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Mathematical Group Cognition in the Anthropocene

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Introduction

> *Euclid presented his classic approach to geometry as a succession of propositions. Here, an approach to geometry education today is offered through a sequence of quite different propositions. They suggest focal points of a philosophy of computer-supported collaborative learning that emerged from research on teaching and learning dynamic geometry. In particular, this chapter proposes that dynamic geometry can provide a model of dependencies in interconnected systems, preparing students to understand mathematical structure of interactions among human and natural systems in the new age of the Anthropocene.*

By providing an illustrative case of educating for the Anthropocene, this chapter suggests that dynamic geometry as taught in the reviewed research project can provide student thinking with a model of dependencies in interconnected systems. Review of this research into the development of mathematical cognition by student groups learning dynamic geometry in online teams elaborates a theory of learning and thinking as "group cognition." This conception of group cognition seems appropriate for designing the teaching and learning of mathematics in the Anthropocene.

Proposition α: The Anthropocene

> *Living in the Anthropocene requires new ways of understanding interactions among countless actors: including human, animal, mineral, technological, computational and Earth-system agents.*

According to many scientists, the world changed significantly with the advent of the Anthropocene epoch about 70 years ago. The atomic bomb, the population explosion, exponential growth of fossil-fuel usage and CO_2 emissions, urban/suburban sprawl and many other socio-economic transformations led to a rapidly increasing influence of human behavior on worldwide natural systems. Our public knowledge systems now have to catch up with these changes so we can comprehend and moderate the new and potentially dangerous processes. The educational system must develop revised approaches to understanding and teaching about this new world. This will require new conceptualizations of knowledge and new approaches to education.

Referring to the present geological epoch as the "Anthropocene" denotes the essential influence of human (anthropological) behavior, industry and consumption upon major systems of the biosphere, including the land, oceans, vegetation, animals, sea life, insects, viruses and climate (Crutzen and Stoermer, 2000, Steffen et al. 2015, Wallace-Wells 2020). The current coupling and interpenetration of cultural and natural evolution (Donald 1991, Donges et al. 2017) requires more than simple mechanistic laws and equations of Galileo and Newton to comprehend, anticipate and influence; it involves thinking in terms of probabilistic formulations of subtle interdependencies (Thomas, Williams and Zalasiewicz 2020, Wiener 1950). Teaching and learning mathematics in our age should provide cognitive tools and perspectives for humanity to survive in this complex setting of climate change and potential extinction (Coles 2017, Gomby 2022).

In response to a major shift in reality, we need to reconceptualize scientific analysis, including its mathematical and cognitive underpinnings (Griscom et al. 2017, Steffen and Morgan 2021). Just as physics has had to consider stochastic and non-linear processes, relativistic and quantum calculations, feedback and observer influences, field and gauge theories or conceptualizations like entropy, strings, entanglement, dark energy and alternative universes, our understanding of the everyday world (environment, biosphere, Gaia) needs to see how things are tied together in surprising ways with exponential growth, feedback loops and tipping points (Kemp et al. 2022, Steffen 2018). New approaches to teaching and learning mathematics are required here as much as in particle physics (Boylan and Coles 2017, Mikulan and Sinclair 2017). This chapter reports on a research project to develop a computer-supported collaborative-learning approach to teaching dynamic geometry as a way of conceptualizing dependencies among objects as a foundation for comprehending interconnections.

Proposition β: Dynamic Geometry

> *Teaching and learning relevant mathematical thinking may be promoted by student exploration of dynamic geometry. This interactive application allows students to investigate the structure*

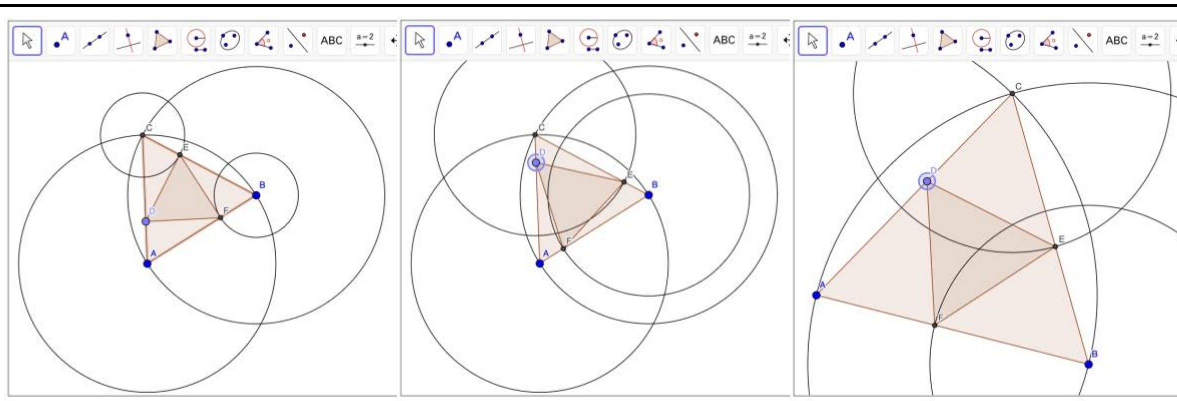

Figure 1. Inscribed equilateral triangles constructed in GeoGebra and dragged to different positions.

and interrelationships of well-defined geometric elements and complexes. This can provide a basis for understanding the complexities of the intertwined Anthropocene world.

Dynamic geometry is a computer-based form of mathematics grounded on Euclidean geometry and implemented in popular applications such as GeoGebra and Geometer's Sketchpad (Sinclair 2008). In Figure 1, an equilateral triangle is constructed in dynamic geometry with side lengths dependent upon circles with equal radii, as specified in Euclid's first proposition. Then an interior equilateral triangle is constructed with vertices equal distances from the vertices of the exterior triangle. Dragging around points of each triangle suggests that the two triangles both remain equilateral regardless of the positions of the specified points.

Proposition γ: Dragging Shapes

Dynamic geometry visualizes the generalization implicit in Euclidean geometry and the dependencies that underlie it by allowing points, lines and figures to be interactively dragged to alternative possible locations. Dependencies that persist despite such dragging reveal underlying causal relationships. They suggest which relationships still hold when locations are generalized from illustrated positions of points to other possible positions.

On a given finite straight line to construct an equilateral triangle.
Let AB be the given finite straight line.
....
Therefore, the triangle ABC is equilateral; and it has been constructed on the given finite straight line AB.
Being what it was required to do.

While the Greek proofs stress deduction, they implicitly assume the generality of their constructions. Digital geometry, by contrast, allows points to be moved around, rearranging related elements in order to maintain dependencies defined by the construction process. This allows a viewer to observe some of the generality of the construction, including effects (constraints) of the dependencies. The relevant dependencies are established by Euclidean constructions when carried out in dynamic geometry.

The implication of Euclid's (300 BCE) text in Figure 2 is that this construction works for any finite straight line and that the construction using the specific line AB in the accompanying diagram is an example of how to do the construction for any similar lines located elsewhere. If this construction is carried out in dynamic geometry as in Figure 1, then one can drag point A, point B and/or line AB to arbitrary other positions and the constructed triangle ABC will still be equilateral. Such dragging, which is typical of dynamic geometry, displays visually that the construction is valid for many lines AB—all those tested with different locations for end points A and B. It also displays the dependencies imposed by the construction that constrain the triangle to be equilateral: namely the two circles of radius AB, which ensure that the lengths of sides BC and AC are each equal to the length of line segment AB, and therefore the triangle's three sides are all equal to each other.

The same applies to Euclid's propositions which are proofs rather than constructions. They are presented as examples of how to conduct proofs for specific diagrams at specific locations, but are intended to be generalized to any diagrams with the same features (Netz 1999). It is because Euclid's constructions and proofs are designed to be generalizable to points, triangles, etc. located anywhere, that his static diagrams translate directly to dynamic-geometry constructions. They are tacitly built around the application of dependencies, such as the length of a line segment being dependent upon a circle of certain radius. These dependencies underlie the proofs, for which diagrams are constructed following Euclid's propositions. An understanding of dynamic geometry in terms of the design of dependencies provides insight into the design of geometric figures—insight that is not always fostered by a traditional presentation of deductive proof.

Proposition δ: Constructing Figures

Construction of dynamic-geometry figures by students can offer them insightful understanding of the elements of associated proof structures. Active construction provides immediate feedback on consequences of design decisions. By actively building up figures, students become aware of the sequentiality and interdependency of constructions related to propositions.

Becoming a skilled constructor of dynamic-geometry figures involves paying close attention to actions that establish dependencies among objects, such as dragging points to make sure that the software has defined those points at intended line intersections. A student's growing explicit concern for establishing and checking effective dependency relationships gradually becomes habitual, a matter of assumed behavior that is henceforth carried out tacitly.

Viewing, understanding and manipulating constructions in terms of their interdependencies provides students with insight into why associated proofs work the way they do (deVilliers 2004). It is because triangle ABC's sides were constructed by radii equal in length to segment AB that the three sides are always necessarily of equal length. The construction of the internal triangle DEF in Figure 1 is more complicated and the proof of the equality of its sides is correspondingly longer, but similarly related to constructed dependencies.

Proposition ε: Dependencies among Objects

Geometry can be viewed as the systematic study of dependencies that are designed into or discovered within complexes of simple objects like points, lines, angles, circles, polygons. The dependencies inherent in dynamic-geometry constructions correspond to characteristics and relationships of figures referenced in their corresponding proofs. The establishment and preservation of *dependencies is fundamental to the logic of Euclid's propositions and to the mechanisms of dynamic geometry's software.*

Euclid's propositions talk about points and lines being placed in the plane, but do not explicitly discuss the dependencies that are implicitly designed into the constructions. The dynamic-geometry software, on the other hand, must keep systematic track of these dependencies behind the scenes. When a point is moved, the software checks for any dependencies involving that point, and moves other points in ways that maintain the dependencies. The dynamic-geometry display thereby provides a model of a geometric structure that obeys sets of dependencies among its elements.

Students exploring dynamic geometry can learn to think about systems of interdependent elements, some of which are completely dependent upon the positions of others, some are constrained (e.g., to move only in a fixed circle around another point) and some are simply free to move anywhere (Hölzl, Healy, Hoyles, and Noss 1994, Jones, 1996). This kind of systems thinking can later be applied to evolutionary models of nature, such as a model of animal populations dependent upon climate, vegetation and interactions among species.

Proposition ζ: Texts Referencing Visualizations

Since the Greek geometers, constructions and their proofs have been communicated among mathematicians and math students through carefully structured texts that reference associated diagrams. Understanding geometry involves reading/writing the specialized language and being aware of previous propositions. Mathematical cognition takes place in such inscriptions: sequential descriptive statements, illustrative figures and specialized symbol systems.

Geometric cognition is embodied in inscriptions – texts coordinated with labelled constructions (such as Figures 1 and 2 above). These are knowledge-building artifacts in the visible material world. Their meaning is shared and based on intersubjective language and cultural traditions. The meaning must be understood and interpreted by trained and capable individuals. Students have to learn how to make careful constructions, but also how to discuss these constructions and their designed dependencies with other people in the precise language of mathematics. These are skills requiring deep understanding and personal engagement, not just rote memorization of terminology and facts.

There is a subtle combination of individual, small-group and community cognition at work in the teaching and learning of mathematics. The knowledge of how to construct an equilateral triangle is expressed in an inscription of Euclid's first proposition. This inscription may be included in a geometry textbook or in a dynamic-geometry exercise. Its meaning is defined by the shared understanding of the mathematical community, including textbook authors, schoolteachers and—to a lesser extent—beginning geometry students.

If a small group of students explores one of Euclid's propositions, the group cognition consists of the shared meaning in the group discourse—issuing from the multiple perspectives and individual linguistic abilities to understand and contribute to the group interaction. The group processes of collaborative learning involve individual capacities to participate effectively. However, while individual cognition is required for group cognition, the group level cannot be reduced to a sum of individual contributions. The collaborative level includes references, anticipations, goals, agreements, decisions and history of the group as such. Individuals in the group are typically not consciously aware of most of these factors and would not be subject to them if not participating in the group interaction

Proposition η: Mediated Cognition

> *In general, high-level cognitive functions of individual human minds are developed first through small-group interactions and may be subsequently further developed as individual skills. Intellectual skills are mediated by language and tools. Mathematical cognition is*

mediated by the terminology, practices, symbols and inscriptions adopted by the worldwide, historical community of practitioners.

The common focus on individual cognition in philosophy, psychology and educational theory is based on introspection by adults and observation of skilled practitioners. As adults, we picture ourselves learning through solitary reading or silent reflection. However, if we observe infants and toddlers learning the basic skills for living in the physical and social world, we can see the central role of interaction with other people, such as parents and siblings. Vygotsky (1930: 57) concluded that cultural development—including formation of concepts—occurs first on a social level. For instance, children in his studies "could do only under guidance, in collaboration and in groups at the age of three-to-five years what they could do independently when they reached the age of five-to-seven years."

Vygotsky's analysis of the development of the pointing gesture (p. 56) provides a clear example of group cognition. The mother does not teach her infant how to point to what he wants; the meaningful gesture is not "enculturated" from existing culture. Rather, it is co-constructed by the participants situated in the setting as an intersubjective meaning-making interaction. The gesture develops as tacitly understood within the intimate mother/infant group and gradually becomes sedimented into a symbolizing artifact through repetitive habituation. The meaning of the pointing finger as a reference to some desired object is mediated by the whole situated interaction involving mutual recognition of agency, observed glances, bodily orientations and physical relations among the actors and intended objects. There is more going on here at the group level of analysis than the coordination of individual mental representations. Deixis, pointing or reference is a fundamental cognitive function. Here, we see how it develops as primarily a phenomenon of group interaction, rather than just individual mental mechanisms.

More generally, Vygotsky concluded that cognition is mediated by language and artifacts. He developed the foundations of a theory of "mediated cognition." Cognition is not a matter of isolated mental functions that individuals develop internally, but a consequence of interaction with the social and physical world, including other people, physical artifacts and spoken language. To study such learning, one must observe early learning in real-world social settings and observe the embodied, intersubjective origin of cognition and learning. To stress the social basis of learning and cognition, we use the term "group cognition" as an alternative to the traditional focus on individual cognition.

Proposition θ: Networks of Interdependent Agents

> *In human cultures — especially advanced technological ones — cognition is mediated by writings, symbol systems, drawings, maps, external memories, computational devices, automated processes, feedback signals, and so on. Cognitive accomplishments come about due to innumerable influences, determinants, factors and considerations. The causation is not mechanical, but dependent upon the nature of the agents and their relationships. Social interactions are matters of understanding, interpretation and ambiguity. Predictions can at best be probabilistic, taking into account tendencies and trends. Understanding human/nature interactions in the Anthropocene world requires similar analysis. Like a butterfly fluttering in the breeze, an emitted CO_2 molecule reflecting a sunray does not cause a storm, but may imperceptibly contribute to its likelihood or magnitude.*

Causation can no longer be considered a simple effect of individual thoughts determining action. First, cognition increasingly takes place within tools, such as sheets of paper, charts, calculators, computer models, spreadsheet analyses. Ideas are posed, worked out, communicated and preserved in these media in ways they could not be in pure thought (Donald 2001). They are also discussed, shared, critiqued, developed and negotiated in small groups. Although people today can internalize some of these aids and alternative perspectives to take them into account to some degree in their own mind, the embodied and interrelated character of situated group cognition remains dominant.

Second, the consequences of individual human intentions and actions are not simple direct results of individual cognition. Latour (2014: 7) points out that the central military outcome in Tolstoy's presentation of *War and Peace* was not simply due to the commander's agency, but was influenced by innumerable peripheral actors. The details of a messenger's wanderings while delivering military orders, a cannonball's bouncing through the enemy's front line, a horse rearing in the calvary line are examples multiplied many times of influencing events. Latour develops a new conceptualization of causation involving potentially huge networks of actors, both human and non-human. Technological artifacts, for instance, can embody inferred human intentionality, such as a spring door closer trying to keep a door shut (Latour 1988).

Third, especially in the Anthropocene, human actions involve and affect natural phenomena. The causal relationships involved are complex and only partially understood. They may involve huge numbers of objects and intricate patterns of interaction, which are not precisely predictable. It is often not possible for people to know the ultimate consequences of their actions based on simple causal relationships; broader dependencies may have to be taken into account.

Dynamic geometry provides a workshop for exploring systems of interdependent objects, where the dependencies can be designed into constructions of multiple objects by students and then consequences of the dependencies can be observed through manipulation of the objects. This can offer a playground for groups of students to learn about the kinds of mathematical relationships that are important for understanding the contemporary world. Such cognitive models are needed in a world in which simplistic common sense is inadequate to understand our dynamic world systems.

Proposition ι: Collaborative Learning

> *The meaning of geometry propositions is a matter of shared understanding within the communities and traditions of mathematicians, articulated and preserved in their documents. Learning geometry involves acquiring the practices of discussing geometry with others, following their constructions and agreeing upon each step in deductions. Mathematics education should incorporate small-group collaborative learning, exploration,*

> *discussion and reflection, organized around the cultural artifacts of the domain.*

The design of computer software to support online collaborative learning is explored through a number of systems and experiments in *Group Cognition* (Stahl 2006). One major concern is that the notion of "meaning making" or the "negotiation of meaning" needs to be better understood. Most earlier analyses of this notion were based on theories of individual cognition, perhaps coordinated by efforts of "common grounding" (Clark and Brennan 1991). In *Group Cognition,* alternative analyses are provided of small groups adopting shared meanings of charts or mathematical problems through discourse, explicit agreement and subsequent tacit usage. The groups are shown to construct shared knowledge through interaction, much as the mother and infant built their shared meaning of the pointing gesture.

The demonstration of a need for more detailed analysis of collaborative learning in *Group Cognition* led to a decade-long research effort: the Virtual Math Teams Project (VMT). This project involved designing and iteratively improving an online environment for small groups of students to explore and discuss mathematics together. Functionality was provided for both textual dialog (chat) and diagrams (whiteboard). Teams of students were recruited through teachers and were provided with challenging mathematical problems, mainly from middle-school combinatorics and geometry curriculum.

Like the infant's pointing gesture, meanings, artifacts, actions and knowledge can be created as the group cognition of online small groups in the VMT setting. The project's collaboration software, dynamic-geometry app and sequenced curriculum provides a setting in which the interaction of the group can evolve mathematical practices. Just as the mother and infant subsequently take frequent advantage of the intersubjectively understood pointing gesture, the students can apply their shared geometry habits together and eventually even use them in individual cognition. Geometric knowledge developed in the small group is aligned with the standards of the larger mathematical community through the automated constraints and feedback of the dynamic-geometry app, questioning by other students, the embedded curriculum and teacher guidance in the encompassing classroom.

Proposition κ: Computer-Supported Teaching

> *Hosting education on networked computer devices not only allows the use of dynamic geometry apps, but can also support collaborative learning beyond face-to-face settings. This can permit many forms of automated support, such as access to online information sources and archiving of activities. Computer support must be designed to enhance individual and group cognition by people, rather than reducing their intellectual roles.*

Unfortunately, most commercial collaboration software and social media are only designed to support the expression of individual thinking and hierarchical management. They reinforce individual opinion rather than stimulating collaborative thinking. The VMT Project experimented with systems of flexible computational support for collaborative interaction, negotiation of meaning, and intersubjective consensus building. *Studying Virtual Math Teams* (Stahl 2009) includes reports of this research by about 40 academics from several countries. It motivates the project, analyzes the data of student interactions and draws implications for the science of Computer-Supported Collaborative Learning (CSCL).

An important aspect of this research is that learning is analyzed at the group level of analysis. It is studied as group cognition. There are no surveys or questions concerning individuals' ideas, reflections, representations or memories. Rather, the data for analysis of learning and knowledge building consists of automated transcripts of the small-group interactions. The VMT system is instrumented to capture all the discourse and construction that took place. The collection of reports includes examples of many approaches that were developed for analyzing this group-level data. The data of group cognition includes discourse sequences consisting of proposals, responses, questions, answers, interpretations, acceptances and other chat postings or interjections that work together to anticipate, expand upon, accept or reject each other.

The effort reported here began to define a science of group cognition and to identify the characteristics and mechanisms of small-group-level cognitive phenomena which can, for instance, contribute to the teaching and learning of mathematics. The computer technology involved in the project not only supports interaction and exploration by student groups, but also facilitates experimentation and analysis by researchers.

Proposition λ: Sedimentation of Geometric Concepts

> *The historical effectiveness of mathematical cognition requires a subtle interweaving of processes at the individual, small-group and community levels of analysis. Even a phenomenological analysis of mathematical cognition in terms of individual subjectivity stresses the centrality of intersubjective concepts and associated shared inscriptions. Conversely, the functioning of cultural traditions like Euclidean geometry requires reactivation of insight by individuals.*

In considering the "crisis of the European sciences," Husserl (1936/1989) felt impelled to investigate "the origin of geometry." As a phenomenologist, Husserl started from introspection on the experience of understanding a geometric proof and asked how an object of individual cognition like a geometric concept could become an ideal object with universally recognized meaning. He described a multi-step process of group cognition in which people collaborated using geometric inscriptions (p. 164). The insights into the necessity of proofs were "reactivated" by the individual participants as they shared the intersubjective meanings "sedimented" in their adopted mathematical language.

The VMT Project represented a systematic attempt to "translate" Euclidean geometry into a form appropriate for the Anthropocene by reactivating its meanings in settings of collaborative learning and by emphasizing the functioning of dependencies. A description of this research in *Translating Euclid* (Stahl 2013) includes chapters detailing multiple aspects of this effort, including: the project vision, history of geometry, guiding philosophy, covered mathematics, developed technology, approach to collaboration, educational research, social theory, curricular pedagogy, analysis of practice and design-based-research methodology.

At this point, the VMT Project developed a unique multiuser version of GeoGebra and integrated it into the online collaboration environment. It also iteratively tested curricula scaffolding student groups to explore the basic concepts, propositions and dependencies of Euclidean geometry. Researchers analyzed the group cognition in which meanings were negotiated, sedimented and tacitly reactivated in their group language and understanding.

Although the VMT software is designed for use by small groups of students collaborating online in real time, the research project stresses the importance of integrating support for the individual students as well as for classroom efforts in addition to the collaborative learning. Group cognition necessarily includes interpretation and contributions from individual cognitive perspectives. It also benefits from a supportive classroom context. The theory of group cognition emphasizes this integration. It recommends that small-group collaborative learning be adopted in coordination with phases of individual and classroom learning. This provides multiple opportunities, formats and processes for the sedimentation of key concepts, the reactivation of mathematical insight and the sharing of knowledge and procedures.

Proposition μ: Group Practices

> *Because learning involves a mix of tacit understanding and explicit interpretation, it is perhaps best to conceive it in terms of practices rather than mental representations. In particular, collaborative learning can be analyzed as the adoption of group practices by the small group. These practices may be derived from pre-existing society-wide cultural practices, and they may be subsequently personalized as individual practices, but they must be adopted by the small group and integrated into its activity and discourse.*

Constructing Dynamic Triangles Together (Stahl 2016) analyzes every chat posting by a particular small group of students who engaged in eight hour-long online sessions in the VMT Project using the collaborative version of dynamic geometry. Through the close analysis of their chat discourse and geometric manipulations, it becomes clear that they were collaboratively negotiating shared meanings and adopting these as group practices. About 60 distinct practices are highlighted in the analysis. Each of these is explicitly discussed in the group discourse and analyzed in the book. The variety of practices reviewed covers needs of collaborative learning, dynamic geometry, computer support, design of dependencies and online interaction, including:

- Group collaboration practices
- Group dragging practices
- Group construction practices
- Group tool-usage practices
- Group dependency-related practices
- Group practices using chat and GeoGebra actions

For each practice, the group went through a process of confronting a problem, discussing action options, agreeing on a path for going forward and then proceeding with putting the practice into action. While this initial response to a problem required explicit discussion and group agreement, subsequently the group could tacitly proceed with the adopted solution without any discussion. The practice was thereby adopted by the group and integrated into its behavior. The practice could have been derived from the larger social context, such as a teacher recommendation based on mathematical tradition or it could have been a suggestion from an individual student, but it had to go through the negotiation process by the group in order to become part of the group's effective behavior or group cognition.

While the cognitive behavior observed in the VMT Project was a mix of individual, small-group and classroom interactions, it is possible to distinguish phenomena at each of these levels of analysis, such as individual habits, group practices and classroom traditions. While it may be possible to define various other levels of analysis, these three are typical of school settings, in which individual students are graded, small groups of students may interact, and teachers orchestrate classroom activities.

Proposition v: Group Cognition

> *Human cognition is not a simple process of rational deduction that operates like the well-defined sequential operation of a computer program executing within a person's head. Rather, it often takes place in group discourse – individual abilities contribute to shared cognitions from multiple perspectives and backgrounds, within complex shared situations. Especially in instances where fundamental learning takes place, there is a mix of individual, small-group and community processes, mediated by a complex historical world of influencing factors and mediating artifacts. Articulated statements aim for future responses by building on the past context in the present situation. The analysis of group cognition in geometry education attempts to reconceptualize the nature of mathematics in minds.*

Cognition takes place expressed in explicit dialog, hidden within tacit practices and preserved in persistent inscriptions. Knowledge building is mediated by and stored in physical knowledge artifacts. These can be internalized or personalized in mental abilities and representations through memory and imagination, but they are not originally purely mental phenomena. Euclid's propositions exist in contemporary texts. Their meaning is not dependent upon the minds of Thales or

Euclid, but upon the current texts and accompanying figures, as well as upon the meanings and practices of the mathematical community today.

When a group of students collaborates on a dynamic-geometry problem in a system like VMT, their group cognition resides primarily in the shared software interface, which displays their group work, including both chat discourse and constructed figures. From observation of these traces of shared work and interaction, researchers, teachers and the participants themselves can infer negotiation of meaning and mathematical reasoning without having to appeal to assumptions about individual mental events behind the scenes. Group cognition can be persistent and observable within physical knowledge artifacts such as textual inscriptions and computer transcripts. The learning of mathematics can be studied by analysis of the development of mathematical group cognition, such as occurred by teams of students using VMT.

Group cognition is a conceptualization appropriate to the Anthropocene. Sciences and theories of the Anthropocene no longer support notions of independent organisms in environments, such as methodological individualism or even man-in-nature. They conceptualize agents as defined by intricate links, interactions and interdependencies. They focus on "complex nonlinear couplings between processes that compose and sustain entwined but nonadditive subsystems as a partially cohering systemic whole... self-forming, boundary maintaining, contingent, dynamic, and stable under some conditions but not others...not reducible to the sum of its parts, but achieves finite systemic coherence in the face of perturbations within parameters that are themselves responsive to dynamic systemic processes" (Haraway 2016: 36).

Analyses of group cognition do not consider the isolated thinker, but look at interactions among multiple agents embedded in rich worlds, especially technological systems. They unfold over time and are subject to the ambiguities of interpreting meanings in shifting historical contexts. The analysis of group cognition is a multidisciplinary undertaking; it often involves forms of conversation analysis, statistical analysis, educational psychology, semantics, video analysis, communication theory, software design, etc.

Theoretical Investigations (Stahl 2021b) brings together two dozen papers on various aspects of philosophic foundations of computer-supported collaborative learning (CSCL). Starting with a meso-level analysis of software design that looks beyond a single app to its whole technological, digital infrastructure, the book goes on to consider technology in terms of its interaction with and adoption by students. This begins to shift CSCL to the kind of science appropriate to the Anthropocene, where minds and technologies increasingly work together. Other papers reprinted from the CSCL journal consider semantic, visual, sequential, temporal and interactional aspects. A pair of studies reflects on transforming whole educational systems in Hong Kong and Singapore to feature collaborative learning.

The second half of the book presents micro-analyses of interaction data from small groups learning mathematics. It includes a wealth of examples of specific aspects of how group cognition unfolds. This includes detailed illustrations of groups constituting themselves as involved in intersubjective understanding, negotiating meaning, solving problems, adopting practices, building knowledge, crafting knowledge objects, refining terminology and learning mathematics. Case studies of problem solving show how teams conduct reconceptualization, visualization, deduction, etc. similar to that commonly performed by individuals, but now accomplished by groups. The analyses reflect the situated nature of such group cognition within shared worlds of embodied and virtual existence – structured and defined by the ongoing interaction. Both successes and limitations of group learning are showcased and evaluated.

The book includes investigations of VMT data that explicate core concepts of group cognition, such as: intersubjectivity, knowledge building, shared meaning making, negotiation of meaning, adoption of group practices, cognitive evolution, knowledge objects, referential resources, instrumental genesis and the co-experienced world. It looks at how words and digital utterances in excerpts from VMT data weave together references to terms, objects and events in the past, present and future to create intersubjective meaning and shared knowledge. Elements of the theory of group cognition emerge from these empirical analyses. Considered as a whole, the volume of investigations points toward a multi-disciplinary science that considers educational issues within a complex environment of interdependencies.

Proposition ξ: Virtual Math Teams

The Virtual Math Teams project provides an educational model for fostering group cognition of digital geometry in the Anthropocene. It developed and tested a dynamic-geometry curriculum for collaborative learning by small groups of teenage students, emphasizing the role of dependencies. This can be used as one educational component of mathematical teaching and learning, to be adapted to diverse educational settings and integrated with individual and community learning.

The VMT Project pursued a vision of students around the world learning mathematics collaboratively by communicating and exploring problems online within virtual math teams. However, it was a research effort, not scaled up for widespread classroom usage. The Covid Pandemic inspired hurried efforts around the world to provide educational resources for online pods (virtual small groups) of students in place of shuttered classrooms. Unfortunately, these transformations rarely took advantage of recent research in the learning sciences or in computer-supported collaborative learning, instead simply using business software (like Zoom) and retaining teacher-centric pedagogy carried over from the physical classroom.

To suggest how to fill the glaring educational gap, the latest version of the curriculum for the VMT Project was made publicly available on the GeoGebra website and as a free e-book: *Dynamic Geometry Game for Pods* (Stahl 2020). It includes a sequence of 50 challenges at increasing levels of expertise. The challenges are designed to stimulate the adoption of many of the group practices required by online collaborative learning of dynamic geometry and for the development of mathematical cognition generally. Each level is demanding enough to benefit from collaboration, as most students would likely get stuck without partners to figure out what was required.

The *Game*'s curriculum is initially targeted to specific practices needed for successful online collaboration and for effective use of dynamic geometry. However, it also includes open-ended challenges where the group has to define a problem, negotiate their approach as well as evaluate their solution. Some later challenges set up open-ended themes for inquiry learning (Dewey 1938/1991, Papert 1980). Then, appendices offer several suggestions of related math domains to explore (sequences of transformations; taxicab geometry, etc.).

For students who do not have access to VMT or working relations with appropriate pod-mates, options are outlined for individual study, for home schooling and for online pick-up teams. In addition, an associated article delineates a proposal for blended learning (Stahl 2021a). It proposes integrating individual, small-group and classroom activities around the game challenges. That paper is included as an appendix to the *Game* e-book.

The VMT Project developed a model CSCL approach to introducing dynamic geometry to groups of students. Extensive trials supported a design-based research effort to develop effective technology, curriculum, pedagogy, analysis and theory. The extensive reporting referenced above characterizes the development of group cognition that took place in many instances.

Conclusion

> *The Game for Pods and the VMT Project leading up to it may offer a glimpse of what could foster the development of group cognition related to dynamic geometry, including an understanding of dependencies. This can provide a CSCL model for learning and teaching mathematics in the Anthropocene.*

Geometry has been a training ground for comprehending the world since Plato and Euclid. The VMT Project explored ways of adapting computer technologies to a CSCL approach to teaching geometry. The pedagogical focus was on the development of group cognition related to analyzing and designing dependencies.

Our new epoch presents multiple challenges to mathematics education. As we have already seen with the impact of the Pandemic on schooling and the influence of climate denial on public acceptance of science, the need for and the urgency of appropriate innovations are rising rapidly. The mathematics education research community should consider how best to support learning and living in the Anthropocene.

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