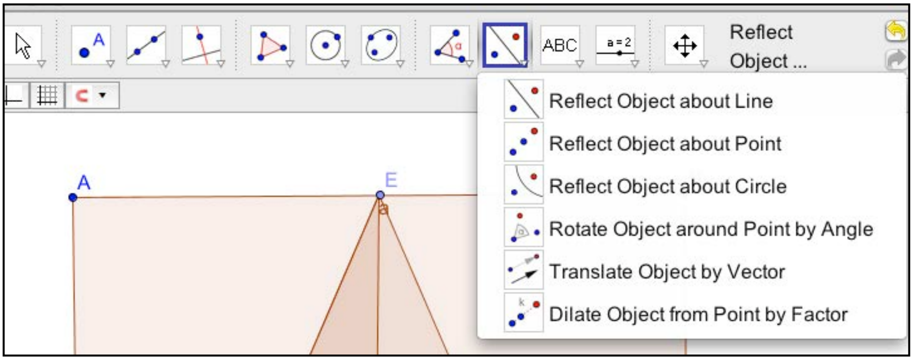
Activity: Transforming Triangles

Goal of the activity

In this activity, you will construct a triangle that always has two sides of equal length (an isosceles triangle). To do this, you will use the "rigid transformations" of translation, rotation and reflection. This will give you a different way of looking at relationships of geometric objects – like different kinds of triangles and reflections in mirrors – which may be new and interesting for you.

Prepare for the activity

The tool bar for this activity should look like the figure:



Tool bar with transformation tools.

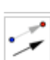
If it does not, then use the GeoGebra menu "Perspectives" | "Geometry". Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked so your points will have their names showing. Note that the tool buttons in this menu have a small pull-down handle in their lower right-hand corner. For instance, the Reflect-Object-about-Line tool allows you to select other tools for reflection, rotation, translation and dilation transformations. All these tool options are also available from the GeoGebra menu system, but these buttons are often handy. You can click on the grid icon in the View menu bar to turn the grid lines on or off in the construction area.

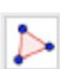
Try it on your own

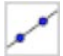
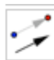
Here are some things you should do before you meet with your group for this activity. Try them in your own GeoGebra tab:

Translation, rotation, reflection

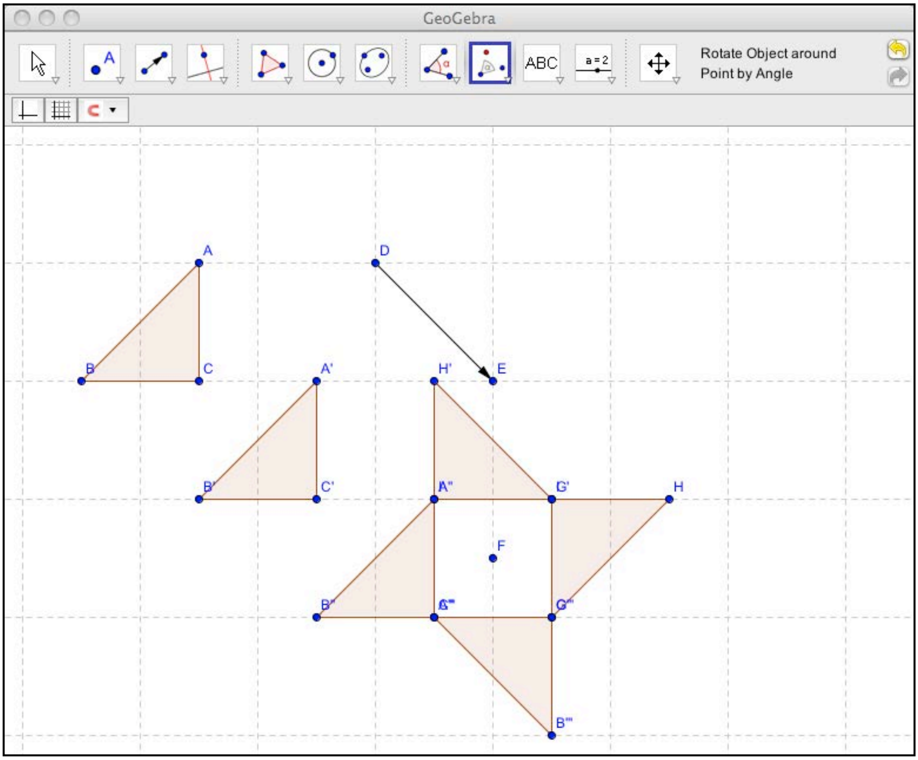
You have seen that dragging a geometric object like a triangle is an important way to explore it in dynamic geometry. Sometimes, dragging a point changes the shape or size of an object, like a segment, circle or triangle. There are three forms of movement or transformation that do not change the shape or size of a geometric object: *translation*, *rotation* and *reflection*. GeoGebra has special tools for translation, rotation and reflection:

Translate object by vector  : *Translation of an object moves that object by a given distance in a given direction.*

1. Use the polygon tool  to make a triangle.

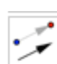
2. Then pull down from the line tool  the vector-between-two-points tool .
3. Decide how you want your triangle to be moved: how far and in what direction. Select the vector-between-two-points tool ; click to make the first point for the vector anywhere; then move your cursor the distance from this point that you want to translate the triangle and in the direction you want it to be translated; click there to create your vector.
4. Now pull down from the Reflect-Object-about-line tool  the Translate-Object-by-Vector tool .
5. Make sure the Translate-Object-by-Vector tool is selected: you will see the help message to the right of the tool bar saying "Translate Object by Vector – Select object to translate, then vector". So first click on your triangle and then on your vector. You should see a translated copy of your triangle appear at a distance and direction from your original triangle corresponding to the length and direction of your vector.
6. With the Translate-Object-by-Vector tool still selected, click on the new translated copy of the triangle and the vector again.

What do you see? Is that what you expected? Notice the labels: if your triangle was labeled ABC, the first translation is A'B'C' and the second translation is A"B"C".

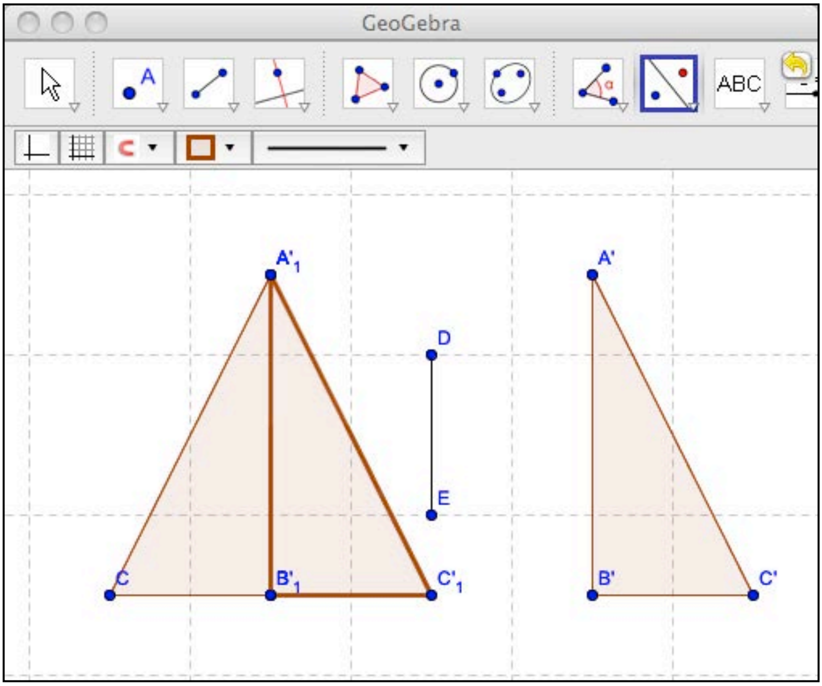


Two translations and three rotations of a triangle.

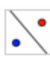
Rotate object around a point by an angle  : *Rotation of an object turns that object around, following a circle whose center is at a given point and around the circle by an amount corresponding to a given angle.*

1. Let's rotate triangle A"B"C" around a nearby point. First create the new point.
2. Then pull down from the Translate-Object-by-Vector tool  the Rotate-Object-around-Point-by-Angle tool  . Select the triangle and then the point. A pop-up will ask you to type an angle and select clockwise or counterclockwise as the direction of rotation. Type in 90 and select counterclockwise. Press OK. You should see a new triangle rotated 90° from the triangle you selected.
3. Select this new triangle and the same point. Do it again: select the newest triangle and the same point.
4. What would happen if you repeated this a fourth time?

Experiment with other triangles and other points, including points inside the triangle. Can you predict what will happen each time?



FigReflections of a right triangle.

Reflect object about a line  : *Reflection of an object flips that object across a given line.*

1. Create a new construction area with "File" | "New".
2. Use the polygon tool to construct a right triangle ABC.
3. Construct a segment DE a small distance from side AB (you may have to pull down the Segment-between-Two-Points tool from the line tool).
4. Pull down the Rotate-Object-about-Line-by-Angle tool  .
5. Click on the triangle and then the segment. You should see a mirror image A'B'C' of ABC.
6. Now reflect ABC about its own side AB. What do you see?

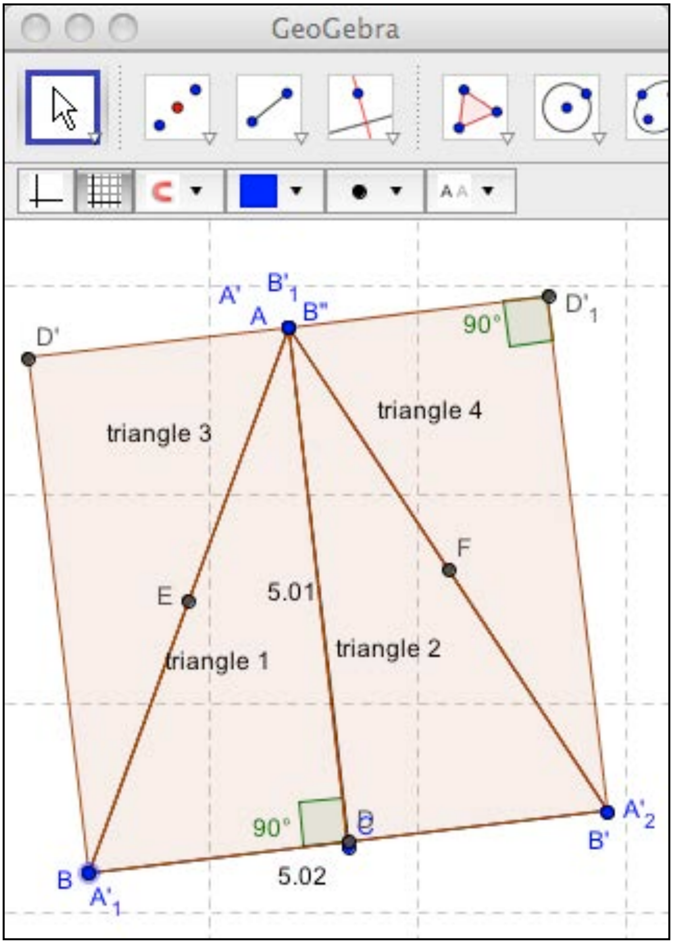
We can use these rigid transformations to construct geometric objects with certain constraints and to explore those objects. Drag triangle ABC; drag segment DE; drag point B; drag point E. What do you notice? Are you surprised by anything?

Challenge: Tessellation

If you know about tessellation, try to create a tessellation pattern starting with a triangle, rectangle, hexagon or other polygon. Use rigid transformations of the original polygon to cover the plane. See how the pattern changes as you drag points of the original polygon. Does it still cover the plane? Can you make "dynamic tessellations" that continue to cover the plane when the original figure is dragged? Are there certain conditions that must be true about the original polygon and/or the transformations?

Explore the isosceles triangle

1. Create a new construction area with "File" | "New".
2. Construct segment AB and segment AC (sharing the first point, A).
3. Reflect segment AB about segment AC.
4. Now construct a segment connecting point B and its reflection, B', to form an isosceles triangle ABB'.
5. Create point D at the intersection of AC and BB'.
6. Use the polygon tool to construct the two symmetric (mirror image) triangles: triangle 1, ABD, and triangle 2, AB'D.
7. Now mark the midpoint of AB as E and the midpoint of AB' as F.
8. Rotate triangle1 around point E by 180° and rotate triangle 2 around point F by 180°.



Exploration of the area of an isosceles triangle.

Check the sizes of the various angles and segments. Do you have a rectangle consisting of four congruent right triangles and an isosceles triangle consisting of two of those triangles? If so, the area of the triangle is half the area of the rectangle. You probably know that the area of a rectangle is its length times its width. Segment AC is perpendicular to the base of the isosceles triangle and goes to its far vertex, so we call AC the altitude of triangle ABB' and we call BB' the base of triangle ABB'. The rectangle's area is this altitude times this base, so the area of triangle ABB' is ½ altitude x base.

You can use the View Bar  below the Tool Bar to display the values of some of the polygons, segments and angles: select the object and use the pull-down arrow at the right end of the bar. You can also use the text tool ABC to label the triangles and other objects. This will help you to explore and discuss the construction.

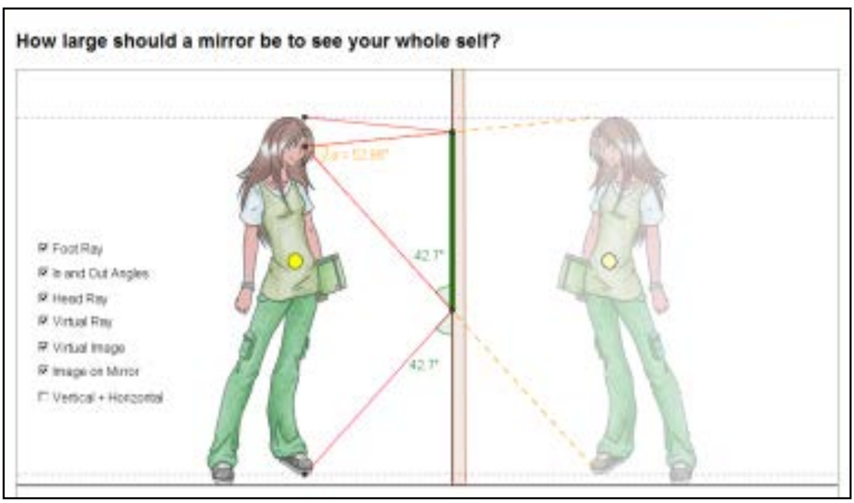
Work together

Here are some things for you to discuss online together with your group:

- Drag points A, B, C and triangle 1. See what dependencies remain in the construction.
- What do you notice about the angles?
- What do you notice about the lengths of the sides?
- What do you notice about the areas?
- Discuss in the whiteboard if these results are what you would expect. Did you notice anything that you were not expecting?
- Discuss in the whiteboard what you learned about isosceles triangles.
- Discuss in the whiteboard what you learned about constructing geometric figures.
- Discuss in the whiteboard what you learned about dynamic math.

Challenge: A reflection problem

Tasja is 5 feet tall. She wants to hang a mirror so that when she stands 5 feet away from it she can see herself from her toes on the floor to the top of her head (6 inches above her eyes. How tall is the shortest mirror that she needs?



Hint: Use the reflection tool. First reflect Tasja about the line of the mirror. Then construct the segments from her eyes to the top of her head and the bottom of her feet. The mirror reflects the light from her head and feet to make it look like they are their "mirror image" behind the mirror. Does this fact let you prove that the "angle of incidence" (between the line from the eye to the mirror and the line of the mirror) is congruent to the "angle of reflection" (between the line from the foot to the mirror and the line of the mirror)?

Tour 7: Creating VMT Chat Rooms

VMT-with-GeoGebra is freely available worldwide for people to create rooms and invite others to collaborate in them. The Lobby includes tools for students, teachers and other people to define their own topics and create VMT chat rooms for exploring and discussing those topics.

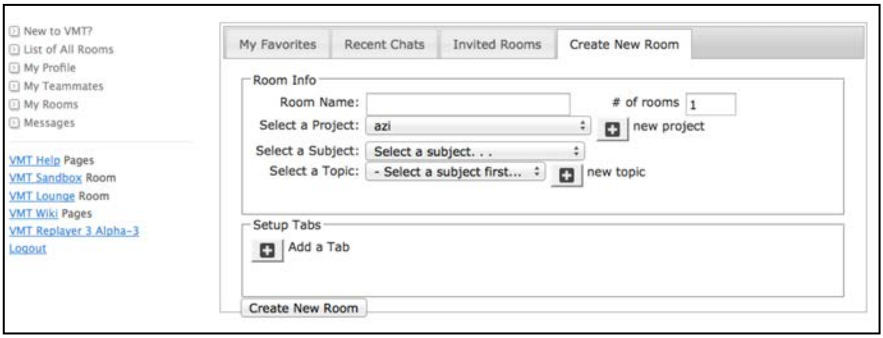
Anyone can create new chat rooms

Anyone who is logged into VMT can create new rooms. There is an expanded interface for teachers. First, we will see how people who are not registered as teachers can create chat rooms.

Enter the VMT Lobby and click on the link "My Rooms" on the left side of the Lobby, as shown in the figure. Select the tab "Create New Room."

The interface to create new chat rooms

The first decision is to define a name for the new room that you are creating. You can create up to ten (10) identical rooms at once. They will be numbered: name_1, name_2, etc. So if you want 3 rooms for 3 teams, you can name the room with a name



The Lobby interface to create a new chat room.

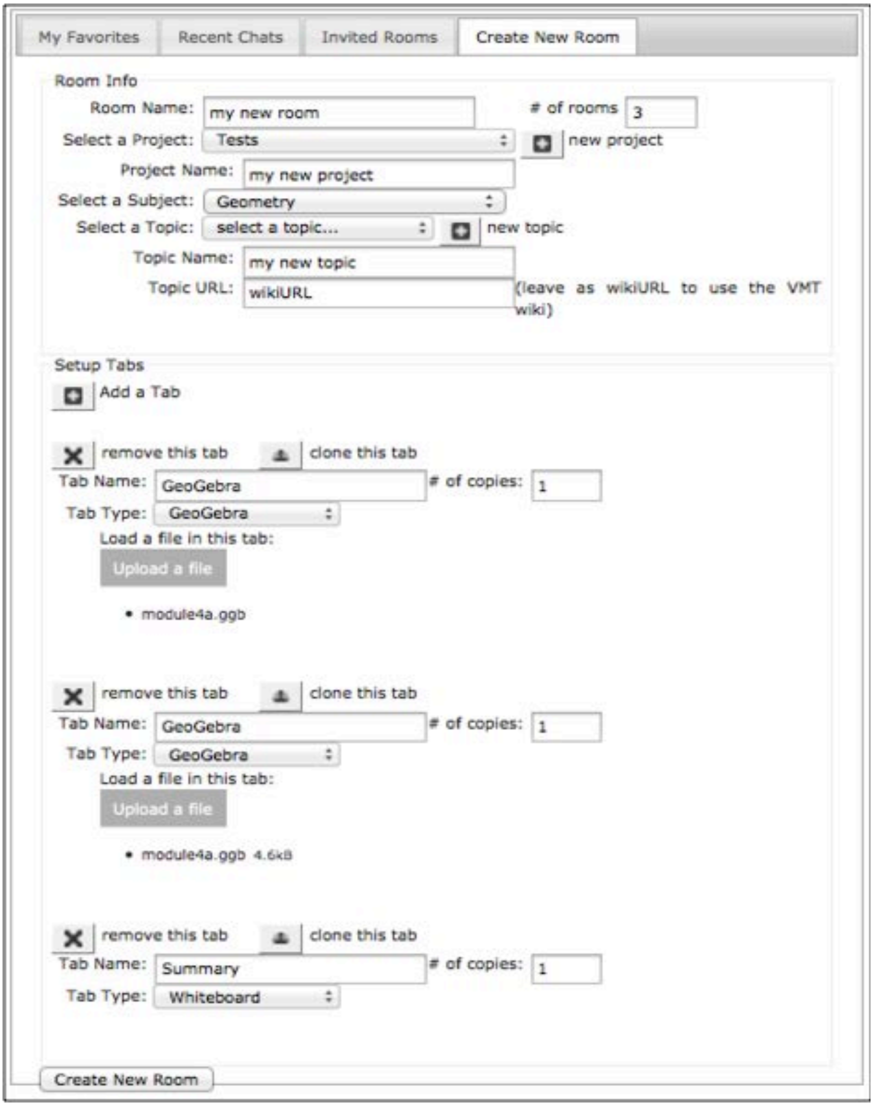
ending in "team" and then the room names will end in "team_1," "team_2" and "team_3."

Rooms are organized by Projects. Within each Project is a hierarchy of Subjects within the Project, Topics within the Subjects, and Rooms within the Topics. So if you are setting up rooms for a course that will have many topics (e.g., one each week for a term), you might want to create a new project for that course, like "Ms Taylor Spring 2012." For our example, we will use the existing Project "Tests" for creating test rooms. It is always good to create a test room, then open it and make sure it works the way you intend. Once rooms are created, they cannot be deleted and their names cannot be reused in the same Subject.

Next select a Subject, like "Geometry." Only people who are registered in VMT as teachers or administrators can create new Subjects. There is already a list of Subjects covering most areas of mathematics.

Now select a Topic. A Topic can have a description associated with it, although this is not necessary. If you create a new Topic, you will have an option to give a URL pointing to a description. This can be an html page on any web server. The description for the topic will appear if someone clicks on the link for that Topic in the Lobby listing of rooms. You can also leave the Topic URL set to its default, "wikiURL." Then the topic description is defined on a wiki page on the VMT wiki associated with the new chat room, and you can go there later to edit that description.

The new chat room has now been defined and it is time to define the tabs that will appear in the room when it is first opened. Press the "+ Add a Tab" icon to add each tab. You can add Whiteboard tabs for textboxes and simple shapes or GeoGebra tabs for constructing dynamic mathematics figures. You can also add WebBrowser tabs for displaying the topic description, wiki pages or websites—however this is a



A form for creating a new chat room.

relatively primitive browser and it is generally better to use browsers like IE, Safari, Firefox or Chrome outside of VMT

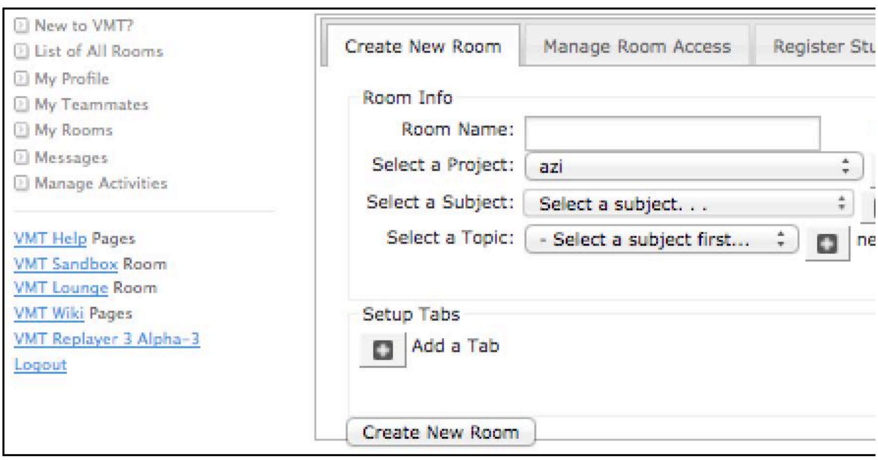
.For a GeoGebra tab, you can upload a .ggb file with a figure already drawn. For instance, you might want four students using the new room to each explore a figure that you have already constructed or that you downloaded from GeoGebraTube. Then you would first construct the figure and save it on your computer desktop or download it from GeoGebraTube to your desktop. Then, when you are creating the GeoGebra tab for the new room, upload the file using the "Upload a file" button. Then you can "clone" that tab so each student will have their own copy to explore. Alternatively, you can specify a number of copies of the tab to have in the room.

When you have defined all the tabs you want, check your entries and press the "Create New Room" button at the bottom. Wait a minute while the rooms are being created. Eventually, you will see a pop-up message that the rooms have been created. Go to the Lobby to find and try your new rooms

The interface for teachers

People who are registered in VMT as teachers or administrators have some extra tools for creating new rooms and registering students. The figure shows the interface for teachers. Click on the link in the Lobby labeled "Manage Activities."

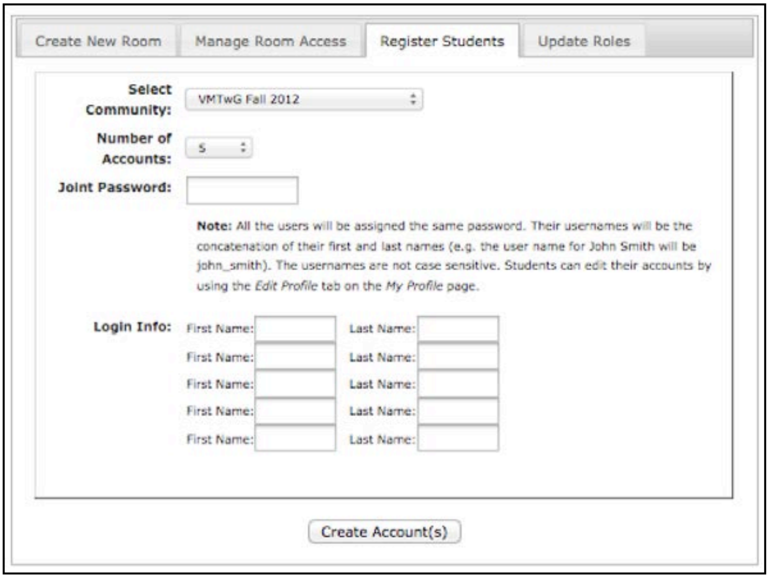
The tab to "Create New Room" is the same for teachers as for everyone else, except



A form for teachers to create new rooms.

that they can define a new Subject as part of the process.

The tab to "Manage Room Access" allows a teacher to ban specific students from entering a particular chat room.

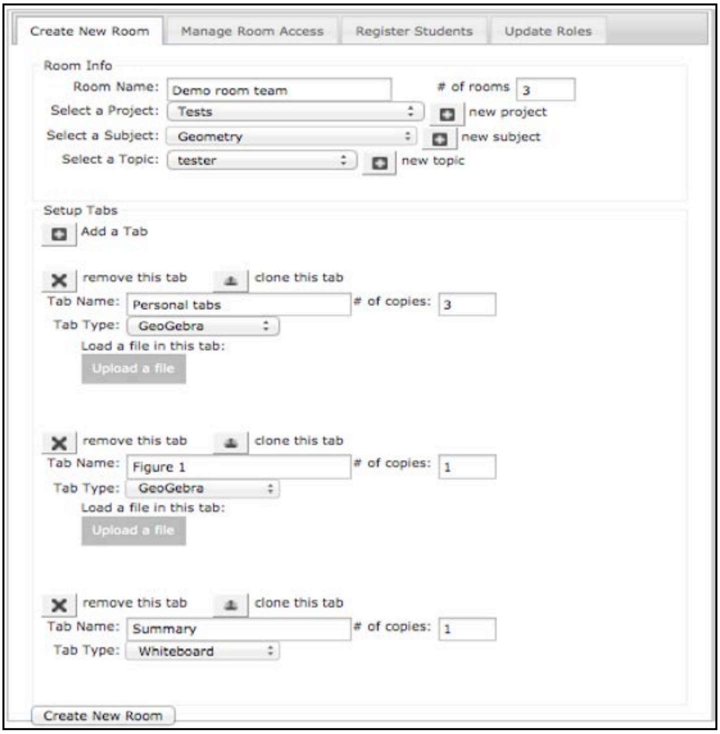


The interface for a teacher to register students.

The tab to "Register Students" is shown in the next figure. It allows a teacher to quickly register up to 5 students at a time by just listing their names. When this registration procedure is used, the teacher's email is associated with each student login and all the students have the same password. As soon as each student logs in to VMT, they should click on the "My Profile" link in the Lobby and change their username, email and password. For security reasons, it is highly recommended that students do not use their regular names as usernames. The teacher might want to keep track of the new usernames, email and password for each student in case the students forget these and in order to track the work of each student in VMT logs.

The tab to "Update Roles" allows someone who is already registered with the role of "teacher" or "administrator" to change the roles of other people. The names of people in a given Project are listed with their assigned role.

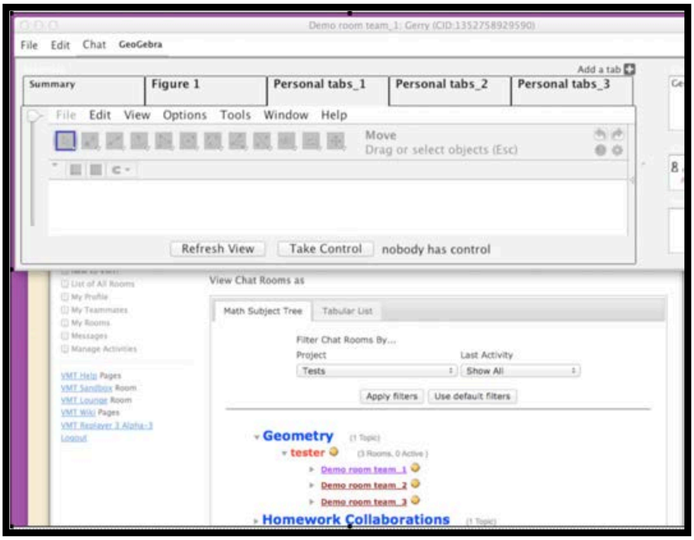
An example of creating test chat rooms



An example of the form for a new room filled in.

The next figure shows the interface filled out to create 3 chat rooms, each having 5 tabs.

The final figure shows one of the 3 chat rooms, with its 5 tabs. Below the room is a view of the Lobby listing the 3 rooms in Project "Tests," Subject "Geometry," Topic "Tester" and rooms "Demo_room_team_1" to "team_3."



A room created in the example and the Lobby listing it.

Activity: Exploring Angles of Triangles

Goal of the activity

In this activity you will explore the measure of angles in a triangle and between parallel lines. You will also explore the symmetries of a triangle. You will see some arguments or proofs for very important relationships among angles based on insights or visualizations that seem obvious once you see them, but that you would never have thought of.

Try it on your own

Here are some things you should do before you meet with your group for this activity:

Sum of angles in a triangle and a straight line

You may have heard that the sum of the three angles in a triangle always add up to 180°. You can use rigid transformations to construct a nice visualization of this theorem:

Step 1. Use the polygon tool to construct a triangle ABC.

Step 2. Use the Vector-between-two-Points tool (pull-down from the lines tool) to make a vector from B to C along the base of your triangle.

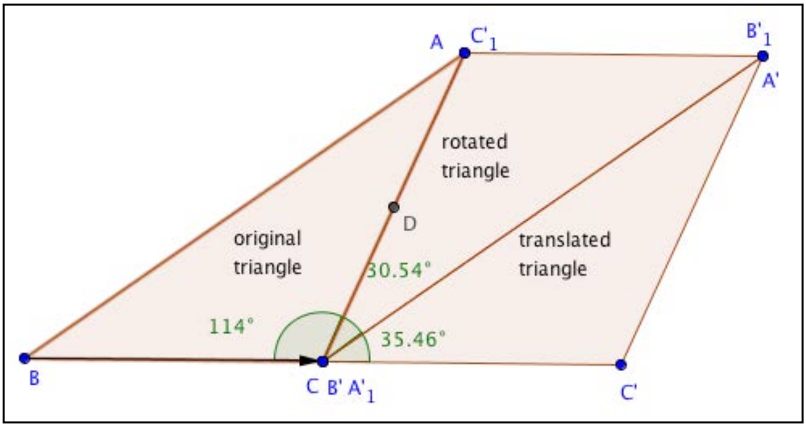
Step 3. Use the Translate-Object-by-Vector tool to translate triangle ABC by vector BC to copy A'B'C'.

Step 4. Use the Midpoint-or-Center tool (pull-down from the points tool) to mark the midpoint of side AC (the side between the original triangle and the translated copy) as point D.

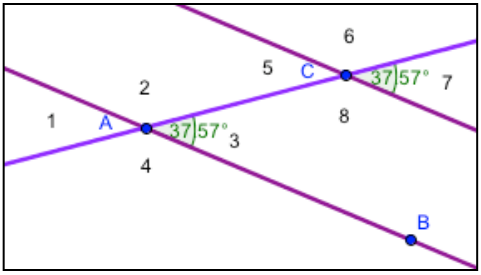
Step 5. Use the Rotate-Object-around-Point-by-Angle tool to rotate triangle ABC around midpoint D clockwise by 180° to form another copy $A'_1B'_1C'_1$.

Step 6. Use the Angle tool to show the values of the angles at point C. Be sure to select the three points forming each angle in clockwise order to display the internal angle measures).

Drag the vertices of triangle ABC to form differently shaped triangles. Do you see where the three angles of ABC have been placed together to form a 180° straight line? Does this combination still add up to 180° when you drag ABC? How do you know that the line ACC' is really a straight line?



Sum of a triangle's angles.



Angles formed by parallel lines.

Parallel lines

You can also use the fact that angles that form a straight line add up to 180° to prove relations among angles formed by a line crossing parallel lines.

Clear the drawing area. Construct a line AB. Construct a line AC that crosses it at A. Use the Parallel-Line tool (pull-down from the line tool) to construct a line through point C parallel to AB. Line AC forms eight angles with the parallel lines; they are numbered from 1 to 8 in the figure.

Angles 1 and 2 form a straight line, so they add up to 180°. Angles 2 and 3 also add up to 180°. If angle 2 is x degrees, then angle 1 is (180-x) degrees. So is angle 3. So angles 1 and 3 are always equal: opposite angles are equal.

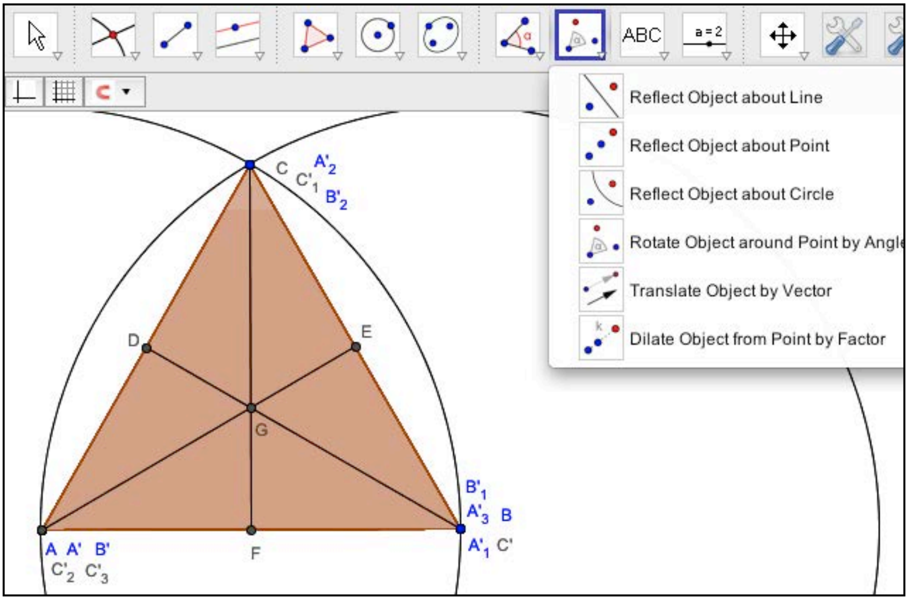
Because they are formed by parallel lines, angles 2 and 5 add up to two right angles, or 180° (by the definition of parallel lines). Angles 2 and 3 also add up to 180°. If angle 2 is x degrees, then angle 5 is (180-x) degrees. So is angle 3. So angles 5 and 3 are always equal: corresponding angles between parallel lines are equal.

Angle 1 = 3 = 5 = 7 and angle 2 = 4 = 6 = 8. Use the Angle tool to show the values of angles (be sure to select the three points forming each angle in clockwise order to display the internal angle measures). Drag point C to change the angles. Are the opposite and corresponding angles still equal?

Symmetry of an equilateral triangle

Clear the drawing area. Construct an equilateral triangle ABC using the segment tool and the circle tool. Mark the midpoints D, E, F of the three sides with the midpoint tool. Construct segments from the vertices to the opposite midpoints. Use the Intersect-Two-Objects tool (pull-down from the Points tool) to mark the meeting point, G, of these segments.

Use the Reflect-Object-about-Line tool to reflect the polygon around each of the segments connecting a vertex with the opposite midpoint. You will not see much change because copies of the polygon will be placed exactly on top of each other. You may notice some additional labels for the vertices. You have demonstrated that an equilateral triangle has at least three lines of **reflective symmetry**. That is, the triangle is symmetric around the three lines of reflection you have constructed.



Symmetries of an equilateral triangle.

Use the Rotate-Object-around-Point-by-Angle tool to rotate the polygon around point G (the "centroid" of the triangle) by 120° (one third of the 360° of a full rotation). You will not see much change because copies of the polygon will be placed exactly on top of each other. You may notice some additional labels for the vertices. You have demonstrated that an equilateral triangle has at least three-fold **rotational symmetry** around its centroid.

Reflective symmetry

Reflect a figure over a line. If it coincides with the original figure, then we say it has "reflection symmetry" along that "line of symmetry".

What lines of reflection symmetry does a square have?

You already demonstrated three lines of reflection symmetry for an equilateral triangle; does it have any others?

Explore the lines of reflection symmetry of other regular (equilateral and equiangular) polygons.

Rotational symmetry

Rotate a figure a certain number of degrees. If it is congruent with the original figure, then we say it has "rotational symmetry" at that angle.

What angles does a square have rotational symmetry?

You already demonstrated three angles of rotational symmetry for an equilateral triangle; does it have any others?

Explore the rotational symmetry of other regular (equilateral and equiangular) polygons.

Work together

Here are some things for you to do while you are online together with your group:

Symmetry of triangles

List the lines and angles of symmetry of different types of triangles in this chart:

Type of triangle	Lines of symmetry	Angles of symmetry
equilateral		
isoceles		
right		
acute		
obtuse		
scalene (generic)		
other		

Can you draw a hierarchy of kinds of triangles based on their lines of symmetry and/or their angles of symmetry? How does this hierarchy compare to one based on other dependencies?

List the chart of lines and angles of symmetry on your group's wiki page.

List the hierarchy of triangles on your group's wiki page.

Activity: Exploring Similar Triangles

Goal of the activity

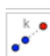
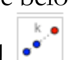
In this activity, you will explore *similar* and *congruent* triangles. You will do this using transformation with *dilation*, as well as translation, rotation and reflection. You will also explore *symmetry* of geometric objects. Understanding similarity, congruence, dilation, translation, rotation and reflection can be essential for solving many practical and intriguing problems involving geometric objects, as you will see in later activities.

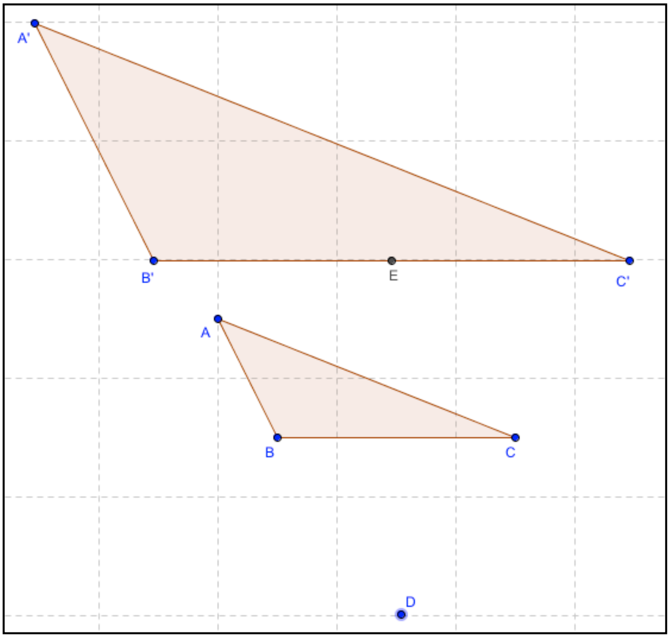
Try it on your own

Here are some things you should do before you meet with your group for this activity:

Dilation and scale

There is another form of transformation of geometric objects that was not discussed in the last activity. It is dilation. GeoGebra has a tool for dilation of an object:

Dilate object from a point by a factor  : *Dilation of an object stretches (or shrinks) that object by a given factor away from a given point.* For instance, use the polygon tool to construct a triangle ABC. Place a point D at a distance below the triangle of about the length of one triangle side. Now use the dilation tool , selecting the triangle and then the point. Enter a factor of 2. A new triangle A'B'C' will appear above the original triangle and twice as big as it. Construct the midpoint, E, of the base of the new triangle.



Similar, proportional triangles.

Drag point D inside of the original triangle and move it toward each vertex. You should see that the dilated triangle is exactly twice as big as the original one and all of its vertex angles are exactly the same size (congruent with) the corresponding angles of the original triangle. You can also see that the original base side is exactly half as long as the dilated base side (up to its midpoint).

We say that the two triangles are "similar" (in the strict geometric sense of the term) because all the corresponding angles are the same. But the two triangles are not "congruent" because the corresponding sides are not the same length. Note that when an object is translated, rotated or reflected in geometry or in GeoGebra, the new copy is congruent (as well as similar) to the original. When an object is dilated, the new copy is similar, but not congruent (unless the dilation factor was exactly 1). When a triangle is dilated, the two similar triangles have equal corresponding angles and the corresponding sides are proportional to each other in the ratio of the dilation factor.

Similar triangles

The study of congruent and similar triangles is very important in geometry. If you know (or can prove) that two triangles are similar, then you know that their corresponding angles are equal and their corresponding sides are proportional. If they are congruent, then you also know their corresponding sides are the same lengths (proportional by 1:1). This can be key to constructing complex geometric objects, solving problems and developing proofs.

Three equal angles (angle-angle-angle or AAA). We saw that when you dilate a triangle by a factor, all the corresponding angles remain congruent even though the lengths of the sides change in proportion to the factor. So two triangles with angle-angle-angle congruent are similar, but not necessarily congruent. In fact, the three angles of a triangle always add up to 180°, as you will soon see. So if two sets of angles are congruent, then all three sets of angles are congruent (how do you know that?), so two triangles with angle-angle congruent are similar, but not necessarily congruent.

Work together

Drag and explore your constructions. Are you convinced about all the cases? Can you explain them to the other people in your group? Do you have any questions about the constructions and what they demonstrate?

Are there any other cases that should be explored? What constructions might be useful for exploring them? What other constructions would you suggest for the six cases above?

Create three text boxes in the Summary tab – one for congruent triangles, one for similar triangles and one for other cases. In each text box, list all the cases that everyone in the group believes belong in that category. Try to think of all the cases of equal corresponding sides and/or angles. Can everyone construct pairs of triangles with dependencies corresponding to each case you list?

Activity: Exploring Congruent Triangles

Goal of the activity

In this activity, you will explore congruent triangles.

Try it on your own

Here are some things you should do before you meet with your group for this activity:

Congruent triangles

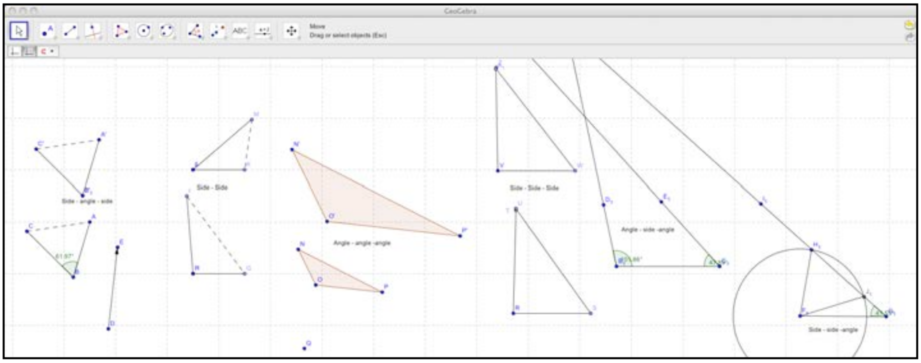
The study of congruent and similar triangles is very important in geometry. If you know (or can prove) that two triangles are similar, then you know that their corresponding angles are equal and their corresponding sides are proportional. If they are congruent, then you also know their corresponding sides are the same lengths (proportional by 1:1). This can be key to constructing complex geometric objects, solving problems and developing proofs.

One way to compare triangles is to see what combination of corresponding sides and/or angles are congruent (of equal measure). Here are some of the possible cases. You can figure out what other cases there are and explore them:

Case 1: Two equal sides and the angle between them (side-angle-side or SAS). Construct an angle ABC and the segments AB and BC on both sides. Construct a vector and translate ABC by the vector. You can see that there is only one possible third side AC. So two triangles with corresponding side-angle-side equal are congruent triangles.

To explore this case further, use the polygon tool to construct the triangle ABC. Now switch to the Move tool and select the polygon. Copy (Command-C on a Mac; Control-C in Windows) and paste (Command-V on a Mac; Control-V in Windows) a new copy of the triangle. Drag the whole triangle (be careful not to drag any of the vertices; that would change the shape of the triangle) on top of A'B'C'. You should see that all of the angles and sides match exactly. If you place B_2 on top of B', then C_2 should be on top of C' because B_2C_2 is constructed to be the same length as B'C' (as specified by SAS) and A2 should be on top of A' because B_2A_2 is constructed to be the same length as B'A' (as specified by SAS). Therefore, the distance from A_2 to B_2 should be the same as the distance from A' to B', making the third sides of the triangle equal and thereby making the triangles similar. This is basically how Euclid proved the SAS case in his fourth proposition, but without the help of GeoGebra.

Case 2: Two equal sides (side-side or SS). What if two sides of the triangles are equal length, but the angle between them is not constrained to be congruent? For instance, construct two “Segments with given length from point” by pulling down this special line tool and setting their lengths to 3 and 4. Start them both from a common endpoint. There are many possible lengths for the third side joining these two sides to form a triangle. So having two sets of equal corresponding sides does not constrain two triangles to be congruent or even similar. By the way, try a third side of length 5; what kind of triangle does this make? What is the range of lengths for the third size that would work? Create a “Segment between Two Points” and display the value of its length as it connects the first two sides while the angle between them changes.



Exploring six cases of triangles with congruent parts.

Work together

Drag and explore your constructions. Are you convinced about all the cases? Can you explain them to the other people in your group? Do you have any questions about the constructions and what they demonstrate?

Are there any other cases that should be explored? What constructions might be useful for exploring them? What other constructions would you suggest for the six cases above?

Create three text boxes in the Summary tab – one for congruent triangles, one for similar triangles and one for other cases. In each text box, list all the cases that everyone in the group believes belong in that category. Try to think of all the cases of equal corresponding sides and/or angles. Can everyone construct pairs of triangles with dependencies corresponding to each case you list?

Activity: More Congruent Triangles

Here are two more interesting cases of congruent triangles.

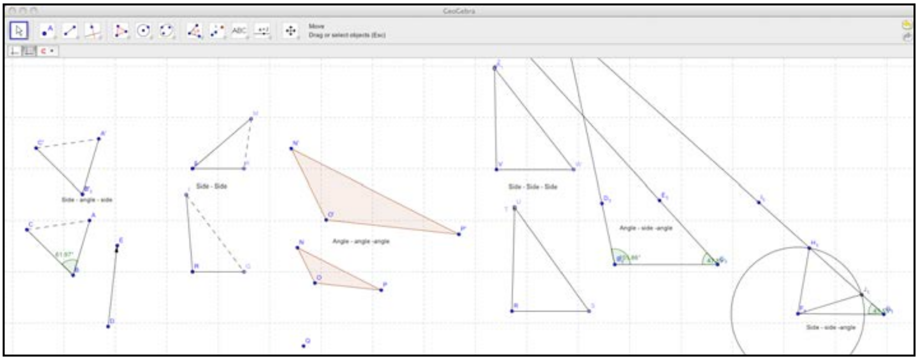
Goal of the activity

In this activity, you will explore *similar* and *congruent* triangles. You will do this using transformation with *dilation*, as well as translation, rotation and reflection. You will also explore *symmetry* of geometric objects. Understanding similarity, congruence, dilation, translation, rotation and reflection can be essential for solving many practical and intriguing problems involving geometric objects, as you will see in later activities.

Try it on your own

Here are some things you should do before you meet with your group for this activity:

More congruent triangles



Exploring six cases of triangles with congruent parts.

Case 5: Two angles and the side between them (angle-side-angle or ASA). Construct a base side and an angle at each endpoint. Construct a ray at each angle. You will see that the rays meet at a unique point, creating the third angle uniquely. So not only are the three angles necessarily congruent, but the lengths of the sides are fixed (for any base segment and two angles at its endpoints), so triangles with ASA are similar and congruent.

Case 6: Two sides and an angle that is not between them (side-side-angle or SSA). This is a subtle case. Construct a segment of fixed length, with an angle of fixed size at one endpoint and a second segment of fixed length at the other endpoint. Construct a ray at the given angle. Now construct a circle whose radius is the length of the second segment. The circle defines all the points that the second segment can reach to form the third vertex. Depending on your construction, you may see that the ray meets the circle at two points. This means that there are two possible triangles with the given SSA, but with different third side lengths and different angles. So SSA does not guarantee congruence or even similarity.

Work together

Drag and explore your constructions. Are you convinced about all the cases? Can you explain them to the other people in your group? Do you have any questions about the constructions and what they demonstrate?

Are there any other cases that should be explored? What constructions might be useful for exploring them? What other constructions would you suggest for the six cases above?

Create three text boxes in the Summary tab – one for congruent triangles, one for similar triangles and one for other cases. In each text box, list all the cases that everyone in the group believes belong in that category. Try to think of all the cases of equal corresponding sides and/or angles. Can everyone construct pairs of triangles with dependencies corresponding to each case you list?

Activity: Exploring Different Quadrilaterals

Goal of the activity

In this activity, you will explore dynamic constructions of quadrilaterals by dragging points on figures consisting of four segments with different dependencies. In previous activities, you have discovered a lot about the geometry of triangles; now you will use your geometry skills to explore polygons with four or more sides.

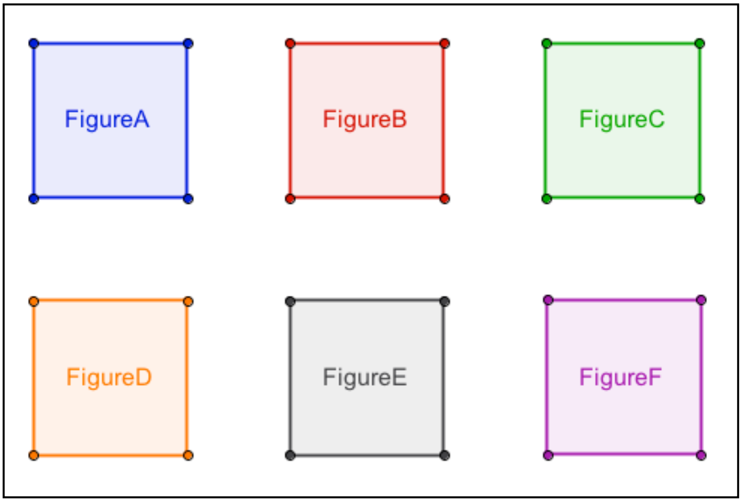
Then, you will construct a quadrilateral that can change its size, position and orientation, but always maintains certain dependencies. Follow the construction steps and then determine what type of quadrilateral it is. In the next activity, you will invent your own construction steps.

Try it on your own

Here are some things you should do before you meet with your group for this activity:

Open the dynamic worksheet

Enter a VMT room for this activity. The GeoGebra tab should contain six quadrilaterals (four-sided figures) constructed in different ways. If it does not, then load the file http://vmt.mathforum.org/activities/six_quadrilaterals.ggb.



Six quadrilaterals.

Exploration

Explore the six polygons by trying to drag their vertices. Use the move tool. Some vertices will not move because they were constructed with special dependencies.

Can you determine what type of quadrilateral each figure is – for instance, square, rectangle, rhombus, or parallelogram?

Can you describe a hierarchy of types of quadrilaterals? Where do the six figures fit in your hierarchy?

Are there more special types of quadrilaterals that are not illustrated in the set of six figures? What are their dependencies? Where would they go in the hierarchy?

Construct a quadrilateral with dependencies

Step 1. Clear the drawing area and construct segment AB.

Step 2. Construct a perpendicular to segment AB through point A.

Step 3. Construct a perpendicular to segment AB through point B.

Step 4. Construct a circle with center at point A going through point B.

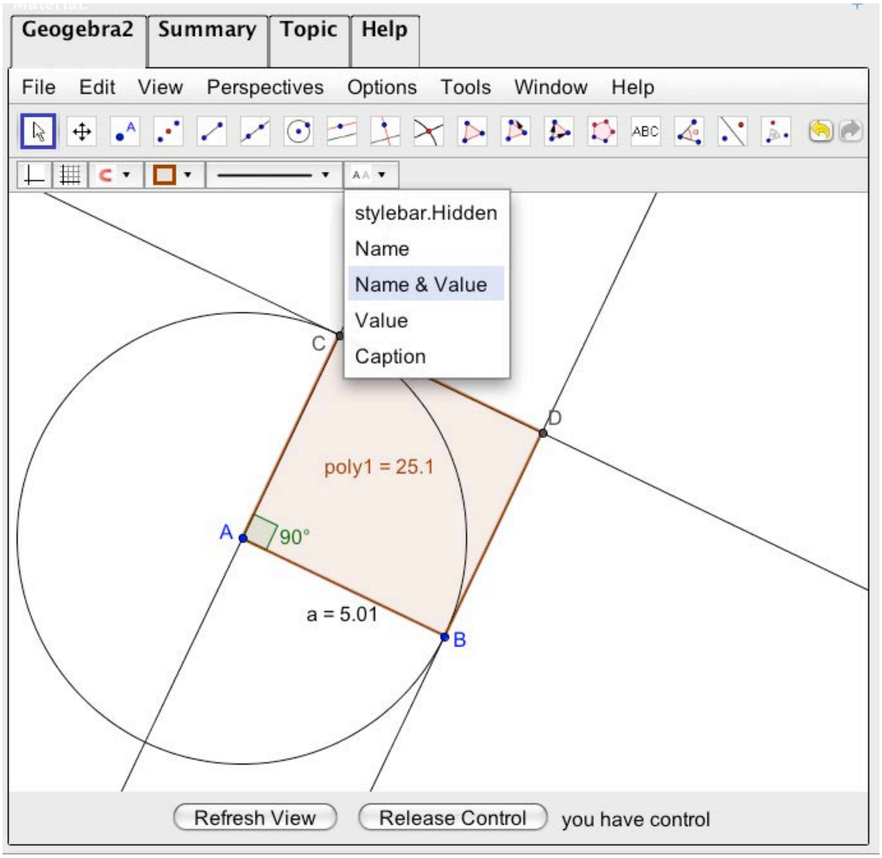
Step 5. Construct a point C at the intersection of the perpendicular through point A and the circle.

Step 6. Construct a parallel line to segment AB going through point C.

Step 7. Construct point D at the intersection of the parallel line going through point C and the perpendicular line going through point B.

Step 8. Construct a polygon through points ABDCA.

Step 9. Construct angle BAC between segment CA and segment AB.



Construction of a quadrilateral with dependencies.

Step 10. Display the values of the area of polygon ABDC, the length of segment AB and the degrees of angle CAB. You can do this by selecting the object and using the pull-down utility menu to show "Name & Value" (see the image of the pull-down tab in the figure).

Step 11. Use the move tool to select the construction lines and circle that you want to hide. (To select multiple objects, hold down the Command-key on a Mac or the Control-key in Windows.) Right-click (in Windows or Control-click on a Mac) or double-click on an object to get its Properties dialog. Un-check the "Show-object" option.

Work together

Here are some things for you to do while you are online together with your group:

Discussion

What is the difference between a traditional *drawing* and a dynamic-geometry *construction*?

What is the "drag test" and why is it important?

Why is it important to *construct* figures instead of just *drawing* them in interactive geometry software?

What do we have to know about a geometric figure before we are able to construct it using dynamic-mathematics software?

Discuss in the chat what constraints were imposed by the construction process.

Apply the drag test to polygon ABDC.

What kind of quadrilateral is polygon ABDC?

How could you change the constraints or eliminate some to create a different type of quadrilateral?

What do you notice about the relationship of the area of the polygon to the length of segment AB? *Relationships* like this are very important in geometry.

Construct your own quadrilaterals

Next, you will construct quadrilaterals that can change their size, position and orientation, but will always display certain dependencies. See if you can figure out how to construct dependencies of the six different kinds of quadrilaterals in the previous activity – and maybe even some other kinds.

Discuss how you will take turns and what kinds of quadrilaterals you each want to construct.

Take turns taking control. Use the chat to decide who goes next and to discuss what each person is constructing.

Take turns constructing different types of quadrilaterals.

Explain in the chat how you constructed the dependencies needed for a particular type of quadrilateral.

Be sure to do a drag test to make sure the figure has the dependencies that you intended it to have.

Challenge

How many different way of constructing a square, a rectangle, a rhombus, a parallelogram, can you come up with – just using the tools in the tool bar?

Can you construct all six of the figures in the previous activity?

Can you construct all of the figures in your quadrilateral hierarchy?

Can you invent a new type of quadrilateral and construct it?

Activity: Types of Quadrilaterals

In this activity you can explore three-sided and four-sided figures. Discuss how to define different types of triangles and quadrilaterals; try to construct them; state their constraints or dependencies; and define them with a minimum number of conditions.

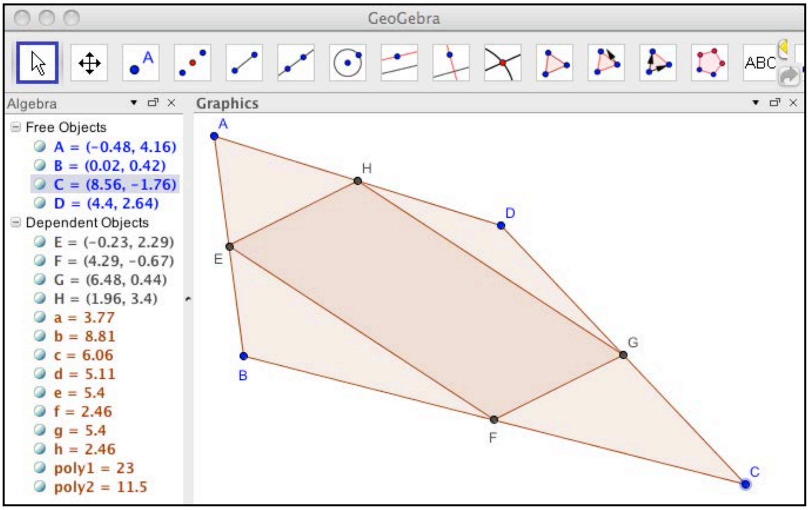
Explore different kinds of center points of triangles and quadrilaterals. Make conjectures about dividing these figures with lines through these center points.

In this activity and the following activities,

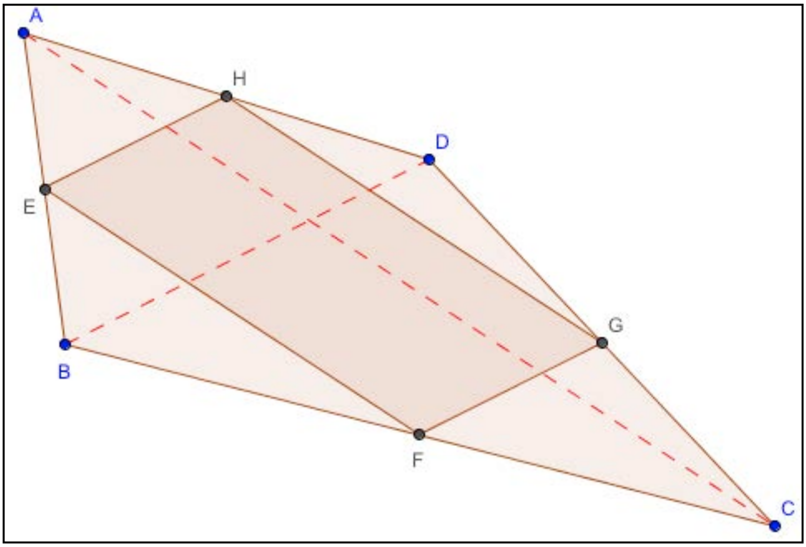
- First try out the different constructions and think about the questions on your own.
- Then get together with your group and work collaboratively on the parts that you had trouble with.
- Discuss the questions in text chat.
- Summarize your group findings in your group's wiki page for the activity.

Connecting midpoints of quadrilaterals

Given a quadrilateral ABCD, connect the midpoints of the sides. What is the ratio of the area of the internal quadrilateral to the area of the external quadrilateral?



Connecting midpoints of a quadrilateral.



Finding areas of quadrilaterals.

Hint: To prove why this ratio holds, connect the opposite vertices and then consider triangle BEF and the larger triangle BAC. Can you prove that they are similar triangles and that their proportionality quotient or dilation factor is 2? What does this imply about the ratio of their areas? Consider the area of the original quadrilateral ABCD and then subtract the areas of the outside triangles like BEF. What remains?

Quadrilateral angle bisectors

We know that the angle bisectors of any triangle all meet at one point. What can we say about the angle bisectors of a quadrilateral? Do they all meet at one point? If not always, then under what conditions do they? Can you inscribe a circle in a quadrilateral? If not always, then under what conditions can you? What is the ratio of the area of the circle to the area of the quadrilateral?

Hierarchy of quadrilaterals

You made a hierarchy of different types of triangles (equilateral, isosceles, right, etc.) based on different relationships and dependencies. Can you do the same for quadrilaterals (square, rectangle, rhombus, parallelogram, etc.)? What do you think is the best way to define these types?

Symmetry of regular polygons

An equilateral triangle is a regular polygon with 3 equal sides and 3 equal (60°) angles. A square is a regular polygon with 4 equal sides and 4 equal (90°) angles. Can you list the lines and angles of symmetry for different polygons? Can you predict how many lines and angles of symmetry each of the many-sided regular polygons have?

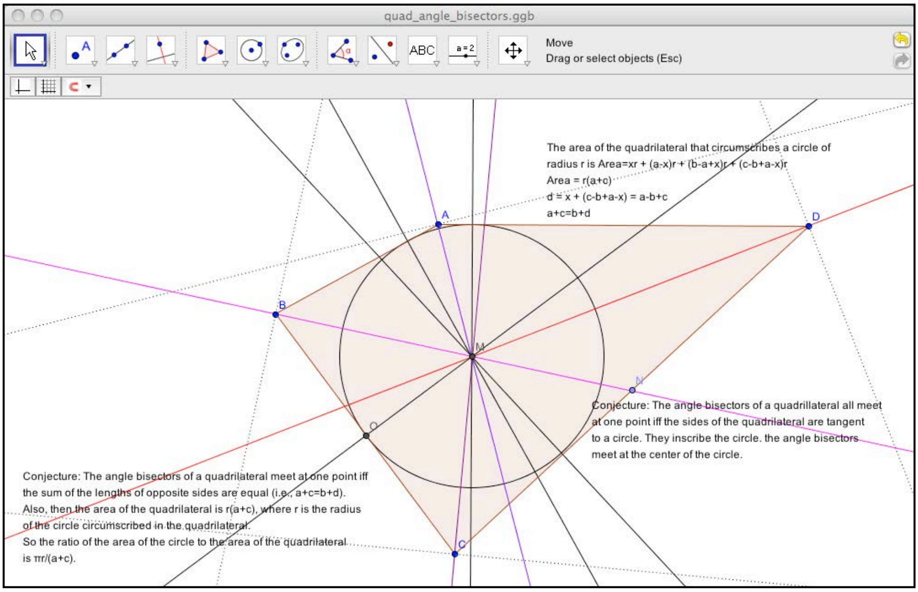
List the lines and angles of symmetry in this chart:

regular polygon	**# sides**	**# lines of symmetry**	**# angles of symmetry**
triangle	3		
quadrilateral	4		
pentagon	5		
hexagon	6		
heptagon	7		
octogon	8		
nonagon	9		
	N		

Shortest paths

You may have already explored the points in a scalene triangle with the shortest total paths to the three vertices or to the three sides. Can you find the optimal point inside a quadrilateral for the shortest total paths to the sides or to the vertices? What about for other irregular polygons?

Challenge: Exploring a quadrilateral with an inscribed circle



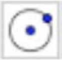
Explorations of a quadrilateral.

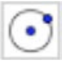
In the figure, people have started to explore a quadrilateral with an inscribed circle. They have formulated some conjectures. One of their conjectures is that pairs of opposite angles add up to 180° for a quadrilateral that circumscribes a circle. Can you help them? Discuss your own conjectures about quadrilaterals with inscribed circles.

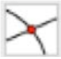
Construct a regular hexagon

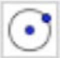
In this activity you are going to use circles to construct a hexagon with equal sides and equal angles – without using the regular-polygon tool. In previous activities, you have explored polygons with three or four sides; now you will start to explore polygons with more sides and see how they compare.

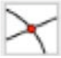
Construction process

1. Draw a circle  with center A through point B

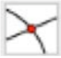
2. Construct another circle  with center B through point A

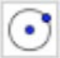
3. Intersect  the two circles in order to get the vertices C and D.

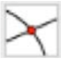
4. Construct a new circle  with center C through point A.

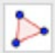
5. Intersect  the new circle with the first one in order to get vertex E.

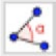
6. Construct a new circle  with center D through point A.

7. Intersect  the new circle with the first one in order to get vertex F.

8. Construct a new circle  with center E through point A.

9. Intersect  the new circle with the first one in order to get vertex G.

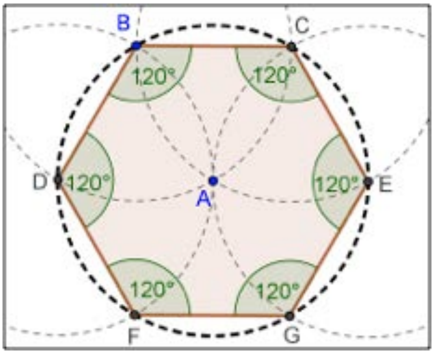
10. Draw hexagon  FGECBD.

11. Create the angles  of the hexagon.

12. Perform the drag  test to check if your construction is correct.

Challenge:

Try to find an explanation for this construction process. *Hint*: What radius do the circles have and why?



A regular hexagon.

Activity: Challenge Geometry Problems

This activity suggests some fun activities involving dependencies, relations and proofs. There are many more. You now have the tools to explore these creatively and to create your own questions. You can investigate your own conjectures and compose your own proofs.

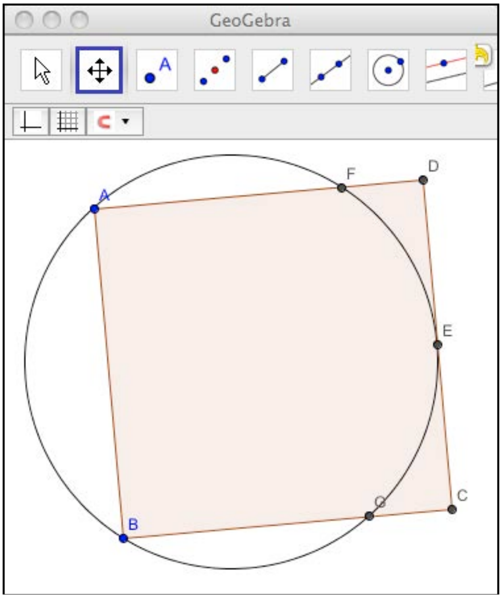
Preparation

Use the Perspectives menu to set the perspective to "Geometry and Algebra".

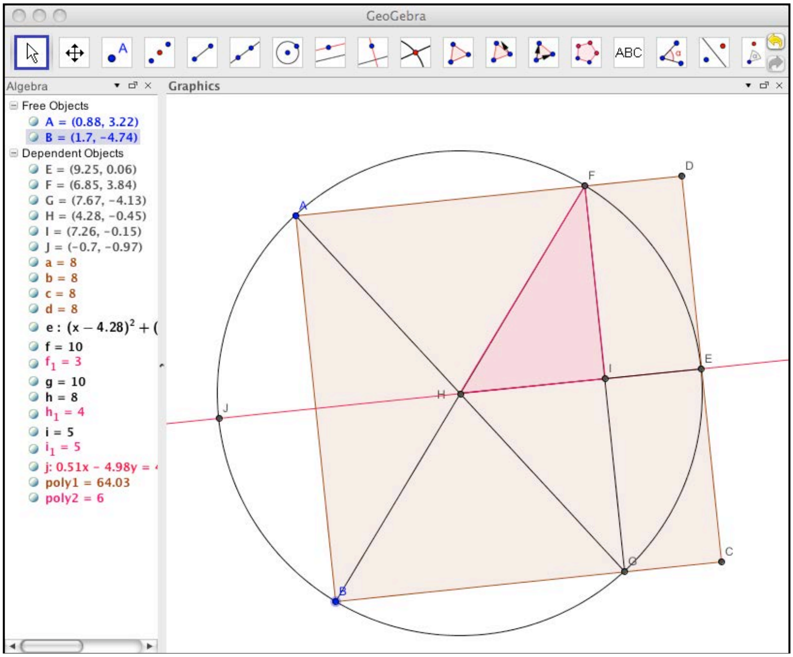
Decide in chat which activity your group should explore next. Discuss how to approach it. Take turns and chat about what you are constructing, what you notice and what you wonder about. Can you state a proof of your findings, using the construction?

Square and circle

The square and circle are tangent at one point and meet at four other points, as shown. If the side of the square is 8 units long, what's the radius of the circle?



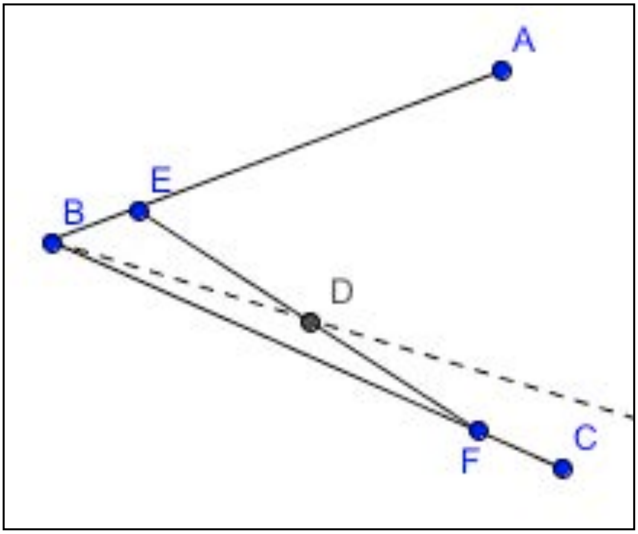
Square and circle problem.



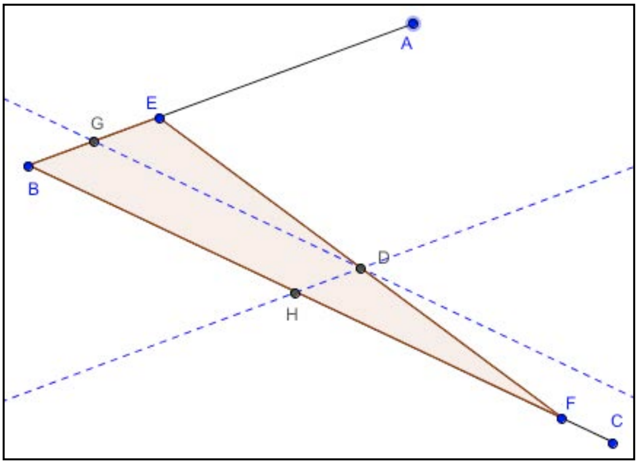
Hint: Add extra lines to show symmetries.

Midpoint of a segment crossing an angle

Given an acute angle ABC and an arbitrary point D inside the angle, how can you construct a segment EF connecting the sides of the angle such that point D is the midpoint of EF, as the figure?



Construct a midpoint spanning an angle.

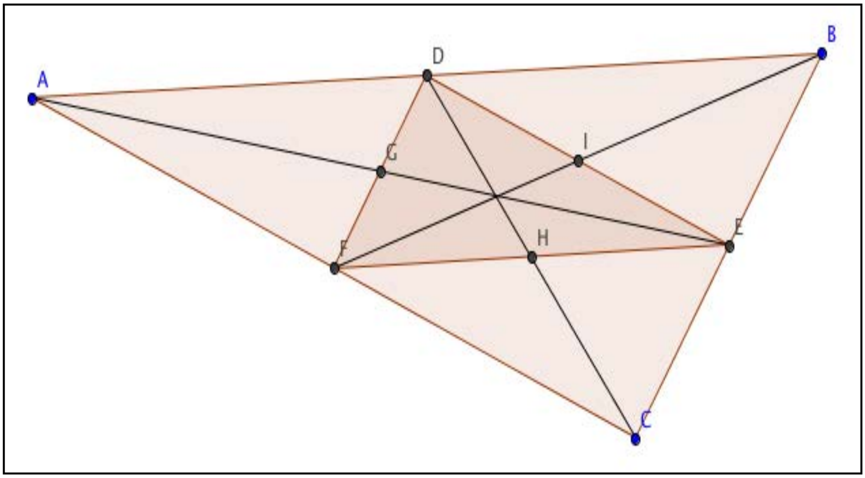


Hint: construct parallel lines.

The treasure near the three trees

According to Thales de Lélis Martins Pereira, legend tells of three brothers in Brazil who received the following will from their father: "To my oldest son, I leave a pot with gold coins; to my middle child, a pot of silver coins; and to my youngest son, a pot with bronze coins. The three pots were buried on my farm according to the following scheme: halfway between the pot with gold coins and the pot with bronze coins, I planted a first tree, halfway between the pot with bronze coins and the pot with silver coins, I planted a second tree, and halfway between the pot with silver coins and the pot with gold coins, I planted a third and final tree." Where should the brothers excavate to find each pot of coins?

A picture of buried treasure.



Hint: draw the two triangles and their centroid.

Further activities

You should now be able to explore the other tools and options in GeoGebra with your groups in VMT. Look in the VMT Lobby to find chat rooms that have interesting topics to explore using GeoGebra. There are also many resources related to GeoGebra available at http://www.geogebra.org and http://www.geogebratube.org.

Activity: Transform Polygons

In this activity you will explore many relationships involving rigid transformations of polygons. You will explore physical models and GeoGebra simulations of different kinds of transformations. You will also compose multiple simple transformations to create more complex transformations. You will apply what you learned to the selection of moving machines in a factory.

In this activity and the following activities,

- First try out the different constructions and think about the questions on your own.
- Then get together with your group and work collaboratively on the parts that you had trouble with.
- Discuss the questions in text chat.
- Summarize your group findings in your group's wiki page for the activity.

Designing a factory

Suppose you are the mathematician on a team of people designing a new factory. In the factory, special machines will be used to move heavy objects from location to location and to align them properly. There are different machines available for moving the objects. One machine can flip an object over; one can slide an object in a straight line, one can rotate an object. As the mathematician on the team, you are supposed to figure out the most efficient way to move the objects from location to location and to align them properly. You are also supposed to figure out the least expensive set of machines to do the moving.

The factory will be built on one floor and the objects that have to be moved are shaped like flat polygons, which can be laid on their top or bottom. So you can model the movement of objects as rigid transformations of polygons on a two-dimensional surface. See what you can learn about such transformations.

Experiment with physical transformations

Take a piece of cardboard and cut out an irregular polygon. Place the polygon on a piece of graph paper and trace its outline. Mark that as the "start state" of the polygon. Move the cardboard polygon around. Flip it over a number of times. What do you notice? Rotate it around its center or around another point. Slide it along the graph paper. Finally, trace its outline again and mark that as the "end state" of the transformation.

Place the polygon at its start state position. What is the simplest way to move it into its finish state position? What do you notice about different ways of doing this?

The other people in your group cannot see your cardboard polygon moving. Explain to them in the chat or in textboxes on the VMT whiteboard what you did and what you noticed. Share what you are wondering about transformations of polygons and discuss these questions.

Now cut an equilateral triangle out of the cardboard and do the same thing. Is it easier to transform the equilateral triangle from its start state to its finish state than it was for the irregular polygon? What do you notice about flipping the triangle? What do you notice about rotating the triangle? What do you notice about sliding the triangle?

Transformational geometry

In a previous activity with triangles, you saw that there were several kinds of rigid transformations of triangles that preserved the measures of the sides and the angles of the triangles. You also learned about GeoGebra tools that could transform objects in those ways, such as:

- Reflect Object about Line
- Rotate Object around Point by Angle
- Translate Object by Vector

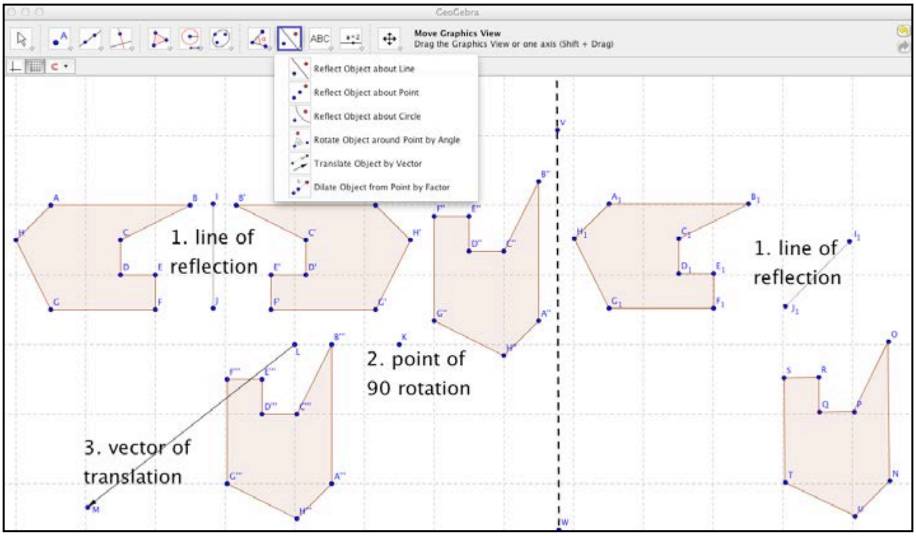
These tools can transform any polygon in these ways and preserve the measures of their sides and angles.

Composing multiple transformations

In addition to these three kinds of simple transformations, you can "compose" two or more of these to create a more complicated movement. For instance, a "glide reflection" consists of reflecting an object about a line and then translating the reflected object by a vector. Composing three transformations means taking an object in its start state, transforming it by the first transformation into a second state, then transforming it with the second transformation from its second state into a third state, and finally transforming it with the third transformation from its third state into its end state. You can conceive of this as a single complex transformation from the object's start state to its end state.

The study of these transformations is called "transformational geometry". There are some important theorems in transformational geometry. Maybe you can discover some of them and even find some of your own.

An example of transformations in GeoGebra



Transformations of a polygon.

In the figure, an irregular polygon ABCDEFGH has gone through 3 transformations: a reflection, a rotation and a translation. A copy of the polygon has gone through just 1 transformation (a reflection) and ended in the same relative position and orientation. There are many sequences of different transformations to transform a polygon from a particular starting state (position and orientation) to an end state (position and orientation). Some possible alternative sequences are simpler than others.

Explore transformations in GeoGebra

Discuss with your group how you want to proceed with each of the following explorations. Do each one together with your group, sharing GeoGebra constructions. Save a construction view for each exploration to include in your summary. Discuss what you are doing, what you notice, what you wonder, how you are constructing and transforming polygons, and what conjectures you are considering.

Exploration 1

What is the minimum number of simple transformation actions needed to get from any start state of the irregular polygon in the figure to any end state? For instance, can you accomplish any transformation with three simple actions: one reflection, one rotation and one translation (as in the left side of the figure)? Is it always possible to achieve the transformation with fewer than three simple actions (as in the right side of the figure)?

Exploration 2

Is it always possible to transform a given polygon from a given start state to a specified end state with just one *kind* of simple transformation – e.g., just reflections, just rotations or just translations? How about with a certain composition of two simple kinds, such as a rotation composed with a translation or a reflection composed with a rotation?

Exploration 3

In a case where you can use just one kind of simple transformation, then what is the minimum number of actions of that kind of simple transform needed to get from a start state to a possible end state?

Exploration 4

Connect the corresponding vertices of the start state and the end state of a transformed polygon. Find the midpoints of the connecting segments. Do the midpoints line up in a straight line? Under what conditions (what kinds of simple transformations) do the midpoints line up in a straight line? Can you prove why the midpoints line up for some of these conditions?

Exploration 5

If you are given the start state and the end state of a transformed polygon, can you calculate a transformation (or a set of transforms) that will achieve this transformation? This is called "reverse engineering" the transformation. *Hint*: constructing the perpendicular bisectors of the connecting segments between corresponding vertices may help in some conditions (with some kinds of simple transformations).

Exploration 6

How would the findings or conjectures from Explorations 1 to 5 be different for an equilateral triangle than they were for an irregular polygon? How about for a square? How about for another regular polygon?

Exploration 7

So far you have only explored rigid transformations – which keep the corresponding angles and sides congruent from the start state to the end state. What if you now add dilation transformations, which keep corresponding angles congruent but change corresponding sides proportionately? Use the Dilate-Object-from-Point-by-Factor tool and compose it with other transformations. How does this affect your findings or conjectures from Explorations 1 to 5? Does it affect your factory design?

Exploration 8

Consider the factory design now. Suppose the factory needs machines for three different complicated transformations and the machines have the following costs: a reflector machine $20,000; a rotator machine $10,000; a translator machine $5,000. How many of each machine would you recommend buying for the factory? What if they all cost the same?

Summarize

Summarize your trials with the cardboard polygons and your work on each of the explorations in your group's wiki page for this activity. What did you notice that was interesting or surprising? State your conjectures or findings. If you did not reach a conclusion, what do you think you would have to do to reach one? Do you think you could develop a formal proof for any of your conjectures?

Post a report of the factory mathematician to the wiki. Include at least one GeoGebra construction for each exploration and describe what it helped you to visualize. Can you make some recommendations for the design of the factory?

Activity: Invent a Transformation

In this activity, you will explore an invented transformational geometry that has probably never been analyzed before (except by other people who did this activity).

An invented taxicab geometry

There is an intriguing form of geometry called "taxicab geometry" (Krause, 1986) because all lines, objects and movements are confined to a grid. It is like a grid of streets in a city where all the streets either run north and south or they run east and west. For a taxicab to go from one point to another in the city, the shortest route involves movements along the grid.

In taxicab geometry, all points are at grid intersections, all segments are confined to the grid lines and their lengths are confined to integer multiples of the grid spacing. The only angles that exist are multiples of 90° — like 0°, 90°, 180°, 270° and 360°. Polygons consist of segments connected at right angles to each other.

Transformational geometry in taxicab geometry

How would you define the rigid transformations of a polygon in taxicab geometry? Discuss this with your group and decide on definitions of rotation, translation and reflection for this geometry.

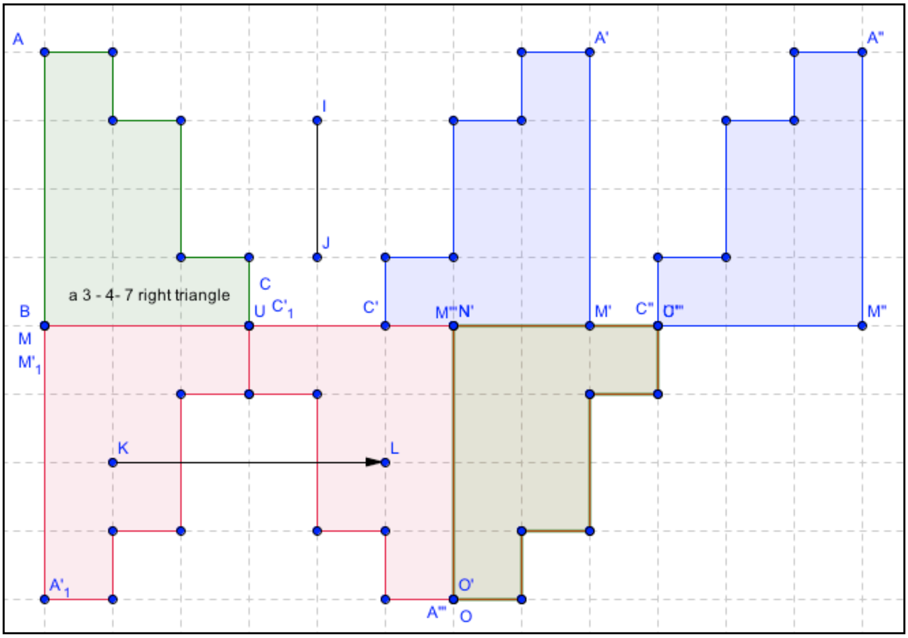
Modeling taxicab geometry

Use GeoGebra with the grid showing (Use the View menu to display the grid; a special tool bar provides a pull-down menu letting you activate "Snap to Grid" or "Fixed to Grid"). Only place points on the grid intersections. Construct several taxicab polygons. Can you use GeoGebra's transformation tools (rotation, translation and reflection) or do you need to define custom transformation tools for taxicab geometry? Rotate (by 90° or 180°), translate (along grid lines to new grid intersections) and reflect (across segments on grid lines) your polygons.

Explore taxicab transformational geometry

Now consider the question that you explored for classical transformational geometry. Can all complex transformations be accomplished by just one kind of transformation, such as reflection on the grid? What is the minimum number of simple transformations required to accomplish any change that can be accomplished by a series of legal taxicab transformations?

In Euclidean geometry, if a right triangle has sides of length 3 and 4, the hypotenuse is 5, forming a right triangle with integer lengths. In taxicab geometry, it seems to have a hypotenuse of 7, which can be drawn along several different paths. In the grid shown, triangle ABC (green) has been reflected about segment IJ (blue), then translated by vector KL (blue), and then rotated 180° clockwise about point C'' (brown). Equivalently, ABC (green) has been reflected about segment BC (red), then reflected about the segment going down from C'_1 (red), and then reflected about segment A'''M''' (brown). Thus, in this case, the composition of a reflection, a translation and a rotation can be replicated by the composition of three reflections.



Transformations in taxicab geometry.

Explore kinds of polygons and their symmetries

What distinct kinds of polygons are possible in taxicab geometry with 3, 4, 5, 6, … sides? Can you work out the hierarchy of kinds of polygons with each number of sides? Do you think this should be done based on congruent sides and angles, symmetries or centers?

Discuss and summarize

What has your group noticed about taxicab transformational geometry? What have you wondered about and investigated? Do you have conjectures? Did you prove any theorems in this new geometry? What questions do you still have?

Be sure to list your findings in the wiki and see what other groups have discovered about taxicab geometry.

Activity: Prove a Conjecture

In this activity, your group will construct two overlapping squares and explore the amount of their overlap. The GeoGebra visualization will suggest a conjecture about the amount of overlap. You will then prove why the conjecture is true.

In this activity and the following activities,

- First try out the different constructions and think about the questions on your own.
- Then get together with your group and work collaboratively on the parts that you had trouble with.
- Discuss the questions in text chat.
- Summarize your group findings in your group's wiki page for the activity.

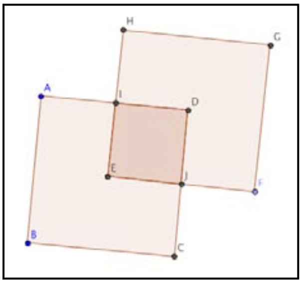
Shiny gold

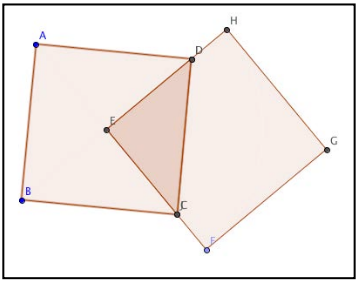
A jewelry maker wants to design a gold broach. She has two identical squares of shiny gold. She has decided to attach one corner of one square to the center of the other. She wants to maximize the amount of gold that shines forth. So she wants to attach the squares at an angle that minimizes their overlap. Can you advise her?

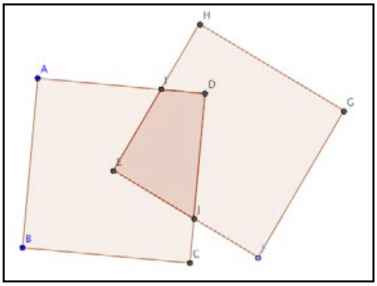
The geometry problem

Given two congruent squares, ABCD and EFGH, where the second square can rotate around the center of the first square, what is the maximum proportion of the first square that the second square can overlap at any time?

Discuss this problem with your group. How do you want to explore the problem? How many different ways of exploring this can your group list? For instance, you might use two squares of graph paper and count the overlapping areas as you rotate one square over the other.







Three cases of overlapping squares.

In these three figures, you can see three cases of the overlap:

1. A special case in which the overlap forms a square.
2. A special case in which the overlap forms a triangle.
3. A general case in which the overlap forms an irregular quadrilateral.

Are there any other special cases that you want to consider?

Can you calculate the area of overlap in each of these cases?

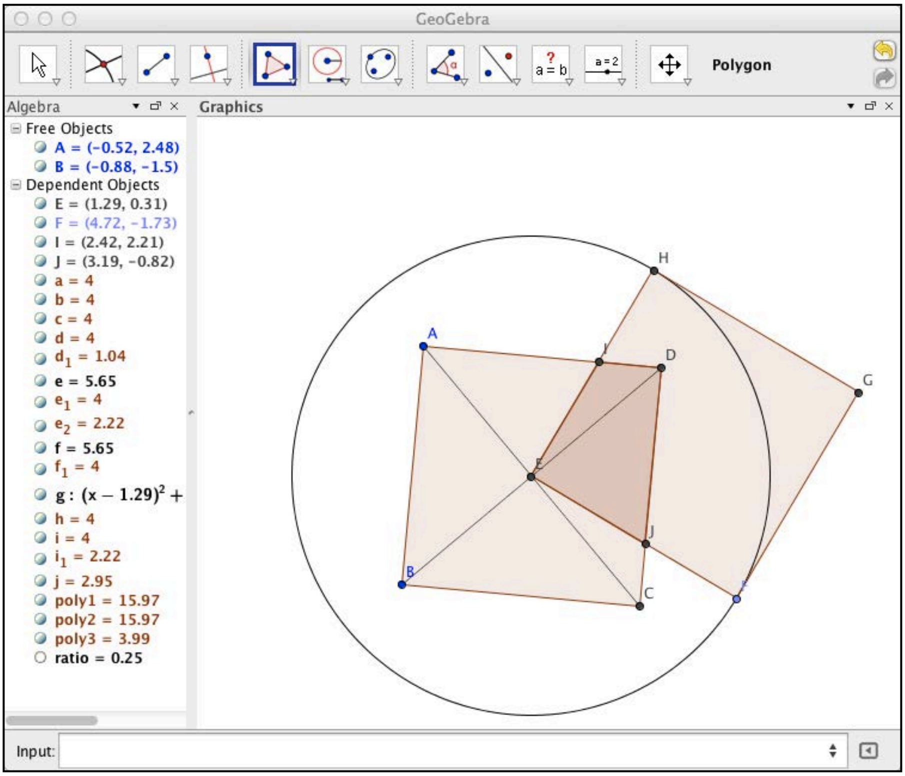
Can you use GeoGebra to help you explore and to calculate the area?

A GeoGebra construction

You can construct a model of the problem in GeoGebra. Here is one way to do it:

- Choose "Algebra and Graphics" from the Perspectives menu. This will display the values of all the geometric objects you create.
- Use the "regular polygon" tool for creating equilateral polygons. Click for point A and point B and then enter 4. GeoGebra will display a square whose sides are all the length of segment AB, with segment AB as one of the sides.
- Now locate the center of square ABCD. Because of the symmetries of a square, its center can be located at the intersection of the lines connecting its opposite vertices. Use the intersect tool to mark the intersection of segment AC and segment BD as point E.
- Use the compass tool to construct a circle around point E (the center of square ABCD) with a radius equal to the side of the square. First click on a side and then locate the center at point E.
- Now use the regular polygon tool again to construct the second square with one vertex at point E and another at a point on the circle. Constructing a side of the square on a segment from the center of the square to a point on the circle will make the new square the same size as the first square because the radius of the circle is the length of a side of the first square.

- The overlap area is a quadrilateral EJDI. You can use the polygon tool to create a polygon with this area.
- If it is not already visible, you can open the Input Form from the View menu and type in “ratio = poly3 / poly1”. Then the variable “ratio” will display the ratio of the area of quadrilateral EJDI to square ABCD.



The overlap of two squares.

Now you can move the second square around by dragging point F with the Move tool. Point F is constrained to stay on the circle and point E is constrained to stay at the center of square ABCD. The overlap area will change as point F moves. Watch the value of the area of quadrilateral EFDI and of ratio as you drag point F.

You can also drag point A. How does everything change when you drag point A?

What happens when you reach the special case of the square overlap and the triangular overlap?

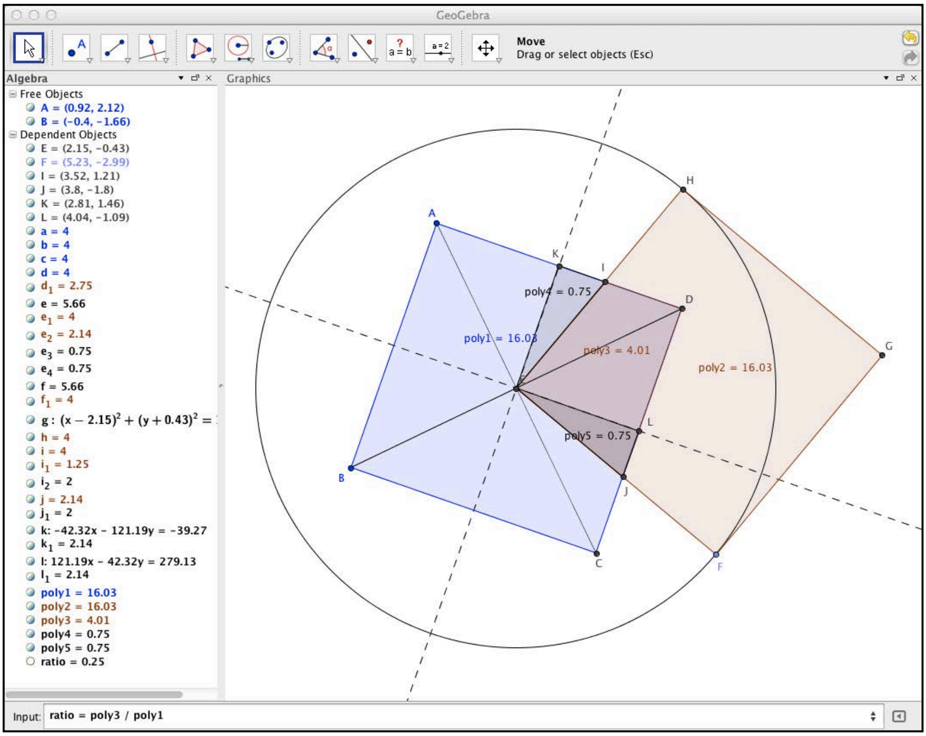
The conjecture

What is your group's conjecture about the maximum overlap and the ratio of its area to the area of the first square?

How could you prove that your conjecture is correct?

Work with your group to compose a clear statement of a conjecture about the overlap of the two congruent squares.

The analysis



Analyzing the overlap of two squares.

Consider the three cases of overlap:

1. A special case in which the overlap forms a square.
2. A special case in which the overlap forms a triangle.
3. A general case in which the overlap forms an irregular quadrilateral; this covers all cases in between the extreme cases listed above.

Cases 1 and 2 are relatively simple to prove. In case 1, where the overlap is a square, it is one of four congruent quadrants that make up the first square. In this case, point

I corresponds with midpoint K and point J corresponds to midpoint L. Therefore its area is ¼ the area of the first square. (How do you know that the four squares are congruent?) In case 2, where the overlap is a triangle, it is one of four congruent triangles that make up the first square. In this case, point I corresponds with vertex D and point J corresponds to vertex C. Therefore its area is ¼ the area of the first square. (How do you know that the four triangles are congruent?)

How can you prove case 3, where the overlap is an irregular quadrilateral?

Hint: Note that as you drag point F, what you are doing is rotating the second square around its vertex at point E. If you rotate it clockwise, then you are increasing angle JEL and angle IEK by the same amounts, which reduces angle CEJ and angle DEI by the same amount. So, as you rotate the second square, the area of overlap remains the same: what it gains on one side it loses on the other.

Be careful not to rotate case 3 beyond the positions of case 1 or case 2. – What happens if you go beyond this range? Why? How could you extend the proof to cover the full range of rotation?

The proof

A more formal proof could use the same kind of argument that Euclid used to prove that opposite angles formed by two intersecting lines are congruent:

Angle KEL is a right angle because it is a vertex of square KELD. Angle IEJ is a right angle because it is the vertex of square EFGH. So KEL = IEJ. If you subtract angle IEL from both of them, the remainders are still equal, so KEI = JEL.

This proves that case 3 has the same overlap as case 1. In the same way, you can prove that case 3 has the same overlap as case 2. So all the cases have the same overlap, namely ¼ of the area of the first square. This proves the conjecture. Intuitively, it shows that as you rotate the second square about vertex E, it loses an area on one side equal to the area it gains on the other side. So the proof makes intuitive sense and confirms your observation when dragging your dynamic geometry model. You have now proved the following theorem:

Theorem. Given two congruent squares, where a vertex of the second square is at the center point of the first square, the second square will overlap exactly ¼ of the area of the first square.

Proof. There are exactly three distinct cases to consider. In each case, it can be proven that the overlap is exactly ¼ of the area of one of the congruent squares. Therefore, the theorem is true.

I wonder

Did anyone in your group wonder what parts of the first and second square will never be overlapped as the second square rotates around its vertex attached to the center of the first square? What is your conjecture about what portion of the area will never be overlapped? Can you prove your conjecture?

Summarize your group work on this activity in your group's wiki page for this activity.

Activity: Invent a Polygon

In this activity, you will analyze a polygon that has probably never been analyzed before (except by other people who did this activity). Mathematics is a creative adventure: you may have to invent new definitions that lead to interesting relationships and conjectures.

An invented polygon

In the time of Euclid, there were no watches, only sundials and hourglasses to measure time. But the Greeks did have plenty of sand since most of them lived near the sea. So they would build an hourglass and put enough sand in it to pass through from the top to the bottom in an hour. The hourglass was a small glass container about the size of a saltshaker. It was symmetric, with one triangle on top and one on the bottom, joined by their vertex, with a hole going through the joint just big enough for the sand to pass through a grain at a time.

An hourglass polygon

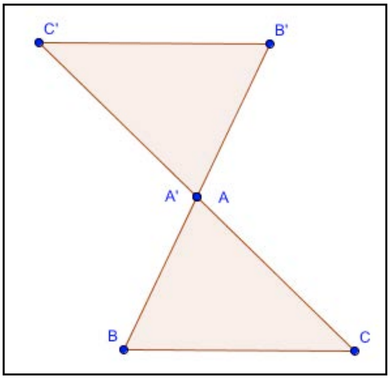
We can model an hourglass with a pair of congruent triangles, where A'B'C' is the mirror image of ABC reflected through point A or rotated about A by 180°. If we treat their joint as a single point, A, then we have a six-sided concave polygon. It is like a triangle in many ways, but different in others. Whereas triangles have been analyzed for centuries, it may be that hourglass polygons have never before been studied. Even the author of this activity has not (yet) explored this polygon.

Explore the hourglass polygon

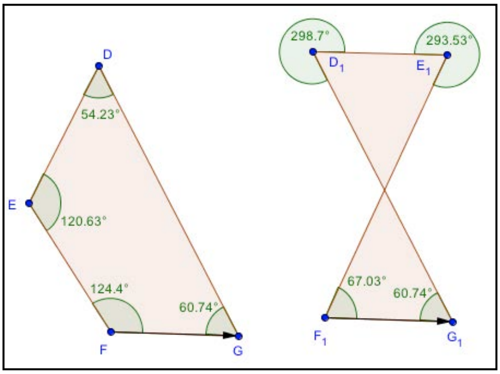
As a group, decide what to investigate, share your conjectures, construct different hourglass polygons, and try to prove some findings about them. You might want to build a hierarchy of different kinds of hourglass polygons, list their rotational and reflective symmetries, circumscribe them, etc. What is the most interesting or surprising thing about this geometric figure? Be sure to list your findings in the wiki and see what other groups have discovered about hourglass polygons.

Other interesting new polygons

Discuss within your group other distinctive polygons that might be particularly fun to explore. What interesting or surprising conjectures do you think you might find about your polygons?



An hourglass polygon.



A crossed quadrilateral.

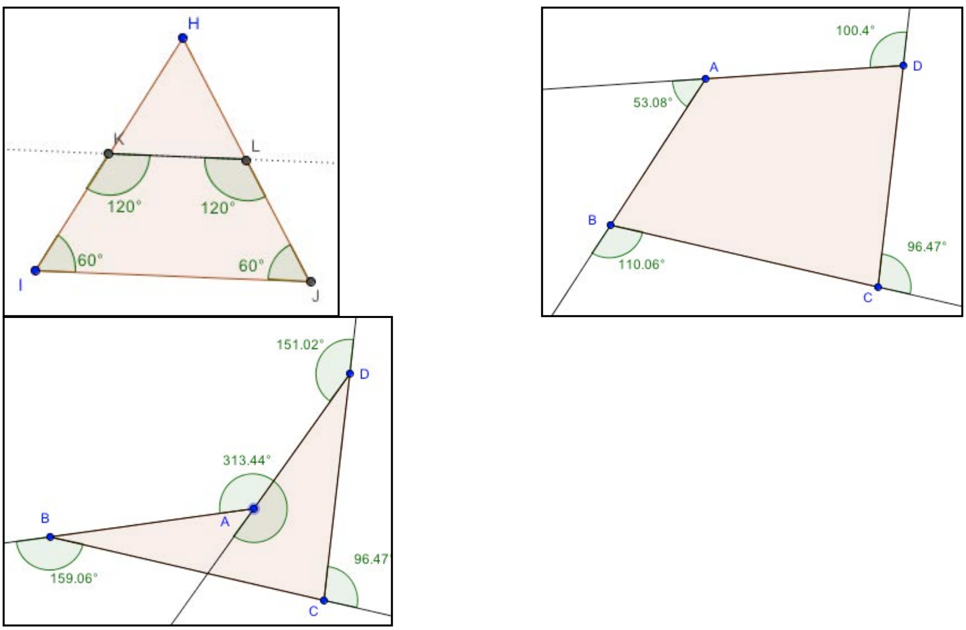
A crossed quadrilateral and its angles

How does the hourglass polygon compare to a crossed quadrilateral? Take a quadrilateral DEFG and drag point E across side DG to form a figure that looks like an hourglass polygon. One question to ask is: What is the sum of the interior angles of each figure?

Use the angle tool to display the size of each interior angle. Click on the three points that define each angle in alphabetical order. Are you surprised? Two of the angles in the crossed quadrilateral seem to now be exterior angles. But the inside of part of the figure has been turned outside and it is hard to say what the "interior" angles are. You can define it different ways. For instance, you can say that as you go around the figure from D to E to F to G to D, the "interior" angle is always on your left (or always on your right for a mirror image polygon).

Proving the sum of the angles

Take an equilateral triangle. What are its three angles and their sum? Now add a segment parallel to its base. What are its seven angles and their sum? Consider just the quadrilateral at the bottom: What are its four angles and their sum? Can you conjecture a general statement for the sum of the angles of a quadrilateral?



The angles of a quadrilateral.

How could you prove your conjecture? Consider quadrilateral ABCD. Think about walking around its perimeter: go from A to B and then turn by the exterior angle shown; go from B to C and turn; go from C to D and turn; go from D to A and turn. You are now facing the way you began your journey. How many degrees have you turned? Does this have to be true for all quadrilaterals? Why or why not? How would you deal with the angles of a concave quadrilateral?

Activity: Visualize Pythagoras' & Thales' Theorems

In this activity you will explore two visualizations of what is probably the most famous and the most useful theorem in geometry.

Pythagoras' Theorem

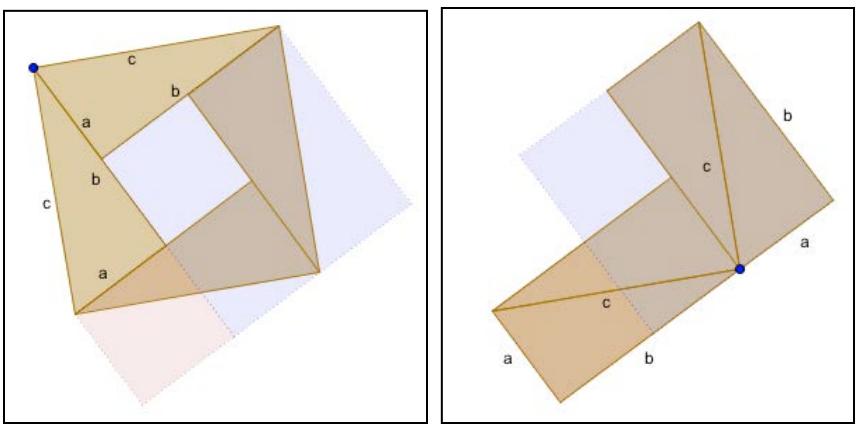
Pythagoras' Theorem says that the length of the hypotenuse of a right triangle (side **c**, opposite the right angle) has the following relationship to the lengths of the other two sides, **a** and **b**:

$$c^2 = a^2 + b^2$$

Here are two ways to visualize this relationship. They involve transforming squares built on the three sides of the triangle to show that the sum of the areas of the two smaller squares is equal to the area of the larger square.

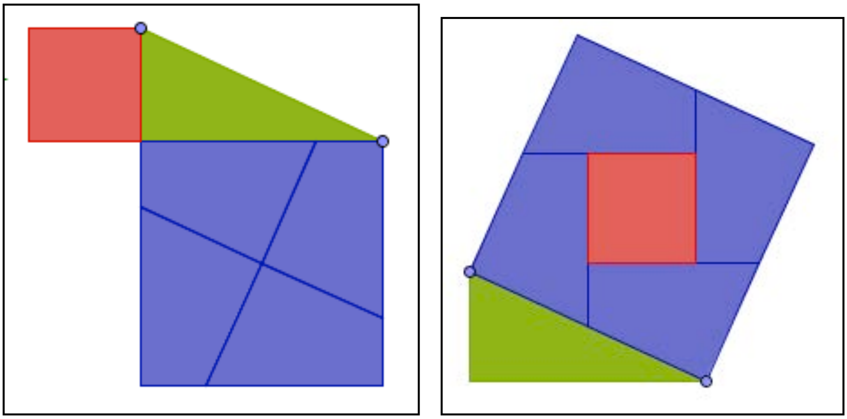
Explain what you see in these two visualizations. Can you see how the area of the c^2 square is rearranged through rigid transformations of triangles (translations, rotations and reflections) into the areas b^2 and c^2 or vice versa?

Visualization #1 of Pythagoras' Theorem



Visualization #1 of Pythagoras' Theorem.

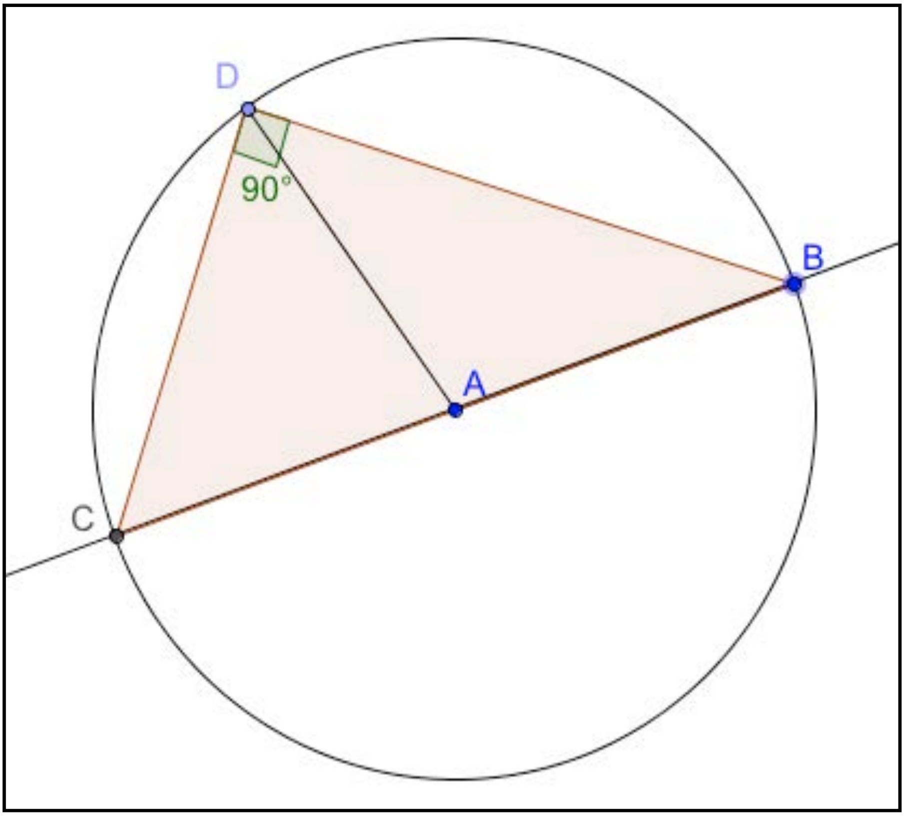
Visualization #2 of Pythagoras' Theorem



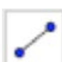
Visualization #2 of Pythagoras' Theorem.

Visualize the Theorem of Thales

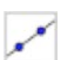
Thales found out about something 2,500 years ago that you will explore in this activity. You will use this result in the next activity.



Construction process

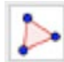
Step 1. Construct segment  AB.

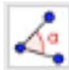
Step 2. Construct a circle  with center at point A and going through point B.

Step 3. Construct a line  going through points A and B.

Step 4. Construct point C at the intersection  of the line and the circle, forming the diameter of the circle.

Step 5. Construct point  D on the circle.

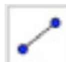
Step 6. Create triangle BCD with the polygon tool .

Step 7. Create the interior angles  of triangle BCD.

Step 8. Drag  point D along the circle. What do you notice?

Challenge:

Try to come up with a graphical proof for this theorem.

Hint: Construct the radius AD as a segment .

Activity: Geometry Using Algebra

In this activity, you will use the Theorem of Thales to construct a tangent to a circle using the geometry tools of GeoGebra that you already know. Then, you will do the same thing in a very different way, using the algebra tools of GeoGebra.

Construct tangents to a circle algebraically

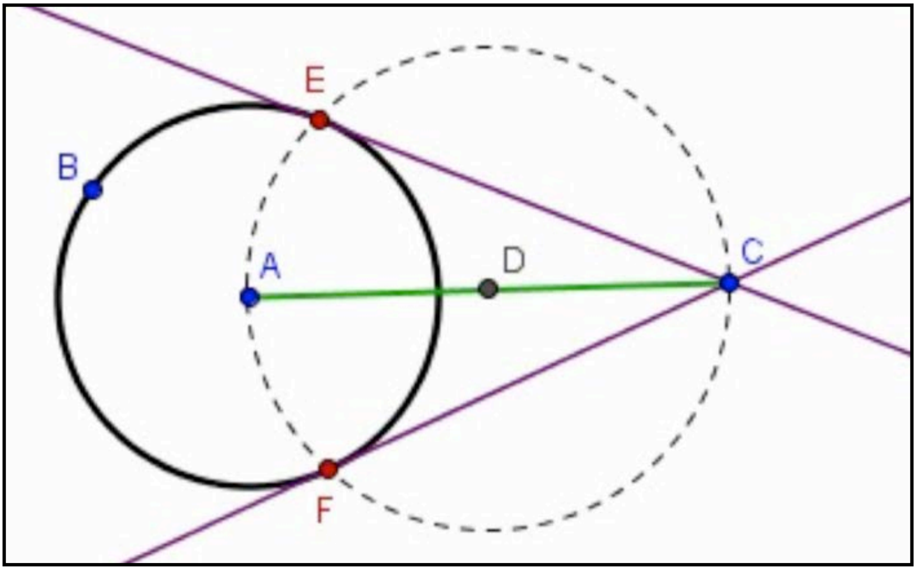
The approach

Given a circle and an arbitrary point outside the circle, construct the tangents to the circle going through the point.

A tangent to a circle touches the circle at one and only one point. The tangent is perpendicular to a radius from the center of the circle to the point of tangency.

You can use Thales Theorem to construct the tangent through a point C to a circle with center A if you construct another circle whose diameter is segment AC. According to Thales Theorem, the angle formed between line CE and a line from A to point E (at the intersection of the two circles) will be a right angle, making line CE a tangent to the circle centered at A.

Discuss in chat what tools to use and how to do the construction. Take turns doing the construction and checking the dependencies.



A geometric construction of tangents.

Explore

Construct a circle with center at point A, going through a point B. Also construct a point C outside the circle.

Then, construct the tangents to the circle, going through point C, as indicated in the image above. Note that there are two tangents and that the diagram is symmetric along AC.

Construct a supplementary segment AE and the angle AEC to check if the tangent is perpendicular to the radius. Drag point C to see if the relationships hold dynamically.

Summary

Explain in your summary what your group observed in this activity. What is the Theorem of Thales and how did it help you to construct the tangent to the circle? State this in your own words and make sure everyone in the group understands it.

Construct tangents to a circle algebraically

In this activity, you will use the Algebra interface of GeoGebra to do the same construction you did in the last activity with geometry tools. This will introduce you to the multiple representations of GeoGebra.

GeoGebra joins geometry and algebra

GeoGebra is dynamic-mathematics software for schools, which joins geometry, algebra, and calculus.

GeoGebra has the ability to deal with algebra variables and equations as well as geometry points and lines. These two views are coordinated in GeoGebra: an expression in the algebra window corresponds to an object in the geometry window and vice versa.

GeoGebra's user interface consists of a graphics window and an algebra window. On the one hand you can operate the provided geometry tools with the mouse in order to create geometric constructions in the graphics window. On the other hand, you can directly enter algebraic input, commands, and functions into the input field by using the keyboard. While the graphical representation of all objects is displayed in the graphics window, their algebraic numeric representation is shown in the algebra window.

GeoGebra offers algebraic input and commands in addition to the geometry tools. Every geometry tool has a matching algebra command. In fact, GeoGebra offers more algebra commands than geometry tools.

Tips and tricks

- Name a new object by typing in `name =` in front of its algebraic representation in the Input Field. Example: `P = (3, 2)` creates point P.
- Multiplication needs to be entered using an asterisk or space between the factors. Example: `a*x` or `a x`
- Raising to a power is entered using ^. Example: `f(x) = x^2 + 2*x + 1`
- GeoGebra is case sensitive! Thus, upper and lower case letters must not be mixed up. Note: Points are always named with upper case letters. Example: `A = (1, 2)`
- Segments, lines, circles, functions… are always named with lower case letters. Example: circle `c: (x – 2)^2 + (y – 1)^2 = 16`
- The variable x within a function and the variables x and y in the equation of a conic section always need to be lower case. Example: `f(x) = 3*x + 2`
- If you want to use an object within an algebraic expression or command you need to create the object prior to using its name in the input field. Examples: `y = m x + b` creates a line whose parameters are already existing values

m and b (e.g. numbers / sliders). `Line[A, B]` creates a line through existing points A and B.

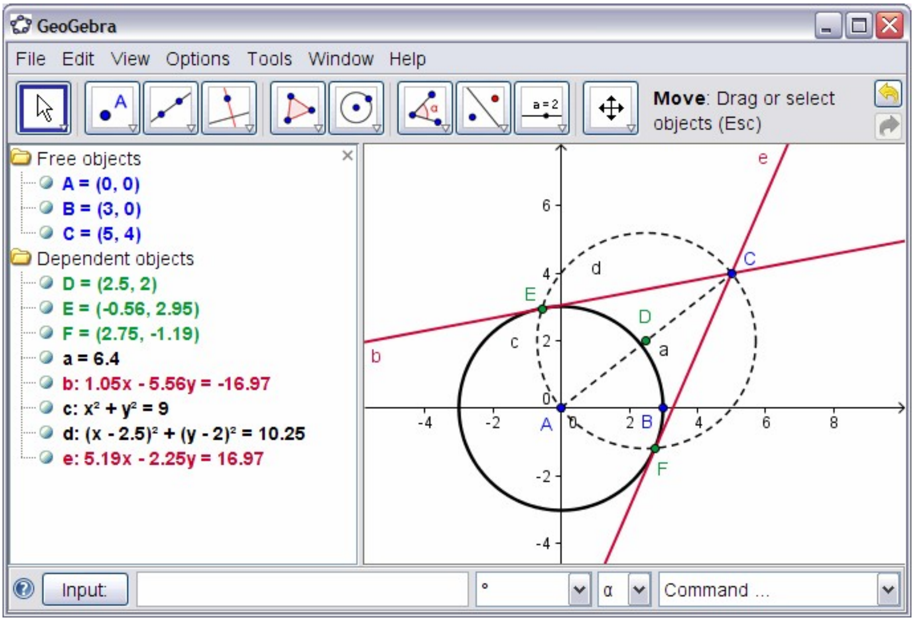
- Confirm an expression you entered into the input field by pressing the Enter key.
- Open the "Input Help" panel for help using the input field and commands by clicking the "?" button next to the input field.
- Error messages: Always read the messages – they could possibly help to fix the problem!
- Commands can be typed in or selected from the list next to the input field. Hint: If you don't know which parameters are required within the brackets of a certain command, type in the full command name and press key F1. A pop-up window appears explaining the syntax and necessary parameters of the command.
- Automatic completion of commands: After typing the first two letters of a command into the input field, GeoGebra tries to complete the command. If GeoGebra suggests the desired command, hit the Enter key in order to place the cursor within the brackets. If the suggested command is not the one you wanted to enter, just keep typing until the suggestion matches.

Algebraic construction

Check out the list of textual algebraic commands next to the Input Help and look for commands corresponding to the geometry tools you have learned to use.

Preparation

Select the Perspective "Algebra and Graphics". Use the View menu to make sure the Input Field, the Algebra window and the Coordinate Axes are all displayed.



An algebraic construction of tangents.

Construction process

Step	Input Field entry	Object created
1	A = (0, 0)	Point A
Hint: Make sure to close the parenthesis.		
2	(3, 0)	Point B
Hint: If you don't specify a name objects are named in alphabetical order.		
3	c = Circle[A, B]	Circle with center A through point B
Hint: Circle is a dependent object		

Note: GeoGebra distinguishes between free and dependent objects. While free objects can be directly modified either using the mouse or the keyboard, dependent objects adapt to changes of their parent objects. Thereby, it does not matter how an object was initially created (by mouse or keyboard)!

Hint 1: Activate Move mode and double click an object in the algebra window in order to change its algebraic representation using the keyboard. Hit the Enter key once you are done.

Hint 2: You can use the arrow keys to move free objects in a more controlled way. Activate move mode and select the object (e.g., a free point) in either window. Press the up / down or left / right arrow keys in order to move the object in the desired direction.

4	C = (5, 4)	Point C
5	s = Segment[A, C]	Segment AC
7	D = Midpoint[s]	Midpoint D of segment AC
8	d = Circle[D, C]	Circle with center D through point C
9	Intersect[c, d]	Intersection points E and F of the two circles
10	Line[C, E]	Tangent through points C and E
11	Line[C, F]	Tangent through points C and F

Checking and enhancing the construction

Perform the drag-test in order to check if the construction is correct.

Change properties of objects in order to improve the construction's appearance (e.g., colors, line thickness, auxiliary objects dashed, etc.).

Discussion

Did any problems or difficulties occur during the construction process?

Which version of the construction (mouse or keyboard) do you prefer and why?

Why should we use keyboard input if we could also do it using tools?

Hint: There are commands available that have no equivalent geometric tool.

Does it matter in which way an object was created? Can it be changed in the algebra window (using the keyboard) as well as in the graphics window (using the mouse)?

Appendix: Notes on the Design of the Activities

These activities have been designed to promote collaborative learning, particularly as exhibited in significant mathematical discourse about geometry. Collaborative learning involves a subtle interplay of processes at the individual-student, small-group and whole-classroom levels of engagement, cognition and reflection. Accordingly, the activities are structured with sections for individual-student work, small-group collaboration and whole-class discussion. It is hoped that this mixture will enhance motivation, extend attention and spread understanding.

Goals

The goal of this set of activities is to improve the following skills in math teachers and students:

1. To engage in significant mathematical *discourse*; to collaborate on and discuss mathematical activities in supportive small online groups
2. To collaboratively *explore* mathematical phenomena and dependencies; to make mathematical phenomena visual in multiple representations; and to vary their parameters
3. To *construct* mathematical diagrams – understanding, exploring and designing their structural dependencies
4. To notice, wonder about and form conjectures about mathematical relationships; to justify, explain and *prove* mathematical findings
5. To understand core concepts, relationships, theorems and constructions of basic high-school *geometry*

The working hypothesis of the activities is that these goals can be furthered through an effective combination of:

1. Collaborative experiences in mathematical activities with guidance in collaborative, mathematical and accountable geometric *discourse*
2. *Exploring* dynamic-mathematical diagrams and multiple representations
3. Designing dependencies in dynamic-mathematical *constructions*
4. Explaining conjectures, justifications and *proofs*
5. Engagement in well-designed activities around basic high-school *geometry* content

In other words, the activities seek a productive synthesis of *collaboration*, *discourse, visualization, construction, and argumentation* skills applied in the domain of beginning geometry.

Development of skills

The set of activities should gradually increase student skill levels in each of these dimensions. The design starts out assuming relatively low skill levels and gradually increases the level of skill expected. There is a theoretical basis for gradually increasing skill levels in terms of both understanding and proof in geometry. Here "understanding" and "proof" are taken in rather broad senses. The van Hiele theory (see deVilliers, 2003, p. 11) specifies several levels in the development of students' understanding of geometry, including:

1. *Recognition*: visual recognition of general appearance (something looks like a triangle)
2. *Analysis*: initial analysis of properties of figures and terminology for describing them
3. *Ordering*: logical ordering of figures (a square is a kind of rectangle in the quadrilateral hierarchy)
4. *Deduction*: longer sequences of deduction; understanding of the role of axioms, theorems, proof

The implication of van Hiele's theory is that students who are at a given level cannot properly grasp ideas presented at a higher level until they reach that level. Thus, a developmental series of activities pegged to the increasing sequence of levels is necessary to effectively present the content and concepts of geometry, such as, eventually, formal proof. Failure to lead students through this developmental process is likely to cause student feelings of inadequacy and consequent negative attitudes toward geometry.

Citing various mathematicians, deVilliers (2003) lists several roles and functions of proof, particularly when using dynamic-geometry environments:

1. *Communication*: proof as the transmission of mathematical knowledge
2. *Explanation*: proof as providing insight into why something is true
3. *Discovery*: proof as the discovery or invention of new results
4. *Verification*: proof as concerned with the truth of a statement
5. *Intellectual challenge*: proof as the personal self-realization or sense of fulfillment derived from constructing a proof
6. *Systematization*: proof as the organization of various results into a deductive system of axioms, major concepts and theorems

In his book, deVilliers suggests that students be introduced to proof by gradually going through this sequence of levels of successively more advanced roles of proof through a series of well-designed activities. In particular, the use of a dynamic-geometry environment can aid in moving students from the early stages of these sequences (recognition and communication) to the advanced levels (deduction and systematization). The use of dragging geometric objects to explore, analyze and support explanation can begin the developmental process. The design and construction of geometric objects with dependencies to help discover, order and verify relationships can further the process. The construction can initially be highly scaffolded by instructions and collaboration; then students can be guided to reflect upon and discuss the constructed dependencies; finally they can practice constructing objects with gradually reduced scaffolding. This can bring students to a stage where they are ready for deduction and systematization that builds on their exploratory experiences.

Practices for significant mathematical discourse in collaborative dynamic geometry

The following set of practices state the main skills that these activities are designed to instill. They integrate math and discourse skills. They are specifically oriented to dynamic geometry and its unique strengths:

a. Visualize: View and analyze constructions of geometric objects and relationships
b. Drag: Explore constructions of geometric objects through manipulation
c. Discourse: Notice, wonder, conjecture, strategize about relationships in constructions and how to investigate them further
d. Dependencies: Discover and name dependencies among geometric objects in constructions
e. Construction: Construct dependencies among objects, and define custom tools for doing so
f. Argumentation: Build deductive arguments, explain and prove them in terms of the dependencies
g. Math Accountability: Listen to what others say, solicit their reactions, re-voice their statements, re-state in math terminology and representations
h. Collaboration: Preserve discourse, reflect on it and organize findings; refine the statement of math knowledge; build knowledge together by building on each other's ideas

These practices can be placed in rough isomorphism with the Common Core math practices:

1. Make sense of problems and persevere in solving them: (b)
2. Reason abstractly and quantitatively: (c)
3. Construct viable arguments and critique the reasoning of others: (g)
4. Model with mathematics: (a)
5. Use appropriate tools strategically: (e)
6. Attend to precision: (f)

7. Look for and make use of structure: (d)
8. Look for and express regularity in repeated reasoning: (h)

It may be possible to organize, present and motivate course activities in terms of these practices. Then pedagogy could be discussed in terms of how to promote and scaffold each of these; formative assessment (including student/team portfolio construction) could also be structured according to these practices.

Appendix: Pointers to Further Reading and Browsing

GeoGebra
http://www.geogebra.org and http://www.geogebratube.org.

Bu, L., Schoen, R. (Eds.) (2011). *Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra*. Sense Publishers.

Geometer's Sketchpad
Bennett, D. (2002). *Exploring Geometry with the Geometer's Sketchpad*. Key Curriculum Press.

deVilliers, M. (2003). *Rethinking Proof with the Geometer's Sketchpad*. Key Curriculum Press.

Geometry
McCrone, S., King. J., Orihuela, Y., Robinson, E. (2010). *Focus in High School Mathematics: Rasoning and Sense Making: Geometry*. National Council of Teachers of Mathematics.

Serra, M. (2008). *Discovering Geometry: An investigative Approach*. Key Curriculum Press.

Common Core State Standards Initiative (2011). *Common Core State Standards for Mathematics*. "High School – Geometry" pp. 74-78. Web: http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Krause, E. (1986). *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*. New York, NY: Dover.

Euclid
Euclid (c. 300 BCE/2002). *Euclid's Elements*. Thomas L. Heath, translator, Dana Densmore, editor. Green Lion Press.

Virtual Math Teams Research Project
Stahl, G. (2009). *Studying Virtual Math Teams*. New York, NY: Springer. Web: http://GerryStahl.net/vmt/book.

Stahl, G. (2013). *Translating Euclid: Liberating the cognitive potential of collaborative dynamic geometry*. Web: http://GerryStahl.net/pub/euclid.pdf.

The Math Forum
http://mathforum.org.

Creative Commons ShareAlike Copyright

The first activities in this document are based largely on: ***Introduction to GeoGebra*** by Judith and Markus Hohenwarter, modified: November 9, 2011, for GeoGebra 4.0

In addition, this document has drawn many ideas from the other sources listed in this section.

Appendix: Fix a Technical Problem

Some common problems while starting up VMT

- **VMT cannot find Java**

Look in your applications library. If you do not have the latest version of Java, download it from the Internet.

- **VMT cannot find Java Web Start**

Look in your applications library. If you do not have the latest version of Java WebStart, download it from the Internet.

Some common problems while using VMT

- **After adding/removing the algebra view or changing the perspective, part of the GeoGebra window is blank**

Press the REFRESH button at the bottom left of the VMT window.

- **Unable to take control even though nobody has control**

Make sure the history slider (on the left) is at the current event (all the way down). You cannot take control while scrolling through the history. On rare occasions the control mechanism breaks, and that tab can no longer be used.

- **When trying to open a VMT chat room, the password field is blank and the logon fails; even if you type in your password it fails**

This can happen when your VMT-Lobby session has expired. Go to the VMT Lobby and logout. Then log back in and try again to enter your room. If that doesn't work, try closing your browser, then logging back into the lobby. As a last resort reboot your computer.

- **Your username is refused when you try to enter a chat room that you recently left**

Sometimes when a chat room crashes, your username is still logged in and you cannot use that username to enter the room again. After a time period, that username will be automatically logged out and you will be able to enter the room again with that username. Alternatively, you can register a new username and enter the chat room with the new username.

- **If your view of the shared GeoGebra construction becomes dysfunctional or you do not think Display problems in VMT**

If your view of the shared GeoGebra construction becomes dysfunctional or you do not think you are receiving and displaying chat messages, then close the VMT chat room window. Log in to the VMT Lobby again and enter the chat room again. Hopefully, everything will be perfect now. If not, press the Reload button if there is one. If all else fails, read the Help manual, which is available from the links on the left side of the VMT Lobby.

Technical requirements to start a VMT chat room

- VMT is a Java Web Start application, so JavaWebStart must be enabled. Note, on Macs you may need to go to the Java Control Panel (or Preferences depending on the version) and explicitly enable JavaWebStart.
- VMT downloads a .jnlp JavaWebStart file, so .jnlp must be an allowed file type to download.
- When VMT starts it will download the needed Java jars from the VMT server. So downloading jars must be allowed.
- If VMT does not start when a .jnlp file is downloaded, then the .jnlp file extension needs to be associated with the JavaWebStart program (javaws).
- It should also be possible to start vmt by finding the .jnlp file in the browser downloads folder and double clicking it.
- The firewall (for instance at a school) must allow vmt.mathforum.org
- The firewall (for instance at a school) may need to open port 8080
- To use the VMT Lobby, javascript must be enabled
- It might be helpful to list vmt.mathforum.org as a trusted site for java downloads

Contact us

Problems or questions? Email us at: vmthelp@mathforum.org .

Notes & Sketches

This space is for your notes. Paste in views of your constructions. List files of constructions or custom tools that you have saved. Jot down interesting things you have noticed, questions you have wondered about or conjectures you might want to explore in the future. Collect more dynamic-math activities here.