How Online Small Groups Co-construct Mathematical Artifacts to do Collaborative Problem Solving

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Murat Perit Çakır

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To my parents, Güler and Nurettin Çakır
for their love, support, encouragement and patience.

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ABSTRACT

How Online Small Groups Co-construct Mathematical Artifacts to do Collaborative Problem Solving

Murat Perit Cakir
Gerry Stahl, Ph.D.

Developing pedagogies and instructional tools to support learning math with understanding is a major goal in math education. A common theme among various characterizations of mathematical understanding involves *constructing relations* among mathematical facts, procedures, and ideas encapsulated in graphical and symbolic artifacts. Discourse is key for enabling students to realize such connections among seemingly unrelated mathematical artifacts. Analysis of mathematical discourse on a moment-to-moment basis is needed to understand the potential of small-group collaboration and online communication tools to support learning math with understanding.

This dissertation investigates *interactional practices* enacted by virtual teams of secondary students as they co-construct *mathematical artifacts* in an online environment with *multiple interaction spaces* including text-chat, whiteboard, and wiki components. The findings of the dissertation arrived at through ethnomethodologically-informed case studies of online sessions are organized along three dimensions:

(a) *Mathematical Affordances*: Whiteboard and chat spaces allow teams to co-construct multiple realizations of relevant mathematical artifacts. Contributions remain persistently available for subsequent manipulation and reference in the shared visual field. The persistence of contributions facilitates the management of multiple threads of activities across dual media. The sequence of actions that lead to the construction and modification of shared inscriptions makes the visual reasoning process visible.

(b) *Coordination Methods*: Team members achieve a sense of sequential organization across dual media through temporal coordination of their chat postings and drawings. Groups enact referential uses of available features to allocate their attention to specific objects in the shared visual field and to associate them with locally defined terminology. Drawings and text-messages are used together as semiotic resources in mutually elaborating ways.

(c) *Group Understanding*: Teams develop shared mathematical understanding through joint recognition of connections among narrative, graphical and symbolic realizations of the mathematical artifacts that they have co-constructed to address their shared task. The interactional organization of the co-construction work establishes an *indexical ground* as support for the creation and maintenance of a shared problem space for the group. Each new contribution is made sense of in relation to this persistently available and shared indexical ground, which evolves sequentially as new contributions modify the sense of previous contributions.

CHAPTER 1. INTRODUCTION

Developing pedagogies and instructional tools to support learning math with understanding is a major goal in Mathematics Education (NCTM, 2000). A common theme among various characterizations of mathematical understanding in the math education literature involves ***constructing relationships*** among mathematical facts, procedures, and ideas (Hiebert & Wearne, 1996). In particular, math education practitioners treat recognition of connections among ***multiple realizations*** of a math concept encapsulated in inscriptional/graphical forms as evidence of deep understanding of that subject matter (Sfard, 2008; Healy & Hoyles, 1999). For instance, the concept of function in the modern math curriculum is introduced through its graphical, narrative, tabular, and symbolic/formulaic realizations. Hence, a deep understanding of the function concept is ascribed to a learner to the extent he/she can demonstrate how seemingly different graphical, narrative, and symbolic forms are interrelated as realizations of each other within specific problem-solving circumstances that involve the use of functions. On the other hand, students who demonstrate difficulties in realizing such connections are considered to perceive actions associated with distinct forms as isolated sets of skills, and hence are said to have a shallow understanding of the subject matter (Carpenter & Lehrer, 1999).

Reflecting on one's own actions and communicating/articulating mathematical rationale are identified as important activities through which students realize connections among seemingly isolated facts and procedures in math education theory (Sfard, 2002; Hiebert et al., 1996). Such activities are claimed to help learners notice broader structural links among underlying concepts, reorganize their thoughts around these structures, and hence develop their understanding of mathematics (Carpenter & Lehrer, 1999; Skemp, 1976). Consequently, learning in peer group settings is receiving increasing interest in math education practice due to its potential for promoting student participation and creating a natural setting where students can explain their reasoning and benefit from each others' perspectives. Nevertheless, despite its benefits suggested by math education theory, implementing small group activities in the classroom to promote learning math with understanding presents practical challenges to both students and teachers. In particular, students need to figure out ways to organize their participation in a collective problem solving activity where they will need to act in accordance with their peers. Such a transition can be difficult to make, especially in traditional math classrooms where mathematical facts are introduced as uncontestable truth, mathematical competence is assessed through individual performance, and collaboration is treated as cheating (Lockhart, 2009). In addition to this, teachers who are willing to incorporate collaborative activities in their curriculum need support for monitoring simultaneously unfolding activities of groups, making each group's work visible to other groups, and designing collaborative tasks that stimulate reflection and communication among peers about the subject matter covered in the classroom.

Recent developments in Information and Communication Technology (ICT) have enabled users across the globe to *communicate* with each other at an increasingly low cost. Forms of ICT-based communication tools such as instant messaging, chat, and social networking sites are especially popular among the youth (Lenhart et al., 2007). Moreover, most ICT environments make persistent records of interactions available for further *reflection*. Therefore, ICT offers a promising opportunity for supporting peer groups to collectively develop their mathematical understanding online and for allowing both practitioners and researchers to study the processes through which such understandings flourish. ***Computer-Supported Collaborative Learning*** (CSCL) is an emerging research paradigm in the field of Instructional Technology that investigates how opportunities offered by ICT can be realized through carefully designed learning environments that support collective meaning-making practices in computer-mediated settings (Stahl, Koschmann & Suthers, 2006).

Multimodal interaction spaces –which typically bring together two or more synchronous online communication technologies such as text-chat and a shared graphical workspace– have been widely employed in CSCL research and in commercial collaboration suites such as Elluminate and Wimba to support collaborative learning activities of small groups online. The way such systems are designed as a juxtaposition of several technologically independent online communication tools not only brings various ***affordances*** (i.e. possibilities for and/or constraints on actions), but also carries important interactional consequences for the users. Providing access to a rich set of modalities for action allows users to demonstrate their reasoning in multiple semiotic forms. Nevertheless, the achievement of connections that foster the kind of mathematical understanding desired by practitioners relies on the extent to which users can utilize these rich resources to relate their actions to each others' and produce shared understandings.

While many educational researchers recognize the potential of small-group collaboration to support learning at the individual, small-group, and classroom levels, there has been little detailed investigation of how this might take place in interaction, especially in computer-mediated settings. For instance, given the novelty of working in computer-mediated environments, students must actively develop and share methods of interaction that are appropriate for enacting the technological affordances and for collaborating on problem-solving tasks. In particular, when working on math topics, online groups must co-construct math artifacts and relate multiple realizations of a math concept in graphical, narrative, and symbolic media. If online collaborative work on math topics is to be successful, participants must find ways to (a) explore solution paths together, (b) coordinate work across different computer-based media, and (c) elaborate multiple realizations of math artifacts.

CSCL environments with multimodal interaction spaces offer rich possibilities for the creation, manipulation, and sharing of mathematical artifacts online. However, the interactional organization of mathematical meaning making activities in such online environments is a relatively unexplored area in CSCL and in Math Education. In an effort to address this gap, we have designed an online environment with multiple interaction spaces called Virtual Math Teams (VMT), which allows users to exchange textual as well as graphical contributions online. The VMT environment also provides additional resources, such as explicit referencing and special awareness markers, to help users coordinate their actions across multiple spaces. Of special interest to researchers, this environment includes a Replayer tool to replay a chat session as it unfolded in real time and inspect how students organize their joint activity to achieve the kinds of connections indicative of deep understanding of math.

This dissertation investigates how small groups *co-construct* ***mathematical artifacts*** in the VMT online environment. The study takes the math education practitioners' account of what constitutes deep learning of math as a starting point, but instead of making inferences about mental states of individual learners through outcome measures, it focuses on the *practices of collective reasoning* displayed by students through their actions in the VMT environment. Through ethnomethodological case studies, the dissertation investigates the ***interactional practices*** through which small groups of students ***coordinate*** their actions across multiple interaction spaces as they collectively construct, relate, and reason with multiple forms of mathematical artifacts to solve open-ended math problems together. In particular, the dissertation focuses on the following research question:

> How do **online** small groups of students (a) **co-construct mathematical artifacts** with **graphical, textual** and **symbolic** resources, (b) **incorporate** them into solution accounts, and (c) achieve a **shared sense** of their joint problem solving activity?

The main research question will be organized around three more specific questions each focusing on a distinct theme:

> RQ1 (Mathematical Affordances): What are the similarities and differences of the different media in VMT (text chat, whiteboard, and wiki) for the exploration and use of **mathematical artifacts**?

> RQ2 (Coordination Methods): How do groups in VMT **coordinate** their *problem-solving actions* across different interaction spaces as they co-construct and manage a shared space of mathematical artifacts?

> RQ3 (Group Understanding): How can collaborating students **build shared mathematical understanding** in online environments? How do they create math artifacts that incorporate multiple realizations?

CHAPTER 2. RESEARCH CONTEXT

2.1. The Role of Math Education in the Age of Participation

Recent advances in computing, telecommunications, and networking technologies have been transforming our society at a mind-boggling pace since the 1950s. The increasing availability of information technologies has fundamentally transformed the ways we access information and the means by which we communicate with each other. We are now living in the so-called "connected age" where physical separation between people is no longer a barrier for communication (Watts, 2003). The network infrastructure that has rendered our world smaller and smaller has made it possible to build technologies that can harness large-scale and low-cost collaboration in the virtual world. This has brought an even more astounding phenomenon, often referred to as "the age of participation," where masses of users around the globe self-organize into communities of shared interests and collaborate in exciting new ways to co-construct innovative commodities online (Tapscott & Williams, 2005). Wikipedia, Linux, the open source movement, the Human Genome Project, and R&D networks such as InnoCentive are among the most vivid products of such massive collaborative knowledge-building efforts that are transforming our scientific, economic, and social practices.

In order to keep up with the fast changing knowledge society, to benefit from the new opportunities it is bringing in, and to meet its present and future challenges, one needs to develop strong communicational, technical, and problem-solving skills in this era

(Sawyer, 2006; Hiebert et al., 1996). The acquisition of static knowledge and skills through methods of drill and practice are no longer considered to be sufficient in the knowledge society, where existing professional practices are constantly being transformed as a result of new discoveries and innovations (Bereiter, 2002). Nowadays companies and research organizations are in dire need of effective methods for managing and improving their organizational resources to catch up with the fast pace of the changing markets (Lytras & Pouloudi, 2006; Weber et al., 2006). Building innovative knowledge through collaborative team work including partners distributed across the globe, and finding means to support and sustain such processes through communication technologies have become key issues in today's high-tech, global, knowledge-based industry (Nonaka & Takeuchi, 1995; Nosek, 2004; Chi & Holsapple, 2005).

The collaborative knowledge-building work performed by teams in professional settings involves making sense of and formulating new problem areas, finding relevant information and resources about the tasks at hand, and incorporating findings into design decisions that yield to innovative solutions/products/services (Laszlo & Laszlo, 2002; Kim & King, 2004). Mathematics is one of the greatest tools that human beings have ever invented to systematically reason about the complexities of Mother Nature. Since ancient times, the influence of mathematics on various key practical aspects of human life such as trade, agriculture, navigation, astronomy, engineering, and computing has paved our way to the modern society we live in today (Wilder, 1965; Ifrah, 2001). When we consider the need for highly creative problem-solving skills in this new era, mathematics education will serve an increasingly important role in helping prospective members of the knowledge society develop such vital skills.

The fact that mathematics is still one of the major components of modern school curriculums ranging from kindergarten to college level shows that its importance is widely acknowledged in educational practice. However, recent calls for reform in mathematics education highlight the great discontent with the so-called traditional methods of math instruction. Recent publications by National Council of Teachers of Mathematics (NCTM) especially emphasize the need for developing pedagogies and instructional tools to support *learning math with understanding* (NCTM, 2000; 1991). This push is motivated by the widespread criticism of existing educational practices for treating the subject-matter as isolated knowledge chunks, and promoting the mastery of each piece through intensive drill and practice activities. The critics argue that such pedagogical approaches encourage students to memorize isolated facts and hinder the development of creative problem solving skills (Lockhart, 2009; Hiebert et al., 1996). Moreover, the teacher-centered classroom discourse is claimed to offer very little opportunity for students to actively participate in collective reasoning activities, especially when the mathematical content is normatively considered as a body of uncontestable absolute facts (Cobb & Bowers, 1999). Another recurring theme in this debate is finding effective ways of incorporating computer-based technologies into math curriculum to better support learning math with understanding. The rapid rise of these technologies and the manifold ways they are influencing the society has been increasing the pressure on policy makers and educational practitioners to employ technological innovations in math education (Cuban, 1986).

The Virtual Math Teams (VMT) Project at Drexel University aims to bring these threads together by investigating innovative uses of online collaboration tools to bring students from different parts of the world together to actively engage them in a collective mathematical discourse outside their classroom environments. The VMT project provides students an infrastructure based on popular web-based communication tools of the "age of participation". The goal of the project is to provide students an environment where they can experience collaborative problem-solving work both as members of small teams by joining VMT Chat sessions, and as members of the broader VMT online community by sharing the ideas they co-constructed during chat sessions via co-authored VMT Wiki pages. This dissertation attempts to contribute to the broader VMT research project by investigating how small teams of students enact the affordances of the VMT online environment to co-construct and reason with shared mathematical artifacts as they work on non-routine, open-ended mathematical tasks.

2.2. Review of Instructional Technology

In the past 60 years, several proposals have been made to incorporate computers in math education. In this section we will provide a brief review of these approaches with a particular emphasis on how they attempt to engage students in mathematics. This section is not intended to provide a full review of all computer-based mathematical applications. Instead, the main goal is to situate the proposed dissertation work within existing approaches for designing computer applications to support math education, and to motivate the interactional perspective it is advocating within the field of Computer-Supported Collaborative Learning (CSCL).

In an influential review of Instructional Technology (IT), Koschmann identified four broad approaches for incorporating computers in educational practice, namely Computer Aided Instruction (CAI), Intelligent Tutoring Systems (ITS), Constructive Learning Environments (Logo-as-Latin), and Computer-Supported Collaborative Learning (CSCL) (Koschmann, 1996). Koschmann contrasted these approaches in terms of their underlying theories of learning, underlying models of instruction, research methodologies, and main research questions. In this section we will narrow our focus on the mathematical applications, and extend Koschmann's analysis by considering the implicit epistemological stance of each area to mathematical practice.

This dissertation adopts the approach of CSCL. To situate this approach historically and theoretically, we will review the sequence of approaches leading up to and including CSCL. Then we will review related methodological issues and the application of CSCL to the domain of mathematics.

2.2.1. Computer-Aided Instruction (CAI)

The earliest educational applications of computers emerged in the 1960s due to the need for providing higher education to the fast growing population of USA in a cost-effective way in the post war era. The PLATO (Programmed Logic for Automatic Teaching

Operations) system developed by the Computer-based Education Research Laboratory (CERL) in Urbana, Illinois, and the TICCIT (Time-Shared Interactive Computer Controlled Information Television) system of MITRE were among the first CAI systems that were designed to deliver instructional materials to a large number of students without the need to train as many teachers (Murphy & Appel, 1978; Kulik & Kulik, 1991). The designers of earlier CAI systems were mainly concerned with hardware challenges to deliver course materials to a large number of users and to transfer paper-based instructional materials into electronic format. As a result, most of these systems came with course authoring tools such as the Tutor of PLATO and the Coursewriter of IBM to help educators to create instructional materials without the need for sophisticated programming skills. Since networking and personal computers were not widely available at that time, students would have to go to labs to use a dedicated terminal to access the course materials hosted at a central mainframe computer. These earlier systems were used to teach standard classes at the high school and college level including mathematics.

The design of the first CAI systems were influenced by B.F. Skinner’s programmed instruction theory that is based on behaviorist accounts of learning. Behaviorism has its roots in Thorndike’s law of effect and Pavlov’s work on classical conditioning, and considers learning as a stimulus-response-reinforcement process that can be systematically measured via manipulations conducted in the lab setting. According to this view, learning takes place when responses of a learner are modified or shaped through appropriate reinforcements through a process called operant conditioning (Skinner, 1968).

CAI systems that were designed along the behaviorist tradition presented course materials to learners as a sequence of small units accompanied by numerous drill and practice activities. Students are encouraged to go through this material at their own pace and revisit the modules and activities as much as necessary based on the correctness of their responses. The underlying theory predicts that students will be able to acquire each individual knowledge piece via operant conditioning, and acquire the subject-matter by assembling the pieces on their own. Since behaviorism favors precise measurement of effects, CAI research is mainly concerned with the measurement of instructional benefits of an introduced technology. Controlled experiments with pre and post tests are popularly employed to measure the instructional efficacy of CAI interventions on individual students as compared to traditional classroom teaching. Such studies generally reported positive results regarding the effects of CAI systems on students' test scores (Kulik & Kulik, 1991).

Behaviorism's distrust of non-public, mentalistic phenomena and its treatment of learning as a measurable difference in observable behavior is a consequence of the influence of positivist philosophy on the behaviorist tradition. Positivism has a realist and absolutist epistemological stance, which treats mathematics as having a real, objective existence in some ideal/platonic realm (Ernest, 1991). According to this perspective mathematical objects are claimed to be transcendental to human consciousness and reasoning. Thus, mathematics is considered as a body of knowledge that needs to be discovered or encountered by students. Educational practices that subscribe to this view are often associated with metaphors of learning such as "delivering", "receiving", and "ready made" to highlight their focus on acquisition of objective mathematical facts (Brown, 1994).

Although CAI systems are among the oldest computer applications for educational use, they are by no means abandoned. The introduction of personal computers with enhanced multimedia capabilities and the proliferation of the Internet have addressed many of the practical challenges that the designers of earlier CAI systems had to tackle. Nowadays educators have access to more sophisticated courseware tools that can incorporate animations, video, voice, and enhanced text processing into their electronic course materials (Yerushalmy, 2005). These instructional units are often organized into learning objects and shared among educators (Polsani, 2003). This opened up some new design issues such as reusability of learning objects at various educational settings, the cost of their production, finding better ways of searching relevant objects and incorporating them into course curriculums (Parrish, 2004). However, the ways students interact with the new generation CAI tools have not fundamentally changed, and the educational focus still remains to be on achieving instructional efficacy in content delivery via carefully designed course materials.

2.2.2. Intelligent Tutoring Systems (ITS)

Intelligent Tutoring Systems appeared as educational applications of the rapidly growing field of Artificial Intelligence (AI) in the 1970s (Wenger, 1987). The information processing theory that treats cognition as a computational process lays the foundation for ITS applications (Fodor, 1975; 1983). This theory considers learning as an acquisition process that reorganizes the cognitive structures through which humans process and store information. Symbolic production systems that can simulate intelligent behavior by following a set of well-defined rules are claimed to serve as explanatory tools to describe human cognitive activities (Newel & Simon, 1972). Such systems served as a basis for theoretical models of learning and cognition such as ACT-R (Anderson & Lebiere, 1998) and SOAR (Newell, 1990) that inform the design of ITS applications.

ITS systems offer improvements over CAI systems by providing more fine-grained, task-specific assistance to students as they go through a sequence of problem solving activities. ITS systems use model tracing and knowledge tracing algorithms to follow a user's problem-solving moves on the interface with respect to an idealized representation of the problem space and a repository of common student misconceptions called buggy-rules (Koedinger & Corbett, 2006). These computational resources allow the system to diagnose the student's actions, intervene with more specific hints, and estimate his/her knowledge level as he/she goes through a sequence of problems.

Some of the mathematical applications of cognitive tutors include the ANGLE Geometry Prover (Koedinger & Anderson, 1990), Cognitive Tutor Algebra, and Cognitive Tutor Geometry packages (Koedinger & Corbett, 2006). These systems offer simple forms of tutoring where students get feedback based on the actions they perform on a structured interface, such as selecting an answer for a sub-problem among a set of alternatives or entering symbols in constrained text-boxes. The main reason for these limitations is to produce machine processable content so that the automated tutor can evaluate the student's moves.

Extending simple forms of tutor-student interactions is an active area of research in ITS. Some of the recent systems attempt to incorporate natural language tutorial dialogue (e.g. Jordan, Rose & Van Lehn, 2001) by taking advantage of the recent advances in machine learning and statistical natural language processing techniques. Such systems provide hints and occasional directive help to students based on the syntactic and semantic structure of their responses to tutor-initiated questions/prompts. The tutor's prompts are based on an ontology of related concepts and action scripts pre-coded in the application, which are prepared in anticipation of specific problematic situations related to the task at hand that may have to be remedied during the course of the tutorial. Despite their limitations for stimulating new ways of thinking about the problems at hand and supporting students' own articulation of related mathematical/scientific concepts (which may deviate from standard terminology), studies show that dialogue-based ITS systems can stimulate student participation with the subject matter and elicit positive learning gains as compared to less interactive learning scenarios like lectures (Van Lehn et al., 2007).

The representative power of logic-based models in characterizing mathematical practice is of fundamental concern in ITS research. Simon and Newell's model of problem solving as a heuristic search process in a problem space heavily relies on such logical structures (Newel & Simon, 1972). Simon and Newell's theory is based on their work on knowledge-lean activities such as chess or checkers that have precisely defined rules and goal states, and that do not require subjects to have any special training or background knowledge to engage in problem solving. Such well-defined problem solving tasks allow the experimenter to model all possible moves in the problem space as a graphical structure, and to trace a player's moves from an initial state to a goal state. Such structures are claimed to be representative of the mental structures framed by subjects as they engage in problem solving work (Greeno, 2006). Simon & Newell acknowledged the challenge posed by ill-defined problems that do not have preconceived correct solutions to their theory, but they treated models devised from well-defined cases as a starting point for understanding problem solving in ill-defined cases.

The computational perspective to problem solving resonates very well with the movement of formalism or logicism among professional mathematicians, which originated in the 1900s in an effort to formulate an interpretation-free, objective, unified mathematics by deriving all mathematical results from a finite set of base axioms and logical operators. Such a logical system would render mathematical proofs routine computational procedures that can be objectively performed (MacKenzie, 2001). However, Gödel's incompleteness theorems flawed these approaches in the 1930s by showing that any logical system defined in this manner would contain undecidable propositions that cannot be deduced from the base rules (Ernest, 1991). In addition to the theoretical limits outlined by Gödel, even in the case of well-defined axiomatic systems like Euclidean Geometry, a modeling approach has to flexibly deal with manifold ways of approaching any given math problem and predicting where the problem solver is heading in an enormous problem space (Koedinger & Anderson, 1990). Consequently, such practical problems have constrained ITS designers to focus on well-defined and relatively narrow problem-solving tasks such as multi-digit addition, evaluation of algebraic expressions, or basic geometry problems where problems can be exactly modeled.

2.2.3. Constructivist Learning Environments (Logo-as-Latin)

CAI and ITS approaches summarized above mainly favor the metaphor of learning by transmission of knowledge, where the system designers' goal is to design tasks and content that will help students to acquire new knowledge. Constructivist learning environments such as Logo and its modern versions (e.g. Microworlds, StarLogo) follow a different approach by actively engaging students to build their own computer models by using simplified programming languages (Papert, 1980). In these environments students learn by teaching the computer how to perform certain tasks by composing simple instructions. For instance, in the case of Logo one typical task is to instruct a turtle character to draw a circle on the screen as a combination of horizontal, vertical, and rotational moves (Kafai, 2006). Since programs are executable entities, in contrast to other standard learning materials, students can observe the consequences of their actions immediately, reflect on their instruction set, make adjustments, and run their models again.

The design of constructivist learning environments is informed by an epistemological perspective that can be traced back to Piaget's theories of knowledge construction in developmental psychology (Piaget, 1971). Constructivism has a relativistic view towards knowledge. This perspective claims that "knowledge is not passively received but actively built up by the cognizing subject" where "…the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (von Glasersfeld, 1989, p. 182). Thus, research on constructivist learning environments focuses on how programming tasks yield to cognitive self-organization. Cognitive benefits of an environment are experimentally studied in terms of the degree of transfer that an individual exhibits between related tasks (Koschmann, 1996).

Constructivism treats mathematical concepts as subjectively constructed[1] mental objects, and hence it is antithetical to Platonism, which places mathematical entities outside the consciousness of an individual (Brown, 1994; Ernest, 1991). Constructivism has a big influence on contemporary math education theory. Existing theories of mathematical concept formation put special emphasis on cognitive processes such as reification (Sfard & Linchevski, 1994; Dubinsky, 1991; Gray & Tall, 1994), reflective abstraction, and assimilation/accommodation (Piaget, 1971), through which individuals are claimed to construct mathematical knowledge. Such processes involve transformation of activities performed in specific problem-solving situations (e.g., the process of addition) into mental objects (e.g., the concept of sum) that have a structure/unity in their own right, and hence can be invoked as a resource to tackle future encounters with related math problems (Dörfler, 2002). In other words, a particular mathematical process is considered

[1] This view resonates with the intuitionist movement in mathematics, which was partly motivated by the challenges Cantor's theory of infinite sets posed to formal logic. Intuitionists consider proofs as a sequence of justification steps rather than a series of mechanistic truth statements, and focus on justifiably constructed statements. This perspective is famous for its rejection of proof by contradiction as a viable proof method.

to be reified into an object when "…the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations." (Cotrill et al., 1996, p. 171).

In addition to Logo, mathematical packages such as ISETL, Derive, Matlab, and Mathematica are frequently used to encourage students to construct and explore mathematical objects in new ways (Dubinsky, 2000). These packages render even the most highly abstract mathematical objects tangible and available for further manipulation. Students are often required to learn how to construct such objects by learning how to write scripts in a special language. Recent software packages that allow visual construction of objects via virtual manipulatives, such as GeoGebra, Geometer's Sketchpad, and Cabri, offer more simplified interfaces and powerful representational capabilities without requiring students to learn a specific programming language (Moyer et al., 2002; Suth et al., 2005). These systems allow students to experiment with geometric constructions and explore their properties inductively (e.g., see if a value remains invariant when the geometric construction is manipulated). Nevertheless, since students may misconceive the illustrations constructed in these environments as formal proofs, it's been also argued that caution needs to be observed in order not to hinder students' development of deductive reasoning skills (Oner, 2008).

2.2.4. Computer-Supported Collaborative Learning (CSCL)

Despite the differences between their epistemological stances and approaches to instructional design, the three perspectives mentioned earlier consider the individual learner as the unit of interest and employ similar experimental methods originating from psychological research. As a relatively recent approach to IT design, CSCL differs from earlier approaches with its focus on computer-mediated social interactions among collectivities. In particular, CSCL is "…centrally concerned with meaning and practices of meaning making in the context of joint activity and the ways in which these practices are mediated through designed artifacts" (Koschmann, 2002). Since the focus is on the social practices of meaning making with computers in both online and face-to-face settings, CSCL research draws upon several disciplines such as linguistics, sociology, social psychology, communications, and anthropology (Koschman, 1996). CSCL also builds on the technological innovations and research methods of Computer-Supported Cooperative Work (CSCW), which has been investigating how groupware applications can be designed to support knowledge sharing and coordination among both virtual and conventional teams in organizations since the 1960s (Grudin, 1994).

2.2.4.1. Social Theories of Learning

Research on collaborative learning predates both CSCL's inception in the late 1980s and the introduction of micro-computers (Stahl, Koschmann & Suthers, 2006). Three main theoretical orientations in collaborative learning research are particularly influential on contemporary CSCL research; socio-constructivist theories that extends Piaget's program of cognitive development (Doise & Mugny, 1984), Soviet socio-cultural theories of learning and development (Vygotsky, 1930/1978), and the theory of situated cognition and learning (Brown, Collins & Duguid, 1989; Winograd & Flores, 1986; Lave & Wenger, 1991; Suchman, 1987).

Neo-Piagetian socio-constructivist research focuses on individual development in the context of social interaction. This perspective emphasizes socio-cognitive conflict for its role in generating breakdowns in cognitive states of individuals, and triggering learning through negotiation of subjective perspectives. In other words, according to this theory learning is achieved through the joint construction of a new state of cognitive equilibrium (i.e., common ground) through productive conflict resolution (Cobb, 1994; Dillenbourg et al., 1995). Social interaction is considered as a background for individual activity or as a context of co-operation through which individuals reorganize their cognitive structures in response to socially provided information (Rogoff, 1998). Hence, the focus of interest is still the individual learner. In socio-constructivist research experimental designs with jigsaw tasks that aim to introduce conflict between dyads who are at the same developmental stage (usually measured with parameters such as age and achievement scores) are popularly employed. Pre/post test measures are used to account for each individual's development as a consequence of their cooperation. Dyads at different developmental stages are not favored on the grounds that establishing equilibrium would be difficult for such pairs due to the projected asymmetry between their cognitive structures (Baker, 2002). In other words, a level of cognitive development is considered to be a necessary pre-condition to enable learning (i.e. construction of new cognitive structures in relation to prior ones). This implicit assumption has been consequential on the way modern school curriculums are structured, where lesson planners aim to introduce concepts that are presumed to be appropriate for the developmental stages associated with certain age groups.

In contrast, socio-cultural theories consider cognitive development as a process driven by an individual's culturally and historically mediated interactions with other members of the society. An individual's participation in cultural activities under the guidance of adults or more capable peers is claimed to result in the internalization of the *tools-for-thinking* shared by that culture. In particular, Vygotsky proposed the concept of *zone of proximal development* to account for the skills that novices can display in the context of socially organized, collaborative activities with more capable peers that they cannot accomplish on their own (Vygotsky, 1930/1978). This phenomenon poses a challenge to the neo-Piagetian camp, because the socio-constructivist perspective suggests that such acts would not be within the reach of the learner as he/she has not developed the necessary cognitive structures yet. Thus, on the contrary to the neo-Piagetian position, socio-cultural theory reverses the relationship between learning and development. Learning is not treated as a byproduct of innate developmental stages associated with internal processes like biological maturation that is presupposed in the Piagetian perspective. Instead, learning is claimed to occur at the intersubjective plane first, and to be located within the activities mediated by cultural and historical artifacts. Therefore, instead of assuming a strict separation between intra-psychological (i.e. thinking) and inter-psychological (i.e. language) phenomena, socio-cultural perspective stresses the dialectical relationship among those phenomena (Vygotsky, 1934/1986).

The situated-learning perspective re-specifies learning as an enculturation process, where individuals develop a sense of social identity among members of a community of practice through their participation in that community's socio-cultural activities (Brown, Collins & Duguid, 1989; Lave & Wenger, 1991; Wenger, 1999). This perspective relocates learning from individual minds to participation frameworks through which newcomers gradually demonstrate increasingly competent conduct as they assume more central positions in a community of practice (Hanks, 1991). In other words, learning is characterized as a process of becoming a member through active participation in a *knowledge-building discourse,* rather than passive acquisition of ready-made domain knowledge (Scardamalia & Bereiter, 1996). Broadening the analytic focus from individuals to situated activities of collectivities in an effort to study learning and cognition implies that one needs to transcend the individual as the sole unit of interest as it is traditionally employed in educational psychology. In particular, post-cognitive theories incepted as a response to the limitations of the cognitive approach to account for social phenomena consider units such as the cultural-historical activity (Leont'ev, 1981; Kaptelinin & Nardi, 2007; Cole & Engestrom, 1993), actor-networks (Latour, 2005), distributed cognitive systems (Hutchins, 1995; Pea, 1993), or small group interactions (Stahl, 2006; Schegloff, 2006) to investigate the participation frameworks in which learning is achieved.

Social theories that motivate CSCL have also informed contemporary theories of mathematical knowledge and practice. In particular, socio-constructivist theories of learning have extended the constructivist theories (cf. section 2.2.3 above) by highlighting the effects of social interaction on the development of mathematical concepts by individuals (Ernest, 1993). This perspective asserts that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge (Ernest, 1998). This point is vividly demonstrated in Lakatos' Proofs and Refutations (1976), where he reviewed the sequence of historical events that led to the development of a proof for a mathematical theorem known as Euler's characteristic formula[2] as a fictional discussion among a teacher and his students. In his book, Lakatos portrayed the proofs offered by several prominent mathematicians (voiced by various actors in his scenario) for the general case of the theorem and their subsequent refutations that limit the scope of the proof claims by offering counter-examples. The refutations did not dismiss the proof statements entirely, but refined them as applicable to rather specific cases. In short, Lakatos put forward the argument that even mathematics evolves through a discourse of proofs and refutations. In other words, mathematics is seen to advance as a field of inquiry through a knowledge-building discourse (Scardamalia & Bereiter, 1996).

Social constructivism treats mathematical results as social facts (e.g., the Pythagorean Theorem) that can be verified with respect to a specific interpretation framework in which the result is situated (e.g., Euclidean Geometry). In other words, the truths of mathematics are defined by social agreement – shared patterns of behavior – on what constitute acceptable mathematical concepts, relationships between them, and methods of deriving new truths from old (Wittgenstein, 1944/1978; Ernest, 1991). Therefore, learning mathematics can be viewed as an enculturation process where students become

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[2] Euler's formula states that for any convex polyhedron, the sum of the number of vertices (V) and faces (F) is exactly two more than the number of edges (E) (i.e. symbolically V-E+F=2). See http://www.ics.uci.edu/~eppstein/junkyard/euler/ for a list of known proofs.

participants in a discourse of proofs and refutations (Sfard, 2000; Dorfler, 2002; Cobb et al., 1997; Meira, 1995; Meira, 1998).

Socio-cultural studies of mundane mathematical reasoning have also contributed to a reinterpretation of mathematics as a form of social practice accomplished through the use of artifacts within specific social contexts to address practical problems. Ethnographic case studies of arithmetic competencies demonstrated by Brazilian street vendors (Nunes et al., 1993), Nepalese shopkeepers (Beach, 1995a), and grocery shoppers (Lave, Murtaugh & de la Rosa, 1984) show that the context in which humans engage in problem solving work and the tools they employ inform their mathematical reasoning. For instance, Brazilian street vendors who were able to competently perform arithmetical calculations to find the sum of various groupings of their products were not able to perform equally well when similar problems were presented to them in the language of formal schooling. The authors explained this difference based on the available cultural tools for problem solving and the way they were interpreted by their users in that context.

2.2.4.2.Evolution of Research Methods for Studying Collaborative Learning

Recent reviews of collaborative learning research identify three stages in the field's history of methodological development, namely (a) the effects paradigm (b) the conditions paradigm, and (c) the interactions paradigm (Dillenbourg et al., 1995; Webb & Palincsar, 1996; Cohen. 1994; Baker, 2002). Initial efforts in collaborative learning research treated collaboration as a black box, and attempted to measure its *effects* on learning via controlled experiments. These studies brought conflicting results, most of which favored collaborative learning over individual learning, but revealed little insight about its nature. Later on, the focus of the field shifted from measuring the effects of collaboration to identifying the main *conditions* under which effective collaboration can be observed. For that purpose several variables that were hypothesized to predict effective collaboration such as group size, task types (e.g., jigsaw designs), group composition (e.g., pairs at same/different developmental levels), and gender were studied. However, these variables turned out to be interacting with each other in complex ways, which made it difficult to design experimental studies that can single out the effects of a given variable and hence aid the interpretation of statistical outcomes.

Recently, alternative methods focusing on the micro-level, moment-to-moment details of *interactions* have been proposed as an alternative to the experimental methods of psychological tradition (Barron, 2000; Sawyer, 2006; Stahl, Koschmann & Suthers, 2006). These studies draw upon discourse analytic and conversation analytic traditions[3] in social sciences, use actual recordings of interactions (e.g. video recordings, computer logs) as data instead of solely focusing on exam scores, and attempt to characterize important patterns in student-student and teacher-classroom interactions. Instead of starting with a preconceived notion of what effective collaboration is and focusing on external measures indicative of it, the new methodological approaches focus on understanding how collaborative learning is *done* as an interactional achievement of collectivities such as small groups or classrooms through ***case studies*** of moment-by-

[3] Despite their focus on interaction, discourse analysis and conversation analysis differ in terms of their epistemological stance with respect to social action. More details about these differences are provided in the methodology section, 3.4.

moment interactions (e.g., Roschelle, 1996; Roschelle & Teasley, 1995; Stahl, 2006; Koschmann, Stahl & Zemel, 2007; Koschmann & Zemel, 2006).

The complicated nature of social interactions both at the small group level and at the classroom level motivated not only the use of naturalistic inquiry (Lincoln & Guba, 1985) methods for analytic purposes, but also the development of iterative approaches to instructional design as part of longitudinal efforts for incorporating pedagogies based on collaborative team work in classroom settings (Cobb et al., 2003). Such iterative approaches in educational research are referred to as Design-Based Research (DBR), where researchers continuously modify their tasks and interventions to facilitate and sustain collaborative knowledge building in the classroom and/or online. Due to this approach's success in investigating learning both at individual and small group levels, DBR has become a prominent methodology for educational research and instructional software development (Barab, 2006).

2.2.4.3. CSCL Applications in Math Education

There are only a few studies that exclusively focus on the use of CSCL applications in the context of math education. Two of these studies involve Nason & Woodruff (2004) and Moss & Beatty's (2006) work that focus on the use of Knowledge Forum to support mathematical knowledge-building activities of math classes in grade levels 4 and 6 respectively. Knowledge Forum is one of the pioneering CSCL applications that embrace the knowledge-building perspective, where students participate in a collective discourse by putting forward theories, and discuss each others' contributions in an asynchronous discussion forum (Scardamalia & Bereiter, 2006). Both studies report that collaborative activities mediated by the Knowledge Forum enhanced and enriched the authenticity of students' mathematical activity and their understanding of mathematical practice as a field of social inquiry. Moreover, Moss & Beatty reported that students were able to inspect multiple ways of approaching the problems they worked on, realize structural similarities between the problems, and improve their initial conjectures and explanations by reflecting on other postings.

In another mathematical application of CSCL, Shaffer (2002) explored the possibility of applying the design-studio model of instruction used by architects to teaching geometry. In this study, students collaboratively constructed special geometric designs by using the Geometer's Sketchpad tool in a design-studio environment. Students were given design tasks that would require them to incorporate geometric concepts such as curvature, rotation, reflection, fractal recursion, etc. into their designs. Students got feedback on their projects during class presentations, expert evaluations, and peer-reviews from teammates. The study reported that participants developed significant transformational geometry skills at the end of the workshop as a result of the reflective and progressive nature of the interactions facilitated by the design-studio model.

Another important application of CSCL in the classroom context is the use of networked handheld devices to facilitate classroom discussion. A recent case study by Ares (2008) focuses on the use of a traffic simulator called Participatory Simulation (Wilensky & Stroup, 1999) in a secondary math classroom. This system allows students to control the behavior of individual traffic lights and observe the impact of their collective behavior on the simulated traffic pattern. The system also provides graphical renderings of average speed, number of stopped cars, and average waiting time. This allows the class to reason with multiple representations and reflect on the consequences of their aggregated actions. Networked systems such as Participatory Simulation or Group Scribbles allow teachers and students to dynamically monitor how the shared task is conceived by the collectivity and give the chance to the teacher to modify the course of the discussion based on emerging outcomes made visible by the system (Roschelle, Patton & Tatar, 2007). The necessity of collecting input that can be aggregated constrains the tasks that can be supported by existing collaborative classroom simulation systems. For example, existing systems support sharing of answers to multiple-choice questions, inputs to traffic lights, or basic diagrams/calculations. However, the increasing ubiquity of mobile platforms, the decreased turn-around time for assessment (as compared to traditional assessment tools such as homeworks/quizzes) and the affordances for incorporating multiple modalities for joint sense making have contributed to the success of these CSCL applications in the classroom context.

In short, existing-math-education related studies in CSCL focus on (a) the use of computational resources to support face-to-face collaboration as part of classroom activities and (b) the use of asynchronous communication tools to mediate mathematical discussion with chiefly textual resources together with static graphical resources. Hence, the possibility of supporting collaborative mathematical problem-solving activities with synchronous online communication technologies is a largely unexplored area in CSCL. In the context of the broader VMT project, this dissertation attempts to address this gap by investigating the affordances of such online environments for supporting collaborative math problem-solving activities. In the next section we will present some arguments regarding the potential of employing synchronous online communication tools to support collaborative learning of mathematics.

2.2.5. Summary of IT Research

The goal of this review is to situate CSCL among other approaches in Instructional Technology that incorporate computers in educational settings. As it is summarized in Table 2.2.1 below, which is adapted from Koschmann (1996, p. 16), there is diversity in the field in terms of approaches to instructional design, research methods, pedagogies, and epistemological perspectives. The ontological and epistemological status of mathematics has been the subject of a 2500 years old controversy in philosophy and mathematics. This dissertation does not aim to settle this historic debate in any way. Instead of going through an *either-or* type argument (Dewey, 1938/97) among existing positions to argue who is right or wrong, it will be more fruitful to focus on what kind of mathematical activities these systems are designed to support and to reflect on their benefits and limitations with respect to the goals math education practitioners aim to achieve.

Table 2.2.1: Summary of Instructional Technology

IT Type	***Theory of Learning***	***Model of Instruction***	***Epistemological Stance w.r.t. Math Objects***	***Research Issue***	***Research Methods***
CAI	Behaviorism	Programmed instruction / instructional design	Platonism	Instructional efficacy	Pre/post test design, with a focus on individuals
ITS	Information processing theory of human cognition	One-on-one tutoring	Logic based modeling, formalism	Instructional competence	Pre/post test design with more fine-grained measures, with a focus on individuals
Logo-as-Latin	Cognitive constructivism	Discovery based learning	Intuitionism, subjective constructions of individuals.	Instructional transfer	Protocol analysis (think -aloud sessions), design evaluations, pre/post test design, with a focus on the development of an individual across different modeling activities
CSCL	Knowledge building, situated learning, social-constructivism	Collaborative learning	Fallibilism, intersubjectivity, social co-construction in situ	Instruction as enacted practice	Design-based research, discourse analysis, conversation analysis, focus on social interaction and practices within collectivities

When we focus on the kinds of activities each paradigm offers to students to support their learning, it is evident that there is a common interest for designing systems that can engage students with the subject matter in interactive ways. CAI and ITS paradigms try to achieve this goal by building instructional behavior (e.g. teacher-led presentations and one-on-one tutoring respectively) into software based on the expectation that students will acquire the knowledge presented to them by interacting with multi-media enhanced representations at their own pace and reflecting on the hints they will get from the system. However, these systems limit students' involvement with the subject matter by following a strict sequence of activities and allowing certain kinds of inputs for the sake of producing machine-processable data. Hence, students do not have much option besides going through a computer-led interaction by reviewing predefined materials and hints programmed into the system. Strong adherence to the curricular sequence leaves very little room for students to be creative and critical about the subject matter they are mastering. The systems are shown to be effective in helping students to develop competencies in arithmetic and algebra in accordance with the curriculum standards, yet the way these systems are designed makes it difficult for students to creatively make use of what they learn in authentic problem-solving contexts, which is fundamentally important in mathematical practice. In short, these systems amplify the problems associated with the standardized curricular approach in teaching mathematics, which is critiqued for its failure to capture the practices that make mathematics meaningful for its practitioners (Lockhart, 2009).

In comparison, constructivist environments offer much richer forms of interaction with the subject matter by allowing students to construct and test their own theories/models of math concepts in the system. Since these models are executable, constructivist systems are more responsive to student input. Yet, such systems often require students to have a much higher level of computer literacy (e.g., how to program scripts, how to use spreadsheets, etc.), so that they can explore the subject matter by creatively constructing models in those environments. Developing such skills is undoubtedly important in an increasingly computerized world, but without appropriate social scaffolding such difficulties and learning curves involved can prevent students from productively engaging with the subject matter. As demonstrated in Shaffer's study (2002), the affordances of such environments for creative engagement of mathematics is perhaps realized better when they are incorporated in a collaborative setting rather than in situations where individuals interact with these tools in isolation.

Instead of engineering student-computer interactions that would mimic productive social interaction, CSCL systems aim to support groups of students to collaboratively explore and make sense of instructional artifacts. Strong inclination towards collaborative learning neither means that all forms of collaboration are equally productive, nor renders CSCL incommensurable with previous approaches. On the contrary, studying how productive collaboration can be elicited and supported with computers is a fundamental research issue in CSCL. Moreover, a CSCL environment can incorporate constructivist tasks or learning objects as part of its design (e.g., Shaffer, 2002). Indeed, most contemporary ITS and constructivist systems are now investigating ways to incorporate small groups in their environments (Noss & Hoyles, 2006; Gweon et al., 2006; McLaren et al., 2005), which shows that there is a strong interest in the field of learning sciences for designing software support for collaborative learning activities.

In the mathematics education community there is a strong interest in establishing classroom cultures in which students analyze and evaluate the mathematical thinking and strategies of others, communicate mathematical thinking coherently and clearly to peers, and formulate and investigate mathematical conjectures (NCTM, 2000). As the studies that focused on the use of Knowledge Forum in math education demonstrated, CSCL environments provide students a medium where they can discuss mathematical arguments with each other outside the typical initiation-reply-evaluation[4] (IRE) organization of teacher-centered classroom discourse, where students expect the teacher to deliver the correct answer at the end (Macbeth, 2004). In contrast, in a collaborative peer group setting students have to decide by themselves whether a mathematical argument they produced or came across holds or not. As we will demonstrate in our case studies, such instances may initiate *episodes of interaction* where proof-like arguments are co-constructed that demonstrate how the proposed answer was derived from what was initially available to the group. Such *solution-accounts* or explanatory proofs are extremely important for an educational program that aims to support math learning with understanding, since such explanations achieve more than demonstrating the correctness of a result by explicitly spelling out the relationships between relevant mathematical objects (Hanna, 1990; Hersh, 1993). Another important advantage of most CSCL environments is the possibility of keeping *persistent records of interactions* in the system for future use. This opens the possibility for students to revisit and build upon what they have accomplished before and observe what others did for similar tasks. Moreover, such records provide valuable data for researchers to analyze interaction as it unfolds naturally during group work (e.g., in contrast to using think-aloud protocols). CSCL environments also have the potential to transcend classroom activities by allowing collaboration between students from different schools and even from different countries. Such activities may bridge and cross-fertilize forms of mathematics practiced in different cultural contexts.

[4] IRE sequences are "…organized by the understanding that teachers routinely know the answers to their questions, and that this is understood by everyone else in the room, whether those others know the answers or not." (Macbeth, 2004, p. 704).

Finally, as our review suggests, the potential of CSCL systems that offer *synchronous communication tools* for supporting collaborative math problem-solving activities is a relatively unexplored area. In addition to all the advantages listed above, focusing on synchronous interactions can provide researchers and practitioners a window into the interactional processes through which students achieve (or fail to achieve) the kinds of reasoning indicative of learning math with understanding. Synchronous tools offer persistent records of collaborative problem-solving activities like asynchronous tools, yet the sense of co-presence established in synchronous environments afford a higher level of interactivity among students as compared to an asynchronous setting where students exchange worked-out solutions. As we will demonstrate in our case studies, synchronous tools can make students' reasoning process available to each other and to the researhers for analysis. Face-to-face interactions mediated by simulations offer a similar lens provided all the relevant actions (talk, gesture, body orientations, drawings, eye gaze etc.) are recorded and transcribed, which is a challenging undertaking. Thus, synchronous collaboration tools provide a promising intermediary platform to support and study interactional organization of collaborative problem-solving activities.

To sum up, what makes CSCL distinctive among other approaches to Instructional Technology design is its emphasis on supporting communication, argumentation and creativity among collectivities with software. Our intention is not to recommend abandonment of other approaches, but to provide opportunities for practitioners to facilitate activities in which students can creatively discuss and explore mathematics by making use of what they have learned and mastered in their classrooms. This dissertation study is an attempt to *contribute to the existing line of inquiry by investigating how small groups of students co-construct mathematical artifacts and collaboratively incorporate them into solution accounts in a CSCL environment* that offers a combination of synchronous and asynchronous communication tools.

2.3. Review of CSCL Studies of Multimodal Interaction

In the previous section we stated that this dissertation adopts the approach of CSCL. In this section we will elaborate on some of the central topics and methodological issues in CSCL related to this dissertation. We will review a representative sample of influential CSCL studies that focus on problem-solving[5] interactions mediated by synchronous online environments comparable to VMT. The review will summarize some of the analytical resources and techniques common to these studies, and highlight methodological limitations associated with them. The goal of this review is to motivate the ethnomethodological case-study approach adopted in this dissertation, which takes the interactional practices observed at the small-group level as its analytical focus.

2.3.1. The Problem of Social Organization in CSCL

A central issue in the theory of collaborative learning is how students can solve problems, build knowledge, accomplish educational tasks and achieve other cognitive accomplishments together. How do they share ideas and talk about the same things? How

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[5] Although none of these studies specifically focus on mathematical tasks, the tasks incorporate comparable reasoning and argumentation elements as compared to the math tasks we used during VMT sessions.

do they know that they are talking about, thinking about, understanding and working on things in the same way? Within CSCL, this has been referred to as the problem of the "attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995), "building common ground" (Baker et al., 1999; Clark & Brennan, 1991) or "the practices of meaning making" (Koschmann, 2002). Our research group at Drexel University has been interested in this issue for some time: (Stahl, 2006) documents a decade of background to the VMT research reported here: chapter 10 (written in 2001) argued the need for a new approach and chapter 17 (written in 2002) proposed the VMT Project, which includes this dissertation study. During the past six years, we have been studying how students in a synchronous collaborative online environment organize their interaction so as to achieve intersubjectivity and shared cognitive accomplishments in the domain of school mathematics.

Knowledge building in CSCL has traditionally been supported primarily with asynchronous technologies (Scardamalia & Bereiter, 1996). Within appropriate educational cultures, this can be effective for long-term development of ideas by learning communities. However, in small groups and in many classrooms, asynchronous media encourage exchange of individual opinions more than co-construction of progressive trains of joint thought. In particular, Hewitt's study (2005) identified (a) the temporal separation between asynchronously posted messages, (b) the single-pass reading strategy widely employed by students to deal with the large content of online discussions and (c) the tendency to read and respond to most recent posts only, as the three main factors that hinder convergence of asynchronous online discussion threads around topics of collective interest. Through case studies, we have found that synchronous interaction can more effectively promote what we term "group cognition"—the accomplishment of "higher order" cognitive tasks through the coordination of contributions by individuals within the discourse of a small group (Stahl, 2006).

In CSCL settings, interaction is mediated by a computer environment. Students working in such a setting must enact or invent ways of coordinating their understandings by means of the *technological affordances* that they find at hand. The development and deployment of these methods is not usually an explicit, rational process that is easily articulated by either the participants or analysts. It takes place tacitly, unnoticed, taken-for-granted. In order to make it more visible to us as analysts, we have developed an environment that makes the coordination of interaction more salient and captures a complete record of the group interaction for detailed analysis. In trying to support online math problem solving by small groups, we have found it important to provide media for both linguistic and graphical expression. This resulted in what is known within CSCL as a dual-interaction space. In our environment, students coordinate their text chat postings with their whiteboard drawings. A careful analysis of how they do this reveals as well their more general methods of social organization.

This dissertation will employ ethnomethodological case studies to investigate collaborative learning in the VMT dual-interaction online environment. In order to motivate our methodological approach we will review other approaches to dual-interaction spaces by important recent CSCL studies. The analytic thrust of these studies is to arrive at quantitative results through statistical comparisons of aggregated data. To accomplish this, they generally have to restrict student actions in order to control variables in their studies and to facilitate the coding of student utterances within a fixed ontology. We fear that this unduly restricts the interaction, which must be flexible enough to allow students to invent unanticipated behaviors. The restrictions of laboratory settings make problematic experimental validity and generalization of results to real-world contexts. Even more seriously, the aggregation of data—grouping utterances by types or codes rather than maintaining their sequentiality—ignores the complexity of the relations among the utterances and actions. According to our analysis, the temporal and semiotic relations are essential to understanding, sharing and coordinating meaning, problem solving and cognition. While quantitative approaches can be effective in testing model-based hypotheses, they seem less appropriate both for exploring the problem of interactional organization and for investigating interactional methods, which we feel are central to CSCL theory.

In the following section we will review studies of dual-interaction spaces in the CSCL literature in terms of their methodological orientation, underlying theoretical background and software features. In the light of the common themes we identify across these studies we will argue that we need to conduct systematic case studies exploring the ways participants organize their interaction across multimodal interaction spaces in order to see how groups work on more open-ended tasks in less restricted online environments.

2.3.2. Approaches in CSCL to Analyzing Multimodal Interaction

Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text chat and a shared graphical workspace—have been widely used to support collaborative learning activities of small groups (Dillenbourg & Traum, 2006; Jermann, 2002; Mühlpfordt & Wessner, 2005; Soller & Lesgold, 2003; Suthers et al., 2001). The way such systems are designed as a juxtaposition of several technologically independent online communication tools carries important interactional consequences for the users. Engaging in forms of joint activity in such online environments requires group members to use the technological features available to them in methodical ways to make their actions across multiple spaces intelligible to each other and to sustain their joint problem-solving work.

In Chapter 4 we will empirically document and describe some of the methods enacted by group members as they coordinate their actions across multimodal interaction spaces of a particular learning environment to collaboratively work on math problems. In this section we first review existing studies in the CSCL research literature that focus on the interactions mediated by systems with multimodal interaction spaces to support collaborative work online. We have selected sophisticated analyses, which go well beyond the standard coding-and-counting genre of CSCL quantitative reports, in which utterances are sorted according to a fixed coding scheme and then statistics are derived from the count of utterances in each category. Our review is not meant to be exhaustive, but representative of the more advanced analytical approaches employed. Unlike the simple coding-and-counting studies, the approaches we review attempt to analyze some of the structure of the semantic and temporal relationships among chat utterances and workspace inscriptions in an effort to get at the fabric of common ground in dual-interaction online environments.

The communicative processes mediated by multimodal interaction spaces have attracted increasing analytical interest in the CSCL community. A workshop held at CSCL 2005 specifically highlighted the need for more systematic ways to investigate the unique affordances of such online environments (Dillenbourg, 2005). Previous CSCL studies that focus on the interactions mediated by systems with two or more interaction spaces can be broadly categorized under: (1) prescriptive approaches based on models of interaction and (2) descriptive approaches based on content analysis of user actions.

The prescriptive approach builds on the content-coding approach by devising models of categorized user actions performed across multimodal interaction spaces. We look at two examples:

(a) Soller & Lesgold's (2003) use of Hidden Markov Models and

(b) Avouris et al's (2003) Object-oriented Collaboration Analysis Framework.

In these studies the online environment is tailored to a specific problem-solving situation so that researchers can partially automate the coding process by narrowing the possibilities for user actions to a well-defined set of categories. The specificity of the problem-solving situation also allows researchers to produce models of idealized solution cases. Such ideal cases are then used as a baseline to make automated assessments of group work and learning outcomes.

The descriptive approach informed by content analysis also involves categorization of user actions mediated by multimodal interaction spaces, applying a theoretically informed coding scheme (Neuendorf, 2002). Categorized interaction logs are then subjected to statistical analysis to investigate various aspects of collaborative work such as:

(c) The correlation between planning moves performed in chat and the success of subsequent manipulations performed in a shared workspace (Jermann, 2002; Jermann & Dillenbourg, 2005),

(d) The relationship between grounding and problem-solving processes across multiple interaction spaces (Dillenbourg & Traum, 2006),

(e) A similar approach based on cultural-historical activity theory (Baker et al., 1999), and

(f) The referential uses of graphical representations in a shared workspace in the absence of explicit gestural deixis (Suthers, Girardeau & Hundhausen, 2003).

(a) Soller and Lesgold's modeling approach involves the use of Hidden Markov Models (HMM) to automatically detect episodes of effective knowledge sharing (Soller & Lesgold, 2003) and knowledge breakdowns (Soller, 2004). The authors consider a programming task where triads are asked to use object-oriented modeling tools to represent relationships among well-defined entities. The task follows a jigsaw design where each group member receives training about a different aspect of the shared task before meeting with other members. The group sessions are hosted in the Epsilon online environment, which includes a text-chat area and a shared workspace. The workspace provides basic shapes that allow users to diagrammatically represent entities and relationships. Participants are required to select a sentence opener to categorize their contributions before posting them in the chat window. The authors manually extract segments from their corpus where each member gets the opportunity to share the unique knowledge element he/she was trained in with other group members. Some of these episodes are qualitatively identified as ideal cases that exemplify either an instance of effective knowledge sharing or a knowledge breakdown, completely based on the results of post-tests. For instance, a segment is considered an effective knowledge-sharing episode provided a chance for demonstrating the unique knowledge element comes during the session, the presenter correctly answers the corresponding questions in both pre- and post-tests, and the explanation leads at least one other member to correctly answer the corresponding question(s) in the post-test. The sequence of categorized actions (including chat postings and workspace actions) that correspond to these ideal cases is used to train two separate HMMs for the breakdown and effective knowledge sharing cases, respectively. An HMM computes the probability of a certain kind of action immediately following another; it thus captures certain aspects of sequentiality. These models are then used to automatically classify the remaining episodes and to assess team performance. However, the method is seriously limited to recognizing connections among actions to those based on immediate sequences of codes. While this can capture adjacency pairs that are important to conversation, it misses more distant responses, interrupted adjacency pairs, temporal markings and semantic indexes. The authors apparently make no specific distinction between workspace and chat actions as they build their HMMs over a sequence of interface actions. Moreover, the relationship between object diagrams constructed in the workspace and the explanations given in chat do not seem to be considered as part of the analysis. Hence, it is not clear from the study how a successful knowledge-sharing episode is achieved in interaction and whether the way participants put the affordances of both interaction spaces into use as they explain the materials to each other have had any specific influence on that outcome. Although they were reported to be successful in classifying manually segmented episodes, HMMs computed over a sequence of categorized actions seem to obscure these interactional aspects of the coordination of chat and workspace.

(b) The modeling approach outlined in Avouris et al. (2003) and Komis et al. (2002) proposes a methodology called the Object-oriented Collaboration Analysis Framework (OCAF) that focuses on capturing the patterns in the sequence of categorized actions through which dyads co-produced objects in a shared task space. The collaborative tasks the authors used in their online study included the construction of database diagrams with well-defined ontological elements such as entities, relationships and attributes. In this problem-solving context the final representation co-constructed in the shared workspace counted as the group's final solution. The OCAF model aims to capture the historical evolution of the group's solution by keeping track of who contributed and/or modified its constituent elements during the course of an entire chat session. The authors not only consider direct manipulation acts on specific elements but also chat statements through which actors propose additions/modifications to the shared diagram or agree/disagree with a prior action. The chat and drawing actions are categorized in terms of their functional roles (e.g., agree, propose, insert, modify, etc.). The mathematical model includes the sequence of categorized actions and the associations among them. The model is then used to gather structural properties of interactions (e.g., how contributions are distributed among dyads, what functional role each contribution plays) and to trace how each action performed in the interface is related to other actions. This modeling approach differs from similar approaches in terms of its specific focus on the objects co-constructed in the shared workspace. The model captures the sequential development of the shared object by keeping track of the temporal order of contributions made by each user. However, it is not clear from the study how the model could deal with the flexibility of referential work. For instance a chat posting may refer to multiple prior postings or to a sub-component of a more complicated entity-relationship diagram by treating several elemental objects as a single object. In other words, a model trying to capture all possible associations between individual actions in a bottom-up fashion may miss the flexibility of referential work and obscure the interactional organization.

(c) Jermann (2002) employs a coding scheme to study the correlation between planning moves in the chat area and the success of subsequent manipulations performed on the shared simulation in the Traffic Simulator environment. The shared task involved students tuning red-green periods of four traffic lights in the simulation to figure out an optimal configuration to minimize the waiting time of cars at intersections. The workspace could be manipulated in specific ways by users. The workspace also includes a dynamic graph that shows the mean waiting time for the cars. The goal of the task is to keep the mean value below a certain level for two minutes. The study included additional experimental cases where dynamically updated bar charts are displayed to provide feedback to users about their level of participation. The logs of recorded sessions are coded in terms of their planning and regulatory content. The nature of the task allowed authors to numerically characterize different types of work organizations in terms of the distribution of manipulations performed on four possible traffic lights. The authors complement this characterization with number of messages posted, number of manipulations done and the types of messages as captured in the coding scheme. The study reported that dyads who coordinated their actions across both interaction spaces by planning what to do next (i.e., task regulation) and discussing who should do what (i.e., interaction regulation) in chat before manipulating the simulation performed better (i.e., achieved the objective more quickly). The interaction meters were not reported to have significant effects on promoting task and interaction regulation. The work of high performance groups are characterized with phrases like "posted more messages," "more frequent postings," "talked relatively more than they executed problem solving actions," "monitor results longer," "produced elaborated plans more frequently" in reference to the tallied codes, frequency of messages and duration of activity. Although the main argument of the chapter highlights the authors' interest in sequential unfolding of regulatory moves, the way the employed quantitative approach isolates and aggregates the actions obscures the temporal connections and sequential mechanisms constituting different forms of regulation moves.

(d) Dillenbourg & Traum (2006) employ a similar methodology to study the relationship between grounding and problem solving in an online environment including a shared whiteboard and a text-chat area. In this study the participants were grouped into dyads and asked to collaboratively work on a murder-mystery task. The authors framed their analysis along the lines of Clark & Brennan's (1991) theory of grounding (at least applied at the micro level of individual utterances) and theories of socio-cognitive conflict. The study reports two kinds of uses of the dual spaces to facilitate grounding during problem solving: a "napkin" model and a "mockup" model. The authors hypothesized that the whiteboard would be mainly used to disambiguate dialogues in the chat window via basic illustrations (i.e., the napkin model). However, the authors report that the dyads used the whiteboard for organizing factual information as a collection of text boxes, and the chat component was mainly used to disambiguate the information developed on the whiteboard (i.e., the mockup model). The authors attributed this outcome to the nature of the task, which required users to keep track of numerous facts and findings about the murder case, and the difference between the two media in terms of the persistency of their contents. Since participants organized key factual information relevant to the problem at hand on the shared whiteboard during their experiments, the authors attributed a shared external memory status to this space and claimed that it facilitated grounding at a broader level by offering a more persistent medium for storing agreed upon facts. The study succeeds in highlighting the important role of medium persistence, even if it does not specify the methods by which students exploited such temporal persistence.

(e) Baker et al. (1999) provide a theoretical account of collaborative learning by bringing together the processes of grounding and appropriation from psycholinguistics and cultural-historical activity theory (CHAT), respectively. In their study they focus on the interactions mediated by the C-Chene software system where dyads are tasked to co-

construct energy models that account for storage, transfer and transformation of energy (Baker & Lund, 1997). The models for energy-chains are constructed in a shared workspace that allows the addition of annotated nodes and directed edges. Participants also have access to a chat area that can be customized with sentence openers, which are claimed to promote reflective contributions, reduce typing effort and minimize off-task discussion. The interface is designed to allow only one user to produce a contribution in a given interaction interval. The users need to press a button to switch between dual interaction spaces. Hence the possibility of parallel or overlapping work (e.g., one user drawing on the board as the other is typing a message) is ruled out on the grounds that this would hinder collaboration. The dyads also could not overlap in typing since they need to take turns to use the dialog box where they type their messages. However, it is possible for a user to interrupt his/her partner through a special prompt, which asks whether it is okay to take the turn. If the partner agrees, then the turn is passed to the other user. The study reported that dyads who used the structured interface exhibited more reflective and focused discussion. The authors point to limitations involved with constraining user actions to fixed categories, but they argued that some of the sentence openers they used correspond to generic speech acts that were used for multiple purposes in the course of interaction.

(f) Suthers et al. (2003) investigate the referential uses of shared representations in dyadic online discourse mediated by the Belvedere system. This environment has a chat area as well as a shared workspace where dyads can co-construct evidence maps to represent their arguments as a set of categorized textboxes linked to each other (Suthers et al., 2001). The study compares face-to-face and online cases to investigate how dyads use the system as a conversational resource in each case as they work on a shared task that involves developing hypotheses about the spreading of a disease at a remote island. Categories for deictic uses such as finger pointing, cursor-based deixis, verbal deixis and direct manipulation of objects are identified and applied to the session logs. Based on the distributions of these categories for each case, the authors report that dyads in the online case made use of verbal deixis and direct manipulation of shared objects to compensate for the limitations of the online environment to achieve referential relationships across dual interaction spaces. Moreover, the study reports that such referential links are more likely to be observed between temporally proximal actions. For instance, a chat posting including a deictic term is likely to be read in relation to a node recently added to the shared representation.

Our review of relevant work in the CSCL literature highlights some common threads in terms of methodological approaches and theoretical orientations. First, the studies we have reviewed all focus on the group processes of collaboration, rather than treating the group context as a mere experimental condition for comparing the individuals in the groups. Second, all studies employ a content-coding approach to categorize actions occurring in multiple interaction spaces. In most cases, representational features like sentence openers or nodes corresponding to specific ontological entities are implemented in the interface to guide/constrain the possibilities for interaction. Such features are also used to aid the categorization of user actions. The categorization schemes are applied to recorded logs and subjected to statistical analysis to elicit interaction patterns.

Despite the accomplishments of these studies, we find that their approaches introduce systematic limitations. Interactional analysis is impossible because coherent excerpts from recorded interactions are excluded from the analysis itself. (Excerpts are only used outside of the analysis, to introduce the features of the system to the reader, to illustrate the categorization schemes employed or to motivate speculative discussion). Moreover, most studies like these involve dyads working on specific problem-solving contexts through highly structured interfaces in controlled lab studies in an effort to manage the complexity of collaboration. The meanings attributed by the researchers to such features of the interface need to be discovered/unpacked by the participants as they put them into use in interaction—and this critical process is necessarily ignored by the methodology. Finally, most of the papers are informed by the psycholinguistic theory of common ground, and are unable to critique it thoroughly.

2.3.3. The Unit of Analysis

For methodological reasons, quantitative approaches generally (a) constrain subject behaviors, (b) filter (code) the data in terms of operationalized variables and (c) aggregate (count) the coded data. These acts of standardization and reduction of the data eliminate the possibility of observing the details and enacted processes of unique, situated, indexical, sequential, social interaction (Stahl, 2006, ch. 10). An alternative form of interaction analysis is needed to explore the organization of interaction that can take place in CSCL settings.

In this dissertation we also focus on small-group interactions mediated by multimodal interaction spaces. However, our study differs from similar work in CSCL by our focus on groups larger than dyads whose members are situated outside a controlled lab environment, and by our use of open-ended math tasks where students are encouraged to come up with their own problems. Moreover, we do not impose any deliberate restrictions on the ways students access the features of our online environment or on what they can say. Our main goal is to investigate how small groups of students construe and make use of the "available features" of the VMT online environment to discuss mathematics with peers from different schools outside their classroom setting. In other words, we are interested in studying interactional achievements of small groups in complex computer mediations "in the wild" (Hutchins, 1995).

Our interest in studying the use of an online environment with multiple interaction spaces in a more naturalistic use scenario raises serious methodological challenges. In an early VMT study where we conducted a content analysis of collaborative problem-solving activities mediated by a standard text-chat tool in a similar scenario of use, we observed that groups larger than dyads exhibit complex interactional patterns that are difficult to categorize based on a theory-informed coding scheme with a fixed/predetermined unit of analysis (Cakir et al., 2005). In particular, we observed numerous cases where participants post their messages in multiple chat turns, deal with contributions seemingly out of sequence and sustain conversations across multiple threads that made it problematic to segment the data into fixed analytic units for categorization. Moreover, coming to an agreement on a code assignment for a unit that is defined a priori (e.g., a chat line) turned out to be heavily dependent upon how the unit can be read in relation to resources available to participants (e.g., the problem description) and to prior units. In other words, the sense of a unit not only depends on the semantic import of its constituent elements, but also on the occasion in which it is embedded (Heritage, 1984). This often makes it possible to apply multiple categories to a given unit and threatens the comparability of cases that are labeled with the same category. More importantly, once the data is reduced to codes and the assignments are aggregated, the sequential relationships among the units are lost. Hence, the coding approach's attempt to enforce a category to each fixed unit without any consideration to how users organize their actions in the environment proved to be too restrictive to adequately capture the interactional complexity of chat (Zemel, Xhafa & Cakir, 2007). Moreover, the inclusion of a shared drawing area in our online environment made the use of a theory-driven coding approach even harder due to increased possibilities for interaction. The open-ended nature of the tasks we use in our study makes it especially challenging to model certain types of actions and to compare them against ideal solutions.

The issue of unit of analysis has theoretical implications. In text chat it is tempting to take a single posting as the unit to be analyzed and coded, because a participant defined this as a unit by posting it as a message and because the chat software displays it as a visual unit. However, this tends to lead the analyst to treat the posting as a message from the posting individual—i.e., as an expression of a thought in the poster's mind, which must then be interpreted in the minds of the post readers. Conversation analysis (Sacks, 1962/95; Schegloff, 2006) has argued for the importance of interactions among participants as forming more meaningful units for analysis. These consist of multiple utterances by different speakers; the individual utterances take each other into account. For instance, in a question/answer "adjacency pair" the question elicits an answer and the answer responds to the question. To take a pair of postings such as a question/answer pair as the analytic unit is to treat the interaction within the group as primary. It focuses the analysis at the level of the group rather than the individual. As we just discussed, in online text chat, responses are often separated from their referents, so the analysis is more complicated. In general, we find that the important thing is to trace as many references as possible between chat postings and whiteboard actions in order to analyze the interaction of the group as it unfolds. As we will see in Chapter 4, it is through the co-construction of a rich nexus of such references that the group weaves its shared understanding.

Relatedly, the notion of common ground as an abstract placeholder for registered cumulative facts or pre-established meanings has been critiqued in the CSCL literature for treating meaning as a fixed/denotative entity transcendental to the meaning-making activities of inquirers (Koschmann, 2002). The common ground that supports mutual understanding in group cognition or group problem solving is a matter of semantic references that unfold sequentially in the momentary situation of dialog, not a matter of comparing mental contents (Stahl, 2006, pp. 353-356). Committing to a reference-repair model (Clark & Marshall, 1981) for meaning making falls short of taking into account the dynamic, constitutive nature of meaning-making interactions that foster the process of inquiry (Koschmann et al., 2001).

Given these analytical and theoretical challenges, we opted for an alternative to the approaches reviewed above that involve modeling of actions and correct solution paths or treating shared understanding as alignment of pre-existing individual opinions. In Chapter 3 we will present an alternative approach informed by insights from Ethnomethodology that focuses on the sequence of actions in which participants co-construct and make use of *semiotic resources* (Goodwin, 2000) distributed across dual interaction spaces to do collaborative problem-solving work. In particular, we will focus on the organization of activities that produce graphical drawings on the shared whiteboard and the ways those drawings are used as resources by actors as they collaboratively work on an open-ended math task. Through detailed analysis of excerpts, we will investigate how actions performed in one workspace inform the actions performed in the other, and how participants coordinate their actions across both interaction spaces. The affordances of the chat and whiteboard spaces will be investigated by documenting the methods enacted by participants to address these interactional matters by using various features of the VMT system.

CHAPTER 3. RESEARCH DESIGN

In this chapter we will first introduce the institutional context in which this dissertation study was conducted. Secondly, we will present the VMT system and the data collection procedure. Finally, we will describe the ethnomethhodological conversation analysis (EM/CA) methodology that we will employ in Chapter 4 to investigate our research questions. The methodology section will begin with a brief historical background to related studies of interaction. Next, we will demonstrate how we appropriated this methodology by presenting an EM/CA analysis of an excerpt obtained from a VMT Chat session. Through this short case study we will also introduce the main analytical concepts contributed by this dissertation, which will be elaborated further in Chapter 4 and will be used to facilitate the discussion of our research questions in Chapters 5 and 6.

3.1. Institutional Context: The Math Forum @ Drexel

The dissertation was conducted as part of the Virtual Math Teams (VMT) Project at Drexel University. The VMT project is an NSF-funded research program through which an interdisciplinary group of researchers from the College of Information Science & Technology, the Math Forum, the Department of Culture & Communication, and the School of Education investigates innovative uses of online collaborative environments to support effective K-12 mathematics learning. The VMT project aims to extend the existing services of the Math Forum to solicit active participation of students to collaboratively discuss math problems and to share their findings with other members of the Math Forum online community.

The Math Forum (http://mathforum.org) has established itself as a leading interactive and inquiry-informed online resource center for improving math learning, teaching and communication since it was founded in 1992 (Renninger & Shumar, 2002). Today the Math Forum hosts over one and a half million pages of resources for the service of an online community that encompasses math students, teachers, parents, professional mathematicians, and enthusiasts. The website receives on average 800,000 visits per month from users across the globe and the content maintained by the Forum is continuing to grow with the contributions of its members. Math Forum offers several services and resources such as:

1. *Ask Dr. Math:* Through this service students can reach expert mentors to solicit help regarding their homework problems and other math related needs. The frequently asked questions section of the service provides an organized archive of selected mentor-student interactions as a resource for other Math Forum users. Mentors guide students either by pointing at related Ask Dr. Math interactions in the archive or by making suggestions for extending what the student has done so far in an effort to stimulate his/her mathematical thinking.
2. *Problem of the Week:* This service provides a new math problem in every 2 weeks through the school year in four broad categories, namely Math Fundamentals, Pre-algebra, Algebra, and Geometry based on the standard math curriculum. These questions are designed to stimulate thinking about the underlying

mathematical concepts rather than rote applications of formulas. Students are encouraged to submit their solutions via PoW's submission system. Each submission gets a comment from the mentors and selected solutions are published on the Math Forum's PoW digital library.

3. *Write Math with the Math Forum* is a standards-aligned program that involves problems selected from the PoW archive based on the specific math curricula set by school districts. As in the case of PoW, students are encouraged to practice writing solutions and to reflect on their problem solving and presentation strategies.
4. *Math Tools* is a community library of technology-based, interactive resources such as Java applets, spreadsheets, dynamic geometry software, and graphing calculators. Users share insights, teaching strategies, and classroom experiences to help each other use these technologies successfully, work through implementation issues, and quickly find tools that may work well in their classrooms.
5. *Teacher2Teacher* is a peer-mentored question-and-answer service operated by volunteer math educators around the world, which mainly serves as a support network through which teachers and parents discuss general issues about teaching math, educational practices etc.
6. *Teacher Exchange* is a service through which teachers share lesson plans for various units and levels.
7. *Math Internet Library* is a searchable, indexed, and annotated library that provides quick access to thousands of math and math education related resources on the web.

In short, Math Forum is both (a) a content site with vast amounts of math information and links to other math-related resources on the web, and (b) an interactive platform promoting interaction, exchange of ideas, discussion, and community building (Renninger & Shumar, 2004). Indexed and annotated links to resources and archived interactions of the community members render the Math Forum one of the richest digital libraries devoted to math education in the world.

3.2. VMT Project: A Design-based Approach to CSCL System Development

Since its inception as an online service, the interactions between the members of the Math Forum online community have been facilitated via asynchronous communication technologies such as email and threaded discussion boards. The VMT project has been developing a new online service for the Math Forum that provides additional interactive resources for students to meet and collaborate with other fellow students across the globe on math problems online. In this section we will describe the design-based research approach adopted by the project to design pedagogical and software support for collaborative learning of mathematics online.

As we mentioned in Section 2.2.4.2 above, conducting experimental studies to investigate how technological and instructional innovations contribute to learning at individual and collective levels presents challenges to educational researchers. In our literature review we highlighted the recent shift of focus in collaborative learning research from experimental studies of variables that predict effective collaboration towards close studies of interactional processes in which participants engage in joint meaning-making activities. Such micro-level processes are embedded in broader contexts of joint activity (e.g., participation norms in a classroom) where individual and collective ways of knowing are reflexively related and emerge together (Cobb, 1994; Cobb & Bowers, 1999). Design-Based Research (DBR) emerged out of the need for addressing this level of complexity by incrementally improving educational interventions and the theory informing their design in naturalistic contexts (Brown, 1992; Collins, 1992). The DBR method follows a *theory-in-action* (Dewey, 1938/1997) perspective that involves progressive improvement of instructional and technological interventions through iterative design cycles (Barab, 2006). In each cycle the intervention is improved further based on a close analysis of the user's experiences with the intervention in *naturalistic contexts*. The DBR Collective (2003, p. 5) describes the focus of DBR methodology as follows:

> Design-based research methods focus on designing and exploring the whole range of designed innovations: artifacts as well as less concrete aspects such as activity structures, institutions, scaffolds, and curricula. Importantly, design-based research goes beyond merely designing and testing particular interventions. Interventions embody specific theoretical claims about teaching and learning, and reflect a commitment to understanding the relationships among theory, designed artifacts, and practice. At the same time, research on specific interventions can contribute to theories of learning and teaching.

Through a sequence of iterative, user-centered design studies in compliance with the DBR perspective, the VMT project has investigated various synchronous online communication tools such as America Online's Instant Messenger, Babylon, and ConcertChat to gather requirements for the kinds of support mechanisms virtual math teams may need.

Table 3.2.1 Design phases of the VMT service

Software Platform	**Math Activity**	**Number of Sessions / Participants**	**Key Software Features**	**Overall Observations on Media Affordances and Co-construction of Mathematical Artifacts**
AOL IM Chat (PowWow 2004)	Problem of the Week (PoW)	19 sessions with 19 different teams Group size: 2-5	1. Text based chat	Absence of shared drawings became an issue especially during geometry PoW sessions. Students occasionally exchanged static drawings with the help of the facilitator. Yet, groups were able to collaboratively discuss math despite the limitations of the tool.
Babylon (2004)	Ask Dr Math SAT Math questions	9 sessions with 9 different teams Group size: 2-4	1. Text based chat 2. Shared Whiteboard	The shared whiteboard turned out to be very useful to co-construct quick shared drawings, but coordinating chat with the drawings required some effort. Drawings were not editable due to bitmap-based design. Deleted drawings could not be recovered.
Concert Chat (VMT Chat v1) (Spring Fest 05)	Taxicab Geometry	18 sessions with 5 teams Group size: 2-5 Each team participated in 4 successive sessions	1. Text based chat 2. Shared Whiteboard 3. Explicit Referencing	References were occasionally used to link postings to drawings and to previous postings. References used more often in larger groups. Students explored a geometric space different from the one they were accustomed to. Teams were encouraged to come up with their own questions, and take up on other group's questions. Groups co-constructed very sophisticated graphical representations, yet some of them had hard time with moving beyond the familiar objects of Euclidean geometry.
VMT Chat v2 (Spring Fest 06)	Pattern Problems	19 sessions with 5 teams Group size: 2-4 Each team participated in 4 successive sessions	1. Text based chat 2. Shared Whiteboard 3. Explicit Referencing 4. Basic Lobby 5. Basic Wiki Support 6. Math markup support	Pattern problems included geometric and algebraic aspects. Most groups used whiteboard and chat in a well coordinated way to illustrate their solutions. Wiki postings did not reflect most of the details discussed in chat. Only one group worked on a problem posted by another group on the wiki. So the use of wiki for posting summaries was limited. Yet, the summarization task stimulated discussions on what the group had achieved in each session.
VMT Chat v3 (Spring Fest 07, Brazil, Singapore, Rutgers)	Probability, Combinatorics, Pre-calculus	Sessions are ongoing[6]	1. Text based chat 2. Shared Whiteboard 3. Explicit Referencing 4. Extra Tabs (browser, summary, wiki view) 5. Math markup support 6. Advanced wiki support 7. Lobby with enhanced social networking support	We project to observe more activity in the wiki space since it is integrated with VMT Chat now. A dedicated tab is created to encourage groups to put together a summary of their findings by reusing their chat content. A couple of probability problems and possible strategies to approach them are posted on the wiki together with worked out examples. Students are expected to work on the problems of their choice by considering different strategies, and share their results with other groups by posting their summaries on the wiki.

[6] The list of ongoing sessions can be viewed at http://vmt.mathforum.org/vmtChat/vmtRoomList.jsp

Table 3.2.1 above provides a summary of the five major design phases the VMT project has gone though since 2004. The table compares the main features of the online environments we used at each phase, and summarizes the tasks and the number of sessions we conducted. Each phase includes 1 to 2 hours long online problem-solving sessions featuring teams of 3-6 student volunteers among Math Forum users. Data collection has been completed under the VMT project's IRB approval with the participation of middle and high school students. After each design phase the underlying CSCL system that hosted prior VMT sessions is improved with new technological and pedagogical features in an effort to address some of the interactional challenges users experienced while they were engaged with collaborative problem-solving online. During these sessions students collaboratively work on non-routine math problems selected from the Math Forum's problem pool. The pedagogical models encourage discourse and explanation of solutions as well as posing new questions and inspecting other group's work.

Through these design studies we not only improved the overall system but also refined our data collection and analysis methods. For instance, our experiments with chat tools including a pixel-based shared whiteboard like Babylon presented difficulties in terms of analyzing relationships between drawings and text, because the system did not allow us to log the production of the drawings. We had to resort to screen captures of the sessions to keep track of the evolution of the whiteboard. We have also experimented with the chat application offered by Blackboard, which includes an object-oriented whiteboard allowing users to manipulate drawings after they are produced. Despite its advanced drawing features this tool also presented us analytical challenges, because the contents of the whiteboard are only included in the logs when one of the users decide to take a snapshot. For that reason, when we eventually started transforming the ConcertChat platform into VMT Chat, we implemented a Replayer tool that allows us to reconstruct a chat session from its log file and replay it at various speeds. Since participants are not located in a lab environment (they join our sessions from different parts of the world), it was not possible for us to video the screens of each user. Although the Replayer does not reveal us all the activity performed at individual terminals, such as mouse clicks or the composition of chat messages, it captures all the actions that change the shared state of the system and hence are viewed by all the group members. Thus, the Replayer offers an innovative solution to the analytical challenges associated with online data collection by offering an acceptable tradeoff between bulky screen recordings and flat log files. Since this dissertation is chiefly concerned with the coordination of actions across whiteboard and chat spaces, the case study presented in Chapter 4 focuses on the replayable data we obtained at phase 4. The data analyzed in this dissertation is taken from VMT Spring Fest 2006 (phase 4), and it's findings contributed to the re-design of phase 5.

3.3. The VMT Online Collaborative Learning Environment

Our most recent iteration of the VMT service now includes a chat-based communication tool called *VMT Chat* (see Figure 3.3.1 below), an integrated wiki component adapted from Media Wiki called *VMTWiki* (see Figure 3.3.2 below), and a basic web portal called

VMT Lobby (see Figure 3.3.3 below) that provides support for social networking and coordination of activities across the chat and the wiki spaces.

The VMT Chat tool has been developed in close collaboration with Fraunhofer's Integrated Publication and Information Systems Institute (IPSI) in Darmstadt, Germany. The system has two main interactive components that conform to the typical layout of systems with dual-interaction spaces: a shared whiteboard that provides basic drawing features on the left, and a chat window on the right. The online environment has features specifically designed to help users relate the actions happening across dual-interaction spaces.

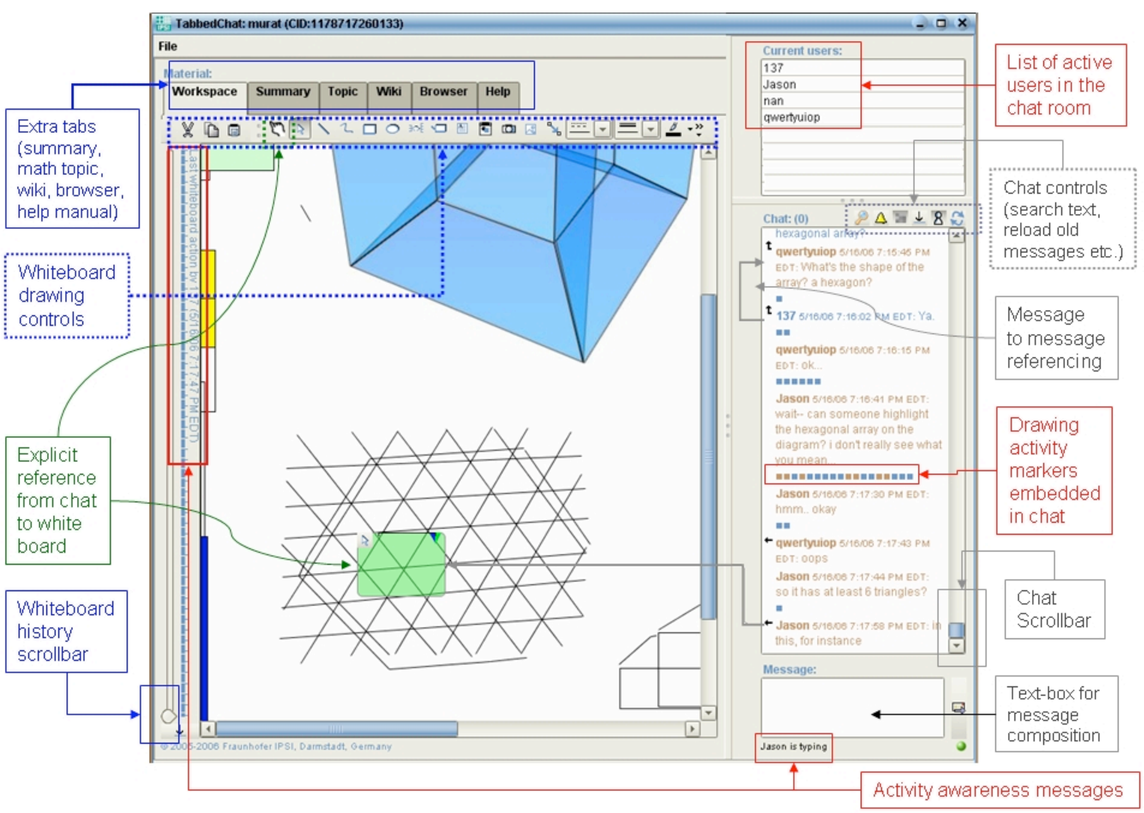


Figure 3.3.1: The VMT Chat Environment (as of Spring 2007)

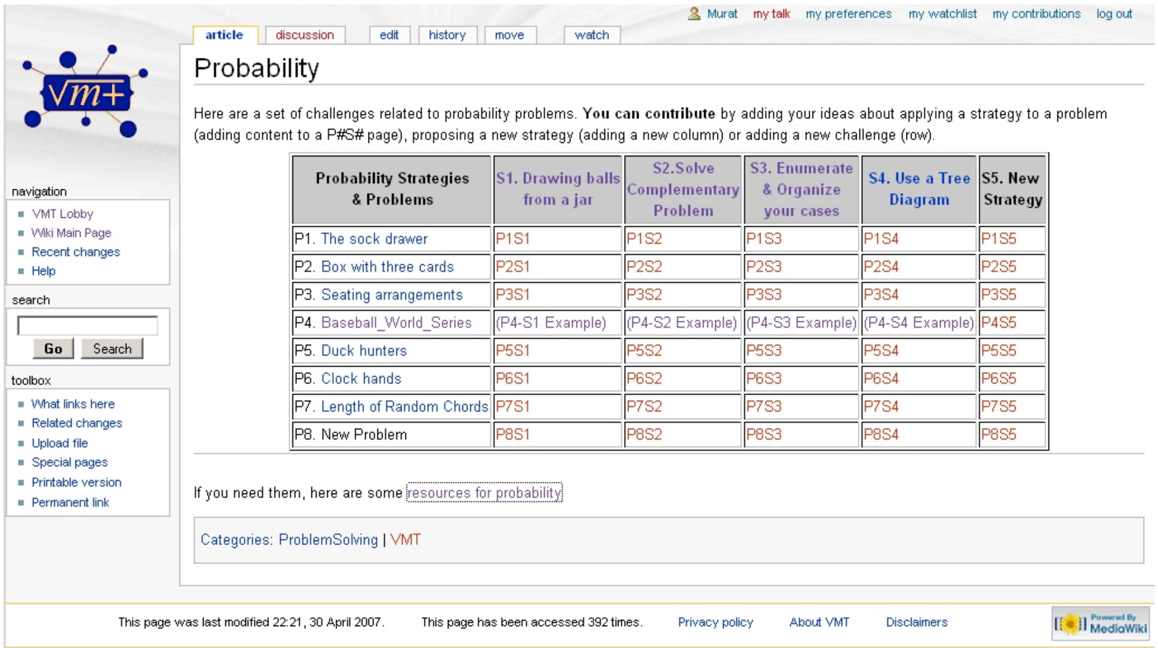


Figure 3.3.2: The VMT Wiki environment

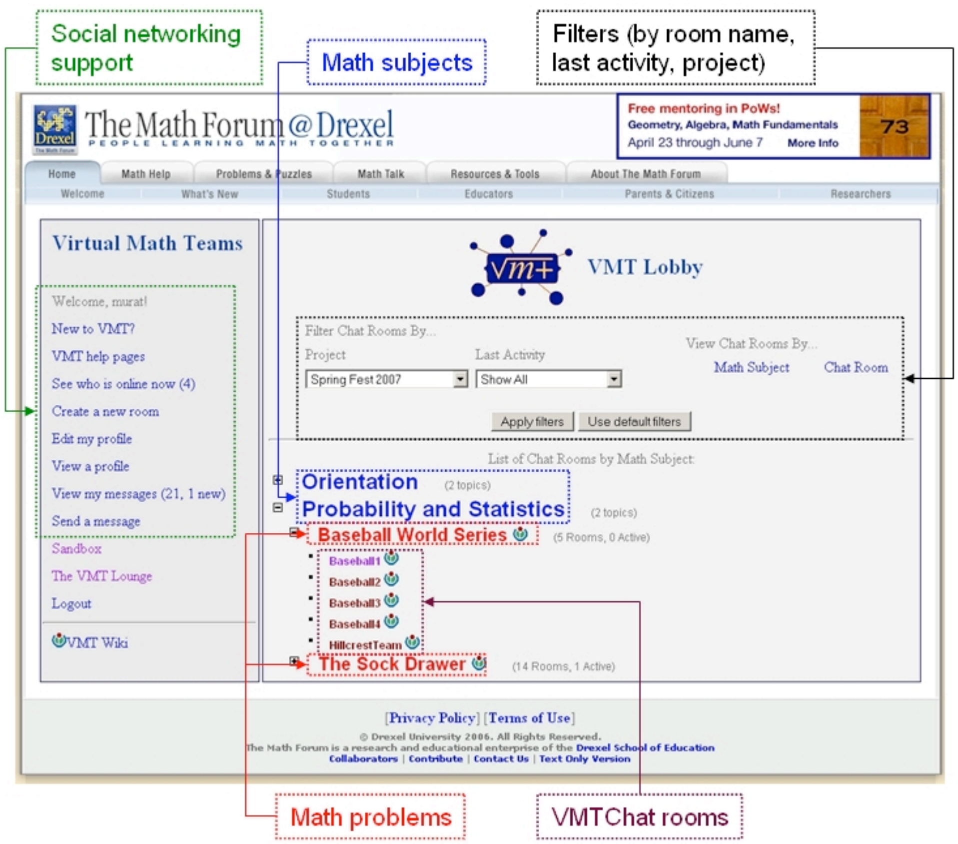


Figure 3.3.3: The VMT Lobby

One of the unique features of the VMT chat system is the *referencing* support mechanism (Mühlpfordt & Wessner, 2005; Mühlpfordt & Stahl, 2007), which allows users to visually connect their chat postings to previous postings or to objects on the whiteboard via arrows. For example, Figure 3.3.4 below illustrates a *message-to-message* reference, whereas Figure 3.3.5 shows a *message-to-whiteboard* reference. The referential links attached to a message are displayed until a new message is posted. Messages including referential links are marked with an arrow icon in the chat window (e.g., see nan's chat posting in Figure 3.3.4). A user can see where such a message is pointing at by clicking on it.

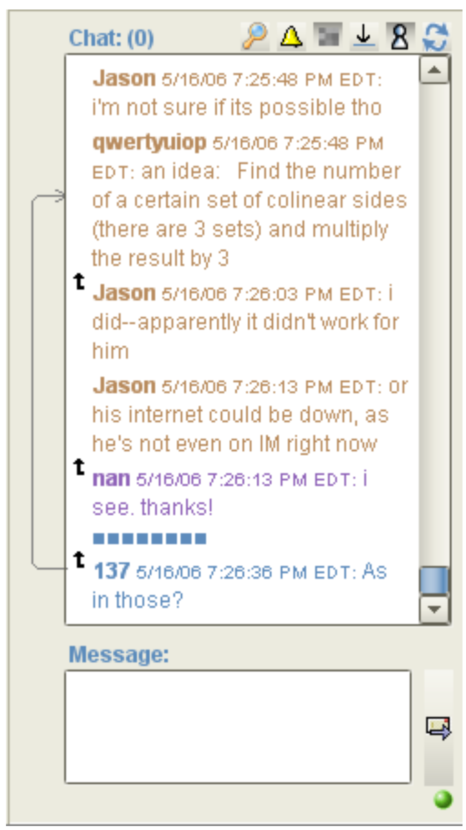


Figure 3.3.4: A message-to-message reference

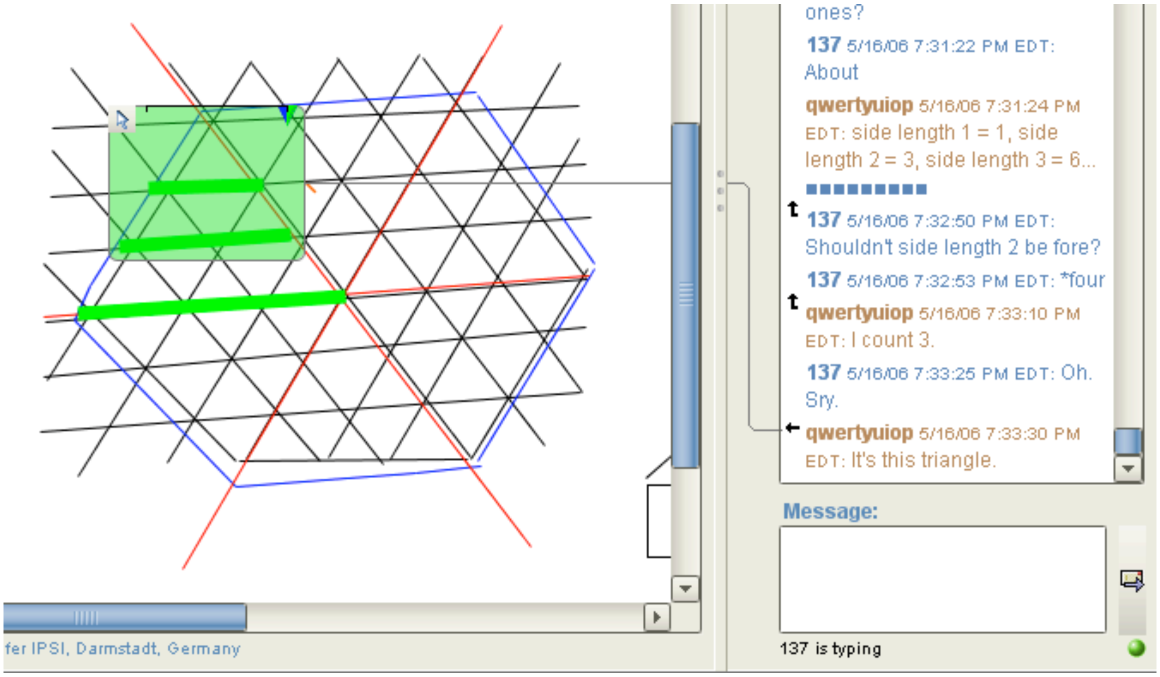


Figure 3.3.5: A message-to-whiteboard reference

In addition to the explicit referencing feature, the system displays small boxes in the chat window to indicate actions performed on the whiteboard. For instance, the blue squares embedded in the chat window in Figure 3.3.5 above indicate whiteboard actions performed by the user 137 at that time. This awareness mechanism allows users to observe how actions performed in both interaction spaces are sequenced with respect to each other. Moreover, users can click on these boxes to move back and forth from the current state to the specific point in the history of the whiteboard when that action was performed.

Chat messages and activity markers are color coded to help users to keep track of who is doing what in the online environment. In addition to standard awareness markers that display who is present in the room and who is currently typing, the system also displays textual descriptions of whiteboard actions in tool-tip messages that can be observed by holding the mouse either on the object in the whiteboard or on the corresponding square in the chat window.

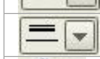
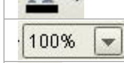
Figure 3.3.6: Whiteboard controls

The shared whiteboard of the VMT system supports basic drawing features such as drawing lines, scribbles, rectangles, and ellipses with different brush thickness and color. A screenshot of the whiteboard tools are provided in Figure 3.3.6 above. Table 3.3.1 below summarizes each function.

Once a drawing feature is selected, a drawing action will change the state of the whiteboard as soon as the user releases the mouse button. For instance, after selecting the straight-line option, the user needs to click and drag the mouse button to extend a line from the point where the initial click is made. Once the mouse button is released the system displays a straight line on all clients' whiteboards. We call whiteboard actions that produce a single change on all users' screens an ***atomic*** action. As we will observe in the case studies, most whiteboard drawings take multiple atomic actions to produce, and this makes the production process available for others to witness.

The contents of the whiteboard and chat spaces are *persistently* maintained by the VMT system. However, one crucial difference between these two spaces, which will be shown to be consequential in our case studies, is that unlike chat messages the whiteboard objects remain in the shared visual field until they are removed by one of the team members. The whiteboard has its own scrollbar that allows users to view the history of actions performed on the whiteboard. In other words, one can see how the drawings on the whiteboard have *evolved* over time step by step, or even bring a previously drawn object back from history by copying and pasting it to the current state of the whiteboard.

Table 3.3.1 : Description of whiteboard features

Icons	**Description**
	Cut
	Copy
	Paste
	Links a chat message to a selected area or to an object on the whiteboard
	Select an object
	Draw a straight line
	Freehand drawing
	Draw a rectangle/square
	Draw an ellipsis/circle
	Add a textbox
	Insert an image
	Take a screenshot
	Link objects
	Add a mindmap
	Add a mindmap node
	Choose line style
	Choose line thickness
	Choose brush color
	Choose fill-in color
	Choose font color
	Zoom
	Lock/unlock the position of selected objects
	Change font and style of text

The most recent version of the VMT chat system is equipped with multiple tabs that allow groups to have an additional whiteboard for summarization work, a topic tab to view the task description, and shared browser tabs to view online resources such as VMT wiki pages, help manual, and the Math Forum's digital library (see Figure 3.3.1 above). The system also supports basic MathML syntax, which can render basic computer algebra notation such as x^2+y^2. For instance, when a user types x^2+y^2 the system displays it as x^2+y^2 when it is posted in a chat message or in a textbox on the whiteboard. Figure 3.3.7 below illustrates some of the symbols recognized by the system.

Input	Output
$(a+b)/c$	$\frac{(a+b)}{c}$
$e^(x+3)$	$e^{(x+3)}$
$pi*r^2$	$\pi \cdot r^{2}$
$root(3,5)$	$\sqrt[3]{5}$
$sum(i=0,10,i^2)$	$\sum_{i=0}^{10} i^{2}$

Figure 3.3.7: MathML examples supported by VMT Chat

3.4. Data

The excerpts we will present in this dissertation are obtained from a problem-solving session of a team of three students called Team C who participated in the VMT Spring Fest 2006. This event brought together several teams from the US, Singapore and Scotland to collaborate on an open-ended math task on combinatorial patterns. Students were recruited anonymously through their teachers. Members of the teams generally did not know each other before the first session. Neither they nor we knew anything about each other (e.g., age or gender) except chat handle and information that may have been communicated during the sessions. Each group participated in four sessions during a two-week period, and each session lasted over an hour. Each session was moderated by a Math Forum member; the facilitators' task was to help the teams when they experienced technical difficulties, not to participate in the problem-solving work.

During their first session, all the teams were asked to work on a particular pattern of squares made up of sticks (see Figure 3.4.1 below). For the remaining three sessions the teams were asked to come up with their own shapes, describe the patterns they observed as mathematical formulas, and share their observations with other teams through a wiki page. This task was chosen because of the possibilities it afforded for many different solution approaches ranging from simple counting procedures to more advanced methods, such as the use of recursive functions and exploring the properties of various number sequences. Moreover, the task had both algebraic and geometric aspects, which would potentially allow us to observe how participants put many features of the VMT software system into use. The open-ended nature of the activity stemmed from the need to agree upon a new shape made by sticks. This required groups to engage in a different kind of problem-solving activity as compared to traditional situations where questions are given in advance and there is a single "correct" answer—presumably already known by a teacher. We used a traditional problem to seed the activity and then left it up to each group to decide the kinds of shapes they found interesting and worth exploring further (Moss & Beatty, 2006; Watson & Mason, 2005)

(1) 4 sticks, 1 square

(2) 10 sticks, 3 squares

(3) 18 sticks, 6 squares

N	Sticks	Squares
1	4	1
2	10	3
3	18	6
4	?	?
5	?	?
6	?	?
...	...	...
N	?	?

Session I

1. Draw the pattern for N=4, N=5, and N=6 in the whiteboard. Discuss as a group: How does the graphic pattern grow?
2. Fill in the cells of the table for sticks and squares in rows N=4, N=5, and N=6. Once you agree on these results, post them on the VMT Wiki
3. Can your group see a pattern of growth for the number of sticks and squares? When you are ready, post your ideas about the pattern of growth on the VMT Wiki.

Sessions II and III

1. Discuss the feedback that you received about your previous session.
2. **WHAT IF?** Mathematicians do not just solve other people's problems - they also explore little worlds of patterns that they define and find interesting. Think about other mathematical problems related to the problem with the sticks. For instance, consider other arrangements of squares in addition to the triangle arrangement (diamond, cross, etc.). **What if** instead of squares you use other polygons like triangles, hexagons, etc.? Which polygons work well for building patterns like this? How about 3-D figures, like cubes with edges, sides and cubes? What are the different methods (induction, series, recursion, graphing, tables, etc.) you can use to analyze these different patterns?
3. Go to the VMT Wiki and share the most interesting math problems that your group chose to work on.

Figure 3.4.1: Task description for Spring Fest 2006

Transcription Conventions

In the VMT Chat environment, as soon as a user enters a character in the message composition box the system displays an awareness message on all users' screens, which indicates who is currently typing a message. The *Time Start Typing* column (*cf.* Excerpt 3.5.1 on page 114) shows the timestamp of the event that a user is first seen as typing by others. Users stay in the typing mode until they either completely erase the contents of their message box or they post the message by pressing the return key. The rows that are marked as *user completely erased the message* in italics correspond to cases where the user completely erases whatever he/she typed without posting it. In the other case, the posted message is displayed under the *Content* column. The timestamp for both cases are listed under the *Time of Posting* column, which gives an indication of how long a particular typing event took.

Since the time-of-posting corresponds to how these actions are displayed on the screen, all the actions are ordered with respect to this column. Hence, it is possible that a chat message x with an earlier time-start-typing value than a message y will appear after y, provided y is completed earlier. Such cases indicate that two or more users' message typing activity have overlapped with each other. As we will observe in the case studies this information can be useful in reading chat and whiteboard contributions in relation to prior contributions.

The whiteboard activities are not always captured in an action-by-action manner. The duration of the drawing activity is given together with a narrative describing what happened in bold italics under the *Content* column. Screenshots of the relevant sections of the shared whiteboard are also provided. Finally, when a user posts a message with a referential link, the number of the message it is referring to is displayed under the *Refers To* column. When a chat message refers to a portion of the whiteboard, then the reference is described in narrative form together with a screenshot.

3.5. Methodology: Ethnomethodological Conversation Analysis

Studying the meaning-making practices enacted by the users of CSCL systems requires a close analysis of the process of collaboration itself (Dillenbourg et al., 1995; Stahl, Koschmann & Suthers, 2006; Koschmann, Stahl & Zemel, 2007). In an effort to investigate the organization of interactions across the dual-interaction spaces of the VMT environment, we consider the small group as the unit of analysis (Stahl, 2006), and we appropriate methods of Ethnomethodology (EM) (Garfinkel, 1967; Livingston, 1986; Heritage, 1984) and Conversation Analysis (CA) (Sacks, 1962/1995; Psathas, 1995; ten Have, 1999) to conduct case studies of online group interaction. Our work is informed by studies of interaction mediated by online text-chat with similar methods (Garcia & Jacobs, 1998; 1999; O'Neill & Martin, 2003), although the availability of a shared drawing area and explicit support for deictic references in our online environment significantly differentiate our study from theirs.

The goal of ethnomethodological conversation analysis is to discover the commonsense understandings and procedures group members use to organize their conduct in particular interactional settings (Coulon, 1995). Commonsense understandings and procedures are subjected to analytical scrutiny because they “enable actors to recognize and act on their real world circumstances, grasp the intentions and motivations of others, and achieve mutual understandings” (Goodwin & Heritage, 1990, p. 285). Members’ shared competencies in organizing their conduct not only allow them to produce their own actions, but also to interpret the actions of others (Garfinkel & Sacks, 1970). Since members enact these understandings and/or procedures in their situated actions, researchers can discover them through detailed analysis of members' sequentially organized conduct (Schegloff & Sacks, 1973).

We conducted numerous VMT Project data sessions, where we subjected our analysis of VMT data to intersubjective agreement (Psathas, 1995; Jordan & Henderson, 1995). The case studies in this dissertation present the outcome of this group effort together with the actual transcripts, so that the analysis can be subjected to external scrutiny[7]. During the data sessions we used the VMT Replayer tool, which allows us to replay a VMT chat session as it unfolded in real time based on the timestamps of actions recorded in the log file. The order of actions—chat postings, whiteboard actions, awareness messages—we observe with the Replayer as researchers exactly matches the order of actions originally observed by the users. This property of the Replayer allowed us to study the sequential unfolding of events during the entire chat session, which is crucial in making sense of the complex interactions mediated by a CSCL environment. In short, the VMT environment provided us a perspicuous setting in which mathematical work is "made visible" (Stahl, 2002) as a joint practical achievement of participants that is "observably and accountably embedded in collaborative activity" (Koschmann, 2001, p. 19).

In the following two subsections we will provide more information about the ethnomethodological conversation analytic approach. The first subsection provides a historical background to this approach in relation to other related methodologies for studying interaction in both face-to-face and computer-mediated settings. This subsection

[7] The replayer application and the data set for the case studies discussed below can be obtained from: http://mathforum.org/wiki/VMT?VMTGroupC

also includes descriptions of some of the key ethnomethodological concepts central to this dissertation, such as indexicality, reflexivity, member methods, and sequential organization. In the next subsection, we will describe how we put this methodology into use by presenting a case study of an excerpt obtained from a VMT session. The case study will also describe some of the key concepts that will be central to the discussion of our research questions, such as math artifacts, co-construction processes, and affordances.

3.5.1. Historical Background

In the social science literature there are a variety of approaches to the analysis of language in interaction originating from disciplines such as linguistics, sociology, and anthropology. Research efforts originating from these distinct disciplines have recently converged along two dimensions; (a) increased attention to how social context influences language use, and (b) characterization of language as a means to perform social action (Drew & Heritage, 1992). This convergence has contributed to the inception of two main methodologies for the analysis of social interaction, namely Discourse Analysis (DA) and Conversation Analysis (CA), in (socio-) linguistics and sociology respectively (Bryman, 2004). Both DA and CA are mainly concerned with "…how coherence and sequential organization is achieved and understood during conversation" (Levinson, 1983, p. 286). However, these disciplines differ by means of their conceptual perspective towards interaction and the methods they employ for analyzing it. In the following paragraphs we will provide a brief overview of each methodology to elaborate on these differences and similarities, and review their applications to the analysis of computer-mediated interactions. Based on our review we will then motivate our choice for a CA-informed approach to perform a micro-level analysis of interactions recorded in the VMT Chat environment to investigate our research questions.

Discourse Analysis is often used as a blanket term that refers to any systematic effort at the analysis of discourse manifested in written text, talk, pictures, symbols, artifacts etc. (Philips & Hardy, 2002). DA has emerged within modern linguistics as a consequence of the 'linguistic turn' often associated with the influential writings of Austin (1962) and Wittgenstein (1953) in linguistic philosophy, which characterize language not only as a representation or reflection of social reality but as a means to perform social actions that constitute social reality (Philips & Hardy, 2002). In particular, Searle's speech-act theory (1969), which elaborates upon Austin's work, has served as an important analytical tool to study human interaction in the DA tradition. Searle's theory focuses on rules and conditions through which a sentence or an utterance is understood as a particular form of action. The apparent links between speech-acts performed by sentences or utterances in written or spoken texts have motivated linguistic investigations of units that are larger than single sentences. Such efforts have been aimed towards generating rules (a) to derive speech acts from linguistic objects deployed in each sentence/utterance, and (b) to devise sequences of sentences/utterances based on the links between the speech-acts they imply (Labov & Fanshel, 1977; Coulthard, 1977/1985). This bottom-up approach to the analysis of interaction was a radical step in linguistics due to its focus on structures that go beyond the traditional sentence boundary (van Dijk & Kintsch, 1983; Stubbs, 1983). The larger units constructed from smaller traditional linguistic units in this fashion are then used to account for inter-textual phenomena such as coherence and sequential organization of discourse.

Most DA studies that follow the rule-based approach outlined above usually start with a set of predefined category terms based on a social theory related to the phenomenon of interest. Then the researcher selects a suitable linguistic unit (usually of fixed size to accommodate assumptions of statistical methods used) to segment the available data, and develops a set of rules (often called a *coding scheme*) to systematically categorize each unit based on its linguistic content. The coded data is then subjected to statistical analysis to devise patterns between assigned codes and to conduct systematic comparisons between datasets under controlled conditions (Neuendorf, 2002).

As we have covered in Section 2.4 above, this rule-driven content-analytic approach is widely employed in CSCL research to analyze computer-mediated discourse, such as email messages and postings contributed to threaded discussion boards (De Wever et al., 2006; Herring, 2004; Strijbos et al., 2006). The rule-based structures and category schemes have also been employed in CSCL studies to automate the analysis of interaction through AI techniques such as machine-learning algorithms and expert systems (Rose et al., 2008; Erkens & Jannsen, 2008). Applications of similar methods on chat data have been a relatively recent development (Strijbos & Stahl, 2007). No matter what the underlying electronic communication medium is, recent publications on methodological issues in CSCL highlight selecting an appropriate unit of analysis, coming up with a segmentation procedure, designing a rule-based categorization scheme, and achieving satisfactory reliability as the central issues in conducting content and rule-driven discourse analytic work (Strijbos et al., 2006).

The difficulties we highlighted in Section 2.4 regarding the use of rule-based formal methods to characterize interaction in terms of relationships between units such as speech-acts are also acknowledged in the DA literature[8] (Levinson, 1983). These problems are mainly attributed to the lack of sensitivity to the social and sequential context in speech-act theory and its rule-based approach to mapping illocutionary acts onto the utterances as they occur in actual contexts (Drew & Heritage, 1992; Maynard & Perakyla, 2003). Following the linguistic tradition, speech-act pragmatics were derived from idealized sentences without any empirical investigation of their use in naturally occurring interaction. This approach reflects the underlying assumption that the illocutionary act achieved by a sentence (i.e., its meaning) emerges from the semantic import of its contents. However, as Levinson (1983) argues there are serious issues with the application of rule-based linguistic methods to the analysis of interaction, because (a) single sentences can perform more than one speech act at a time, (b) non-linguistic vocalizations (e.g. laughter, silence) can perform appropriate responses to prior utterances based on their sequential placement in talk, and (c) a seemingly unrelated or ill-formed sequence of utterances can be quite meaningful when that sequence is

[8] Our characterization of DA in this section is by no means a complete one. There are numerous forms of DA studies that we could not cover, such as Critical DA that focus on power relations that are implicit/hidden in discourse by not specifically following the linguistic approach we summarized here (e.g. Fairclough & Wodak, 1997). There are also DA studies that focus on the construction of social reality and are informed by CA's sensitivity to social context and sequential organization (e.g. Potter, 1996). Yet this does not mean that DA and CA have fully converged into a single methodology. For instance some of the recent work in DA critiqued the limitations implied by the strictly data-driven methodology advocated by CA (e.g. Hammersely, 2003).

considered as a whole within the immediate social context in which it is embedded in[9]. In short, the rule-based, bottom-up approach appropriated from linguistics struggles with explaining how speakers bring individual acts together to achieve interaction. Most importantly, processes where improvised, creative deviations from the rules lead to the realization of new meanings are not within the reach of a rule-centric analysis of meaning-making. As we will observe in our case studies, such deviations can be highly consequential in mathematical practice.

Ethnomethodology (EM) and Conversation Analysis (CA) originated in the late 1950s as a reaction to similar rule-based approaches in sociology influenced by Weber, Durkheim, and Parsons who characterized social order based on a system of shared values, meanings, and rules determined by a pre-existing culture. Influenced by Alfred Schtütz's (1932/1967) phenomenological analysis of intersubjectivity, Garfinkel (1948/2006; 1967; 2002) criticized rule-based theories of social order for not giving empirical consideration to the *methods* by which social order is achieved in practice by the members of a society. The EM position does not deny the relevance of rules and norms to social conduct (Garfinkel, 1963), but it rejects the treatment of rules as the cause or determining factor of social action on the grounds that such a position ignores human agency at best, and treats humans as cultural dopes who blindly follow the rules imposed on them at worst. Instead, EM treats rules as *resources for action* that are invoked by the members based on

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[9] Levinson (1983, p292) provides the following exchange to illustrate this point.

A: I have a fourteen year old son
B: Well that's allright
A: I also have a dog
B: Oh I am sorry

When considered in isolation this sequence seems quite bizarre, yet when we view it as part of a conversation where A lists a series of potential disqualifications for apartment rental to the landlord B it will seem natural and meaningful as a whole.

their common-sense understandings of the circumstances they are situated in (Suchman, 1987). This is based on the observation that people, in the course of their everyday life, engage in the kinds of practical sociological reasoning to figure out what other people mean, and in turn figure out how to act in order to get things done (Dourish, 2001). For that reason, ethnomethodology proposes a shift in analytical focus in sociological inquiry "…away from the search for causes of human conduct and toward the explication of how conduct is produced and recognized as intelligible and sensible" (Pomerantz & Fehr, 1997, p. 65). Focusing on the conduct itself allows analysts to discover the rules as they figure in actors' own practices of reasoning and ways of organizing a social setting (Maynard & Perakyla, 2003).

A central theme in ethnomethodological inquiry is that "…mundane social encounters rely on detailed ***indexical*** understandings of what might be happening right now, what just happened, and what will likely happen next in some particular, located routine activity" (Jacoby & Ochs, 1995, p. 174). In linguistic pragmatics, indexicality is a term referring to tokens whose interpretation requires identification of some element in the context in which they are uttered (Levinson, 1983). The indexical nature of words and the variety of languages around the world that achieve similar/equivalent kinds of practical actions with linguistic units of their own imply that any word could mean anything (Garfinkel, 1948/2006). Hence, indexicality refers to a broader phenomenon and it is not limited to disambiguation of spatial and temporal deictic terms such as this, that, now, or there in relation to a contextual ground. The multiplicity of meanings attributed to a word presents a mapping problem between the words and the actions they imply in speech-act theory, where the mappings are treated as rules (Maynard & Perakyla, 2003). A key contribution of the EM perspective to the resolution of this problem is the observation that members' sequential organization of their conduct is the glue that holds isolated acts together and constitutes them as meaningful *inter*action. The chains of actions have a ***reflexive*** character where each next thing done or said reflects back on what is prior. The meanings ascribed to words and objects are specified through the way they are "indexed" within a sequence of reflexively organized actions. Therefore, the reflexively organized sequential chain of actions forms the fundamental order of sense making (Rawls, 2008).

Indexical and reflexive character of language use in interaction had also received attention from other related sociological and anthropological work of the time, such as Goffman's (1981) analysis of frames and Gumperz and Hymes' (1972) notion of contextualization cues as part of their work in ethnography of speaking. Hymes and Gumperz's work highlights how interactants use various features of language as *contextualization cues* to indicate which aspects of the immediate context are relevant in interpreting what has been said. Likewise, in Goffman's analysis the notion of *frames* focuses on how participants define the social activity they are a part of (e.g., what is going on in this particular situation, what are the roles assumed by participants). Human conduct is interpreted in the context of interactants' understanding of what frame their activity is situated in, which is demonstrated by what they do and say in interaction. Goffman used another concept he called *footing* to characterize the dynamic nature of frames and the ways participants move between different frames as their mutual understandings of the unfolding activity evolves in interaction. In short, this literature has highlighted the mutually constitutive relationship between human conduct and social context, where human conduct is shaped by the social context, and conversely, human conduct modifies the relevant features of the social context as it contingently unfolds in a moment-by-moment basis.

Conversation Analysis (CA) has utilized the insights of Ethnomethodology and context-sensitive accounts of social interaction to form a rigorous methodology. CA originated in Harvey Sacks' collaborative inquiries with Emanuel Schegloff and Gail Jefferson in the mid 1960s within sociology as an approach to the study of social organization in everyday conduct (Sacks, 1962/95; Goodwin & Heritage, 1990; Garfinkel & Sacks, 1970). In particular, CA shares with ethnomethodology the assumptions that (a) everyday human conduct is sensible/meaningful, and (b) meaningful conduct is produced and understood based on shared methods and procedures (Pomerantz & Fehr, 1997). CA is concerned with the social organization of naturally occurring 'conversation' or 'talk-in-interaction' (ten Have, 1999). The main goal of conversation analytic work is to *describe* the shared methods participants use to produce and make sense of their own and each others' actions (i.e. the shared methods that make interaction possible). The CA literature has made important contributions to our understanding of how interacting individuals create and sustain social order in everyday encounters. In particular, earlier work in CA has pointed out fundamental mechanisms of talk-in-interaction, such as turn organization, adjacency pairs, repair structures, insertion sequences, and preference organization (Sacks, Schegloff & Jefferson, 1974; Schegloff, Jefferson & Sacks, 1977; Schegloff, 1988; Schegloff, 2006).

In the CA tradition, analysis is conducted on detailed transcripts of utterances that capture additional interaction-relevant information such as intonation, pauses, bodily orientations, and gestures performed by speakers (Psathas, 1995). In other words, non-verbal aspects of interaction are also taken into account in an effort to deal with the indexical nature of talk in interaction. In contrast to most DA approaches, CA employs an inductive approach to data analysis where sequences of interaction are not investigated in terms of fixed linguistic units and a priori theoretical constructs. Instead, analytic units emerge from the analysis of sequences of interactions based on actors' orientations to each other and their vicinity. The sense of each action in a sequence is analyzed in relation to the temporal context set by prior actions as well as the features of the broader context of the occasion to the extent participants explicitly orient to such features in their utterances. Each analyzed encounter is treated as a unique case study that is not replicable or directly comparable. Analysis is geared towards discovering/describing the ways in which order is produced by the members during those encounters. However, generalizations can be drawn concerning the structure of the interactions once the structures of social action are discovered and described. Scientific rigor is maintained by subjecting the analysis to inter-subjective agreement among researchers through collaborative data sessions and sharing access to unreduced data sources (ten Have, 1999).

There are a number of studies that employ CA methods to analyze interactions mediated by text-based chat tools. These studies emphasize the quasi-synchronous nature of chat, which transforms the turn organization structure as compared to face-to-face interaction

(Garcia & Jacobs, 1999). In online chat environments participants interact with each other by exchanging textual artifacts where actions that overlap in time (e.g. typing) are artificially ordered by the chat system (Zemel, 2005). This often brings the issue of "chat confusion", where postings produced in parallel are treated as responses to one another by the interlocutors (Garcia & Jacobs, 1998; Fuks et al., 2006). Yet, like face-to-face conversation, chat also evolves sequentially where postings or actions build upon and/or respond to earlier ones. Moreover, chat participants develop methods to deal with the issue of parallelism by taking advantage of the persistent nature of postings and various interactional cues to make sense of each other's actions (O'Neil & Martin, 2003).

Applications of CA methods to the analysis of electronic forms of communication we have covered so far focus solely on text messages exchanged between users. However, as we will illustrate in our case studies, users may have access to additional interaction spaces besides a chat interface. Thus, an EM/CA-informed analysis of interactions mediated by such interfaces requires the analyst to focus on the temporal unfolding of actions distributed into multiple interaction spaces. For instance, in order to unpack what was going on in the excerpts presented in Chapter 4 below, we relied on various cues such as temporal relationships derived from timestamps, the ways students used explicit references, etc. to construct a particular way of reading the postings in relation to the drawings on the whiteboard (Livingston, 1995; Zemel, Cakir & Stahl, 2009). Reconstructing the sequence of actions is especially important for analyzing online math discussions since the sketches/calculations performed on the whiteboard and mentioned in chat set the immediate context for the ongoing problem-solving work where indexical terms such as “hexagonal array”, “6 smaller triangles”, etc. make sense (Livingston, 2006). Moreover, as we will demonstrate in Chapter 4, the ways participants animate the contents of the whiteboard can be highly consequential for problem-solving chats. Analysis of such actions require a careful inspection of the ways objects are moved around and spatially organized with respect to each other in the course of a VMT Chat session. Thus, we will need to extend existing applications of CA to online chat to account for the complex nature of interactions taking place in multimodal interaction spaces. Our extension will be informed by relevant work in EM/CA and pragmatics, which focuses on the organization of talk with and around physical objects in face-to-face settings (e.g. Streeck & Kallmeyer, 2001; Goodwin, 1995; Goodwin, 2000), yet our focus on the organization of interaction in a disembodied online setting will differentiate our work from this line of work.

3.5.2. Illustration of EM/CA Methodology

In this subsection we will demonstrate the main aspects of our EM/CA-informed approach to analysis of chat data over a short excerpt obtained from Team C’s first chat session in Spring Fest 2006 (see 3.4 above for details about participants and the task description), which seeks to describe the *methods* students use to interpret their own and other group members’ actions in the VMT environment. In the mean time, we will also establish a conceptual framework for the discussion of our three research questions by describing important terms such as math artifacts, co-construction process, and affordances. The interactional methods or practices suggested by the analysis presented in this subsection will be developed further with a more comprehensive case study in

Chapter 4, which will cover an entire problem-solving session. The collection of these case studies will altogether serve as the main evidence for the consequentiality of the identified interactional methods on joint mathematical meaning-making activities online.

3.5.2.1. Mathematical Artifacts

Our focus on the *co-construction of mathematical artifacts* is motivated by three converging lines of inquiry that we identified in our literature review: (a) the linguistic turn in social science that characterizes language as *action* rather than as rational description based on rules of logic/grammar, (b) the shift from absolutism to fallibilism in philosophy of mathematics that problematizes the eternal truth status ascribed to mathematical theorems, and emphasizes the historical and cultural argumentative processes in the development of mathematics, and (c) post-cognitive theories of social action that treat learning as a fundamentally social phenomenon situated in and accomplished in social interaction among people mediated by meaningful artifacts. These trends motivated our interest in the organization of joint mathematical practices where artifacts are co-constructed, put into use, and become meaningful for those who engage with them.

By definition, an *artifact* is a man-made thing crafted to fulfill a practical purpose[10]. Artifacts are embodiments of the capability of mankind to impose functionality on the objects in the world (Searle, 1998). In the VMT Chat environment participants interact by exchanging graphical and symbolic artifacts in electronic media. They do this by using

[10] This definition of artifact is adapted from http://www.merriam-webster.com/dictionary/artifact

the features made available to them by the online system, which is a technological artifact designed by a team of educational researchers and computer scientists for the purpose of supporting collaborative learning online. As participants make use of these features to act in this online environment, they collectively construct a sequentially unfolding stream of actions oriented towards the common goal of solving a math problem. The participants do not have access to anything else but this stream of shared visual displays that temporally unfold (and are persistently stored) on their computer screens and the descriptive information (e.g., problem statement, feedback) provided by the Math Forum mentors. Therefore, a fundamental concern from an EM/CA perspective is to explicate the methods participants enact in an online environment like VMT for interpreting their own and each others' actions in a shared space of graphical, narrative, and symbolic artifacts.

3.5.2.2. Identification of Members' Methods

Pomerantz (1990) argues that an EM/CA study that analyzes the interpretive work done by members makes at least three types of claims: namely (a) characterization of the actions, (b) proposals of method(s), and (c) proposals of sequential and interactional features of the method(s). These claims are described as follows:

> One type [characterization of the action] is to assert that interactants are "doing" particular social actions, identities, and/or roles. For example, we may assert an interactant is "agreeing", "rejecting an invitation", "fishing for information", "being an expert", etc. A second type [proposed method] of claim is when we offer analyses of methods that interactants use in accomplishing particular actions, roles, or identities. The third type [proposed features] of claim is when we propose how methods work: their sequential features and interactional consequences. (Pomerantz, 1990, p. 231)

In an effort to describe the kinds of claims the EM/CA approach makes on data and to illustrate the underlying reasoning that lead to their derivation, we will provide an EM/CA analysis of on an excerpt obtained from a VMT Chat session. The following sequence of drawing actions (Figures 3.5.1 to 3.5.5) is observed at the beginning of the very first session of Team C in the VMT environment. Shortly after a greeting episode, Davidcyl begins to draw a set of squares on the shared whiteboard. He begins with drawing three squares that are aligned horizontally with respect to each other, which is made evident through his careful placement of the squares side by side (see Figure 3.5.1 below). Then he adds two more squares on top of the initial block of three, which introduces a second layer to the drawing. Finally, he adds a single square on top of the second level, which produces the shape displayed in the last frame in Figure 3.5.1 below.

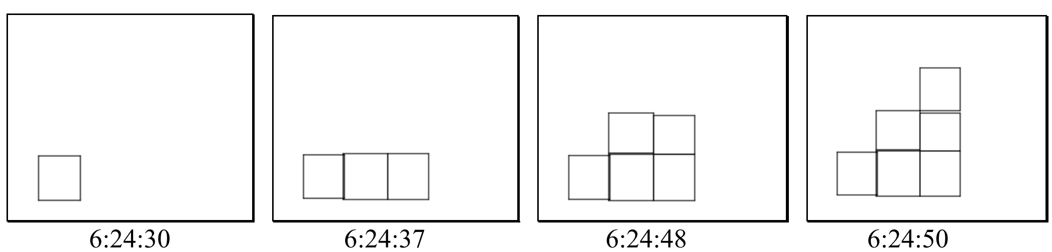


Figure 3.5.1: First stages of Davidcyl's drawing activity

Next, Davidcyl starts adding a new column to the right of the drawing (see Figure 3.5.2 below). He introduces a new top level by adding a new square first, and then he adds 3 more squares that are aligned vertically with respect to each other and horizontally with respect to existing squares (see second frame in Figure 3.5.2). Then, he produces a duplicate of this diagram by using the copy/paste feature of the whiteboard (see the last frame in Figure 3.5.2).

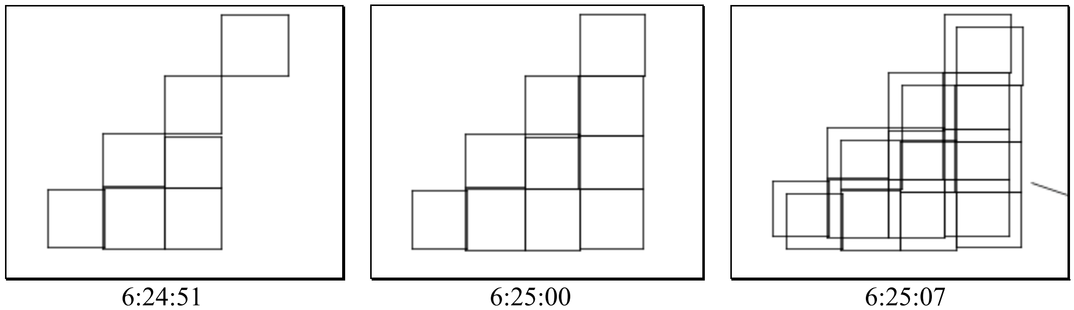


Figure 3.5.2: Davidcyl introduces the 4th column and pastes a copy of the whole shape

Davidcyl moves the pasted drawing to an empty space below the copied diagram. As he did earlier, he adds a new column to the right of the prior stage to produce the next stage. Nevertheless, this time he copies the entire 4th column, pastes a copy next to it, and then adds a single square on its top to complete the new stage (see Figure 3.5.3 below).

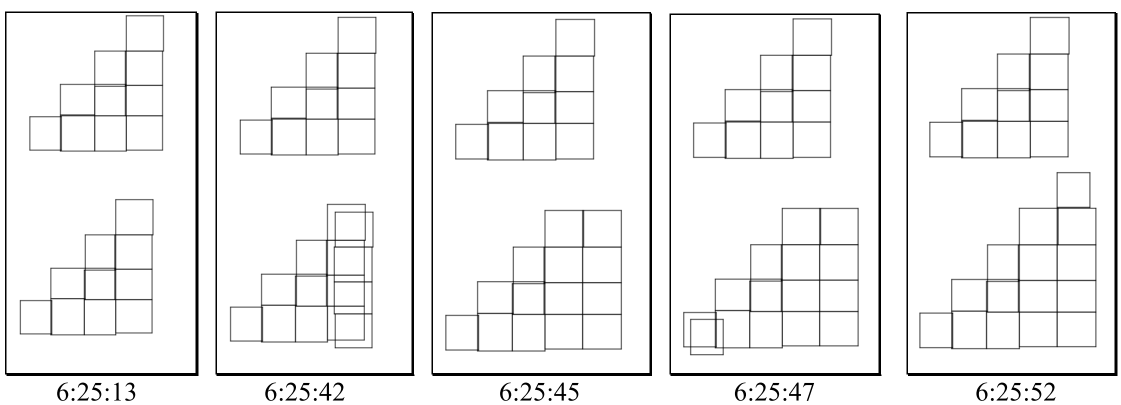
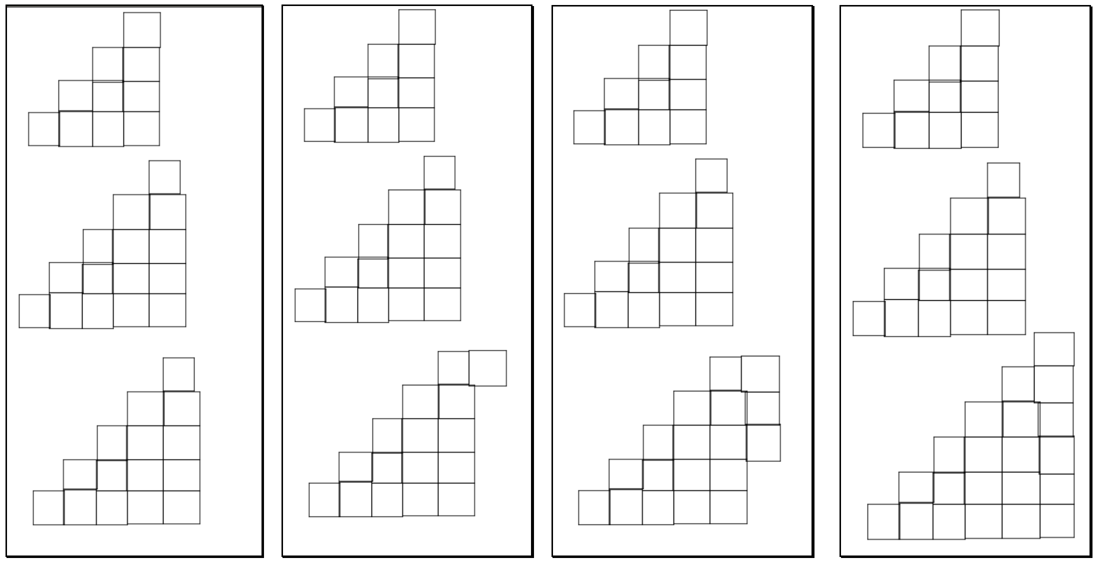


Figure 3.5.3: Davidcyl uses copy/paste to produce the next stage of the pattern

Next, Davidcyl produces another shape in a similar way by performing a copy/paste of his last drawing, moving the copy to the empty space below, and adding a new column to its right (see Figure 3.5.4 below). Yet, this time the squares of the new column are added one by one, which may be considered as an act of counting.



6:26:04 6:26:10 6:26:14 6:26:20

Figure 3.5.4: Davidcyl's drawing of the 6th stage

Shortly after his last drawing action at 6:26:20, Davidcyl posts a chat message stating that "ok I've drawn n=4,5,6" at 6:26:25. Figure 3.5.5 below shows the state of the interface at this moment. The "ok" at the beginning of the message could be read as some kind of a transition move[11]. The next part "I've drawn" makes an explicit verbal reference to his recent (indicated by the use of past perfect tense) drawing actions. Finally, the expression "n=4,5,6" provides a symbolic gloss for the drawings, which specifies how those drawings should be seen or treated. Once read in relation to the task description, Davidcyl's recent actions across both spaces can be treated as a response to the first bullet under session 1, which states "Draw the pattern for N=4, N=5, and N=6 in the whiteboard" (see Figure 3.4.1. above for the task description).

[11] See Beach (1995b) for an in-depth CA analysis of what kind of interactional work is achieved with the use of "okay" in talk. Beach's explication of methodic uses of okay include Davidcyl's case as well, where his use of "ok" served as a projecting device revealing a transitional movement (p. 154)

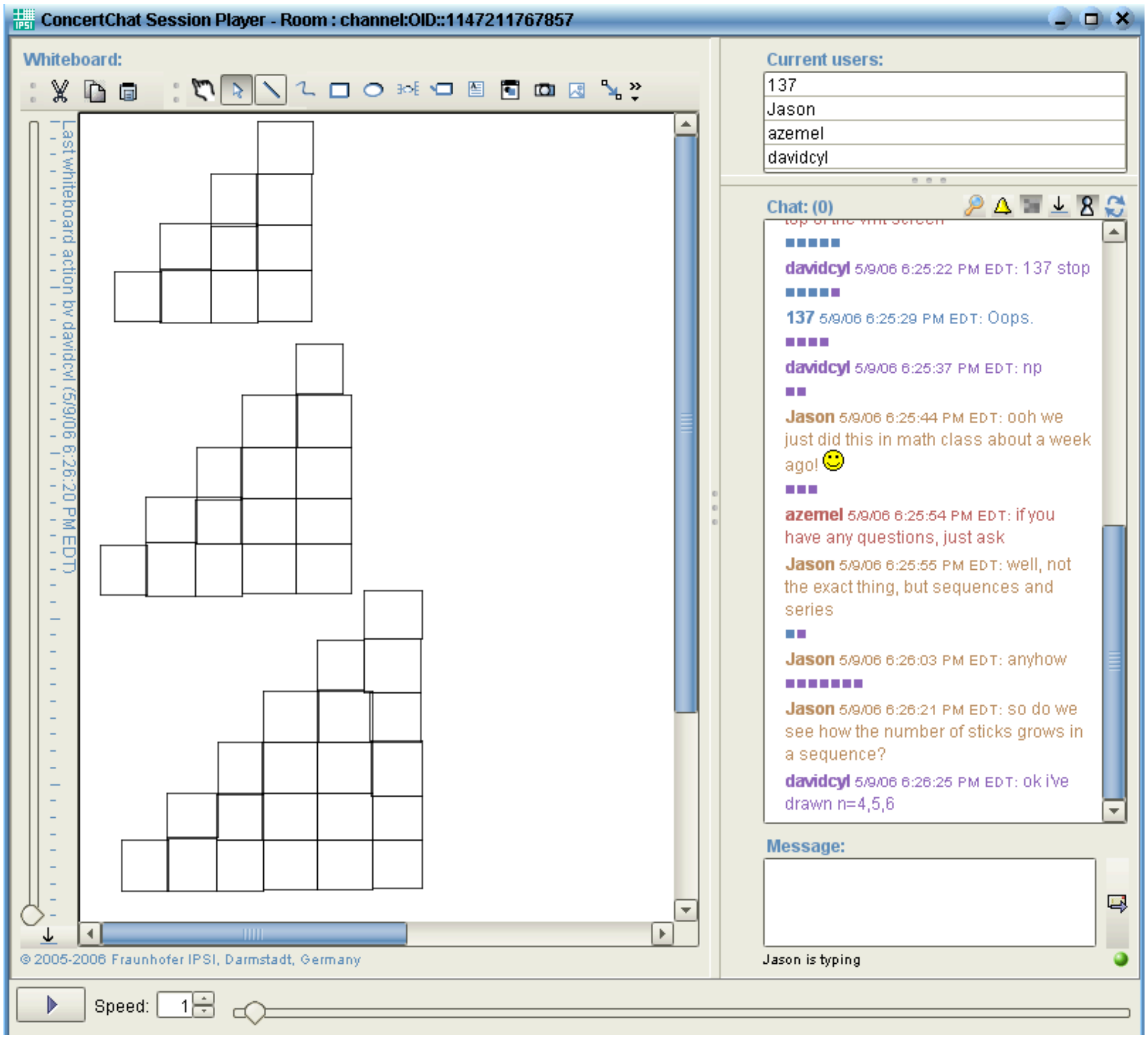


Figure 3.5.5: The state of the VMT environment when Davidcyl posted his chat message.

According to Pomerantz's terminology the narrative we provided above is a characterization of Davidcyl's actions including his drawing work and his subsequent chat message. As language users with comparable competence with respect to the participants, we interpreted Davidcyl's actions as an uptake of a specific section of the instructions for their shared task. Nevertheless, this does not mean that the researcher is put in a privileged interpretive position. On the contrary, we as analysts, who are members of the same linguistic community as the participants, resort to the same sense-making apparatus when we characterize the actions we observe. More importantly, our characterizations rely upon the participants' own characterizations of their work as well as their orientations to each other and to shared artifacts, which are made explicit through their actions. In short, EM/CA informed analysis begins with a characterization of the actions that are deemed relevant to the ongoing activity by the participants.

As Pomerantz (1990) argues, characterizations are not equivalent to analyses yet, but they serve as a stepping-stone that leads to analyses. One of our research questions is about the coordination of actions across the whiteboard and chat spaces. The characterization above includes a situation where a user who has been active in the whiteboard moves on to the other interaction space and posts a message referring to his prior work. Hence, one kind of coordination mechanism that is of interest to our research question is enacted in this excerpt. The chat message sequentially followed the drawings and hence presumed their availability as a shared referential resource, so that the interlocutors can treat the message as an intelligible next move. Moreover, the chat posting reflexively gave further specificity to the drawing work by informing everyone that the diagrams should be seen as specific cases of the staircase pattern described in the problem description. Hence, based on this observation, we propose a *method* that users of VMT employ to relate a drawing to a narrative/symbolic account through what one may call *verbal referencing*.

Once a method such as verbal referencing is identified in the data, the next task of EM/CA analysis is to explicate how the method works by describing its sequential and interactional features. In the excerpt above we observe an explicit orientation to timing or sequencing as evidenced by the use of the past perfect tense and the temporal positioning of the message immediately after the final step of the drawing. This suggests that temporal proximity among actions can serve as a resource for participants to treat those actions in reference to each other, especially when the actions are performed across different interaction spaces.

In an effort to expand this empirical observation further we will need to analyze more instances where participants do similar referential work across dual spaces. The case study presented in Chapter 4 includes numerous instances where participants make use of additional resources (such as the explicit referencing tool, mentioning color-names in chat etc.) to methodically achieve referential relationships between shared diagrams and chat messages. Due to their recurrent appearance as a practical concern for the participants in this dual media online environment, we refer to the collection of these methods as *referential practices*. Referential practices are of particular importance to this dissertation since they are enacted in circumstances where participants explicitly orient to the task of achieving relationships between textual and graphical resources; a phenomenon that is given significance in the math education literature to characterize mathematical understanding (Kaput, 1998). Hence, we will address our research questions by identifying relevant methods or practices enacted by the participants through empirical case studies of this genre.

3.5.2.3. Communicative Affordances

The analysis of sequential/interactional features displayed in the excerpt above is by no means complete yet. Davidcyl's use of a verbal reference at this moment in interaction is also informative in terms of respective limitations of each media and their mutually constitutive function for communication. Davidcyl's chat message not only provided further specificity to the recently produced diagrams, but also marked or announced the completion of his drawing work. This is revealing in terms of the kinds of *illocutionary acts* (Austin, 1962) achieved by users in this dual media environment. In particular, although a drawing and its production process may be available for all members to observe, diagrams by themselves cannot fulfill the same kind of interactional functions achieved by text postings such as "asking a question" or "expressing agreement". In other words, whiteboard objects are made interactionally relevant through chat messages that either (a) project their production as a next action, or (b) refer to already produced objects. This can also be seen as members' orientation to a limitation of this environment as a communication platform; one can act only in one space at a given time in this online environment, so it is not possible to perform a simultaneous narration of a drawing as one can do in a face-to-face setting. Therefore, each interaction space as a communicative medium seems to enable and/or hinder certain kinds of actions, which we refer to hereafter as the *communicative affordances* (Hutchby, 2001) of dual-interaction spaces.

Affordances provided by physical/technological artifacts for users to accomplish specific kinds of actions have been of interest to a broad set of disciplines, such as design (Norman, 1989), human-computer interaction (McGrenere & Ho, 2000; Gaver, 1991; 1996), CSCL (Suthers, 2006; Kirschner, Martens & Strijbos, 2004; Dohn, 2009), and communication (Hutchby, 2001). Before we continue our analysis of the affordances of the VMT environment evidenced in the excerpt presented above from an EM/CA perspective, we will provide some background on the origin of the concept and the way it has been appropriated in design-related fields.

The concept of affordances originated in Gibson's (1979) seminal work on perception in ecological psychology. Gibson proposed this concept to characterize "…what the environment offers the animal, what it provides or furnishes, either for good or ill…" (Gibson, 1979, p. 127). In other words, affordances are dispositional properties of the world that enable particular kinds of interactions between actors and objects. The ecological perspective emphasizes that affordances exist as *relative* to an organism equipped to act in certain ways, so the same material or object may afford different kinds of actions for different organisms. The theory also asserts that affordances have a natural existence independent of organisms' desires, knowledge, or perceptual abilities. In other words, an affordance does not change as the needs and goals of the actor changes. In short, the theory of affordances stresses the coupling between the environment and the actor by characterizing physical invariants (i.e., objective features of the world) while taking actions enacted by actors as a frame of reference (i.e., attributes relative to organisms). The following statement by Gibson indicates that affordances can be treated as a deliberate attempt to transcend the objective/subjective dichotomy:

> …an affordance is neither an objective property nor a subjective property; or it is both if you like. An affordance cuts across the dichotomy of the subjective-objective and helps us to understand its inadequacy. It is equally a fact of the environment and a fact of behavior. It is both physical and psychical. An affordance points both ways, to the environment and to the observer. (Gibson, 1979, p. 78)

Gibson's ecological approach to perception is motivated by the difficulties associated with cognitive theories of perception. The cognitive approach asserts that actors only have direct access to sensations, and describes perception as a process through which sensory data is integrated into a symbolic representation of the environment located in the memory. It is through the processing of these symbols that agents are claimed to achieve goal-oriented action in their environment. The role of perception in producing action is limited to extracting symbols from the environment by assembling atomic sensory stimulus. In other words, the cognitive perspective introduces a dualism between physical and mental worlds, and characterizes action as an exclusively mental phenomenon. Therefore, this theoretical position makes it challenging (if not impossible) to account for the ability of actors to engage in practical activity in the world, since it dismisses the coupling/reciprocity between actors and their environment (Dreyfus, 1992).

In contrast, Gibson argues that organisms perceive their environment *directly* (i.e. without mediation by retinal or mental pictures) in terms of the affordances it provides for action. Hence, perception is characterized through active, embodied engagement of the organism with its environment, not through disembodied processing of symbols presumed to take place exclusively inside the head. Affordances do not cause behavior, but they constrain or enable behavior based on the ways organisms attend to them as features of their environment (Gibson, 1982). In short, aligned with related arguments raised by Merleau-Ponty's phenomenological analysis of perception (1962) and Wittgenstein's philosophy of ordinary language (1953), Gibson's ecological approach offers a non-representationalist account of meaning by locating it within uses enacted in the world:

> The meaning or value of a thing consists of what it affords. Note the implications of this proposed definition. What a thing affords a particular observer (or species of observer) points to the organism, the *subject*. The shape and size and composition and rigidity of a thing, however, point to its physical existence, the *object*. But these determine what it affords the observer. The affordance points both ways. What a thing is and what it *means* are not separate, the former being physical and the latter mental, as we are accustomed to believe. The perception of what a thing is and the perception of what it means are not separate, either. To perceive that a surface is level and solid is also to perceive that it is walk-on-able. Thus we no longer have to assume that, first, there is a sensation-based perception of a thing and that, second, there is the accrual of meaning to the primary percept (the "enrichment" theory of perception, based on innate sensations and acquired images). The available information for the perception of a certain surface layout is the same information as for the perception of what it affords. (Gibson, 1982, pp. 407-408).

In the software design literature this concept is appropriated as "perceived affordances" in reference to those properties or cues designed into artifacts that suggest how they should be used (Norman, 1989; 1999; McGrenere & Ho, 2000). Nevertheless, the term has been predominantly used as a conceptual tool to talk about design features in relation to action capabilities of "potential users", without empirically studying how actual users enact the affordances of designed artifacts while they put them into use to address their practical concerns (Dohn, 2009). Dismissing the relational nature of the original Gibsonian notion of affordances in this way implies an essentialist position that leads to technological determinism, where actions that are possible with or around artifacts are completely determined by their design.

In response to the essentialist point of view the sociological perspective highlights the dialectical relationship through which technological artifacts and the social structures/practices organized around those artifacts mutually shape each other (Orlikowski, 1992). On the one hand, motivations underlying the design of specific technologies are inherently have a social nature, because such design efforts are oriented towards addressing specific issues that have some kind of social significance. On the other hand, technologies can be appropriated in ways that were not specifically intended by their designers, and hence bring in new possibilities for action. For instance, Grint and Woolgar (1997) remark that the telephone was invented originally as a device to broadcast concert music. The realization that the technology affords two-way communication at a distance has led to a deviation from the way this technology was initially conceived by its designers. Based on similar historical and sociological analyses of technological artifacts, Grint and Woolgar propose the "technology as text" metaphor where technologies "written" in certain ways by their designers are "read" by their users through a process of interpretation and negotiation. Nevertheless, Hutchby (2001) cautions that this should not motivate a radical constructivist position, which implies that designed artifacts can be made to mean anything. The material properties of the medium of communication and the artifacts influence the ways people organize their activities by enabling specific kinds of readings and constraining others.

In the light of the discussion summarized above, we consider affordances in reference to constraining as well as enabling materiality of artifacts for social interaction. Although social interaction and users' interpretations of technology are variable and contingent, they are "…constrained in analyzable ways by the ranges of affordances that they possess" (Hutchby, 2001, p. 193). Hence, instead of treating affordances as intrinsic properties of designed environments, we investigate them empirically by analyzing how actors organize their interactions with and around textual and graphical artifacts that they collectively construct in the VMT environment. In other words, we attempt to avoid a solely relativist, materialist, or mentalist approach by focusing on the *practices* (Latour, 1990). Thus, we consider affordances as methodic uses of software features enacted by the participants to produce actions that are meaningful to them. For instance, affordances of VMT for referential work refer to methodic uses of features like the explicit referencing or locational pronouns to achieve relationships between whiteboard objects and chat postings. Likewise, representational affordances refer to the spatial and temporal organization of whiteboard actions that produces shared diagrams, which simultaneously gives further specificity to the mathematical artifacts that the team has been working on.

The way Davidcyl has put some of the features like dragging and copy/paste of the whiteboard into use in the episode described above demonstrates some of its key affordances as a medium for producing shared drawings. In particular, we have observed how copying and pasting is used to avoid additional drawing effort, and how collections of objects are selected, dragged, and positioned to produce specific stages of a geometric pattern. Such possibilities for action are supported by the object-oriented design of the whiteboard. Davidcyl's drawing actions show that, as compared to other physical drawing media such as paper or blackboard, the electronic whiteboard affords unique ways to construct and modify mathematical diagrams.

An important concern for this dissertation study is to investigate how students make use of the technological features available to them to express/articulate mathematical ideas in a dual media online environment like VMT. Drawing features such as copy/paste, dragging, coloring, etc. are important affordances of the shared whiteboard not simply because of their respective advantages as compared to other drawing media. The mathematical significance of these features rely on the way single actions like copy/paste or dragging are sequentially organized as part of a broader drawing activity to construct a shared mathematical artifact. Through such a sequence of drawing actions, Davidcyl demonstrated to us and to his peers (a) how to construct a staircase pattern as a spatially organized assemblage of squares, and (b) how to derive a new stage of the staircase pattern from a copy of the prior stage by adding a new column of squares to its right. The availability of these drawing actions as a sequence of changes unfolding in the shared visual space allows group members to witness the *reasoning* process that lead to their construction. In other words, the sequentially unfolding details of the construction process provide specificity (and hence meaning) to the mathematical artifact that is being constructed. We use the term *mathematical affordances* in reference to the kinds of mathematical actions supported by this environment through which participants display their reasoning to each other as they deploy, produce, and manipulate shared mathematical artifacts.

3.5.2.4. Co-construction of Math Artifacts

The ways mathematical artifacts are deployed in the shared space implicate or inform what procedures and methods may be invoked next to produce other mathematical artifacts, or to modify existing ones as the discussion progresses towards a solution to the task at hand. In an effort to explicate this point further we will make use of Excerpt 3.5.1 displayed below, which immediately follows Davidcyl's drawing activity.

Excerpt 3.5.1

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
26	18:27:13	18:27:32	davidcyl	the nth pattern has n more squares than the (n-1)th pattern	
	18:27:30	18:27:47	137	*[137 has fully erased the chat message]*	
	18:27:47	18:27:52	137	*[137 has fully erased the chat message]*	
27	18:27:37	18:27:55	davidcyl	basically it's 1+2+..+(n-1)+n for the number of squares in the nth pattern	
	18:27:57	18:27:57	137	*[137 has fully erased the chat message]*	
28	18:28:02	18:28:16	137	so n(n+1)/2	
29	18:27:56	18:28:24	davidcyl	and we can use the gaussian sum to determine the sum: n(1+n)/2	
30	18:28:27	18:28:36	davidcyl	137 got it	

Davidcyl's posting at line 26 is stated as a declarative, so it can be read as a claim or assertion. The references to "n" (i.e. not to a particular stage like 2nd or 5th) give the message a general tone. Moreover, the use of the clause "more…than" suggests a comparison between two things, in particular the two cases *indexed* by the phrases "nth pattern" and "(n-1)th pattern" respectively. In short, Davidcyl makes a claim about how the number of squares changes between the (n-1)th and nth stages of the pattern at hand.

The two cases compared in the posting correspond to two subsequent stages of the staircase pattern. Davidcyl's prior drawing work included similar transitions among pairs of particular stages. For instance, while he was drawing the 4th stage, he added a column of 4 new squares to the right of the 3rd stage. In the next line Davidcyl elaborates on his description by providing a sum of integers that accounts for the number of squares required to form the nth stage. In particular, the expression “1+2+..+(n-1)+n” suggests a method to count the squares that form the nth stage. Since Davidcyl made his orientation to columns explicit through his prior drawing work while he methodically added a new column to produce a next stage, this expression can be read as a *reification* of his column-by-column counting work in symbolic form. In other words, Davidcyl achieves a transition from the visual to the symbolic media, which seems to be informed by his methodic construction of specific stages of the staircase pattern.

As Davidcyl composes a next posting, 137 posts a so-prefaced math expression at line 28, "So n(n+1)/2" that (a) shows 137 has been attending to the organization of Davidcyl’s ongoing exposition, (b) displays 137's recognition of the next problem solving step projected by prior remarks, (c) offers an algebraic realization of the procedure described by Davidcyl, and (d) call on others to assess the relevance and validity of his claim. Davidcyl’s message at line 29 (which is produced in parallel with line 28 as indicated by the typing activity markers) is a more elaborate statement that identifies how his prior statements, if treated as a Gaussian sum, yielded the same expression 137 put forward at line 28 (viz. "n(n+1)/2"). Given that 137 anticipated Davidcyl’s Gaussian sum, Davidcyl announces in the very next posting that "137 got it,” which treats 137’s production of the Gaussian sum as evidence that he/she had competently understood Davidcyl's recent remarks.

137's competent contribution to Davidcyl's sequentially unfolding line of reasoning (which has been displayed through his drawings and chat messages) illustrate the organization of actions that we refer as *co-construction* of mathematical artifacts. The *co* prefix for the term co-construction signals our interest in artifacts that are intersubjectively constructed and used by groups rather than individuals. As we have just observed in the excerpt above, *intersubjectivity* is evidenced in the ways participants organize their actions to display their relevance to each other. 137's anticipation and production of the next relevant step in the joint problem-solving effort serves as strong evidence of mutual understanding between him and Davidcyl. Moreover, the term *construction* signals that mathematical artifacts are not simply passed down by the mathematical culture as ready-made platonic entities external to the group. Once enacted in group discourse, culturally transmitted artifacts such as "Gaussian Sum" need to be made sense of and appropriated in relation to the task at hand in a constructivist way. Hence, our use of the combined term *co-construction* implies an interactional process of sense-making by a group of students.

When co-construction takes place in an online environment like a chat tool, the construction process must take place through observable interactions within technical media. This requires students to invent new methods to co-construct mathematical artifacts, yet also makes it possible for them to explicitly reflect on the traces of their co-constructions by investigating the persistent content provided by the technology. Likewise, the persistent records of interactions also allow the researchers to analyze the co-construction process as it unfolded in real-time as we have just demonstrated. In

Chapter 4, we will continue to empirically discover in these records the ways in which mathematical artifacts are (a) appropriated by students from historically developed cultural tools, and (b) emerging from their own ways of languaging and symbolizing within their local communities.

CHAPTER 4. CASE STUDY OF A VIRTUAL MATH TEAM

In this chapter we will provide an EM/CA analysis of a sequence of excerpts obtained from a single VMT Chat session of the same team we covered in section 3.5.2 above. The excerpts are obtained from this team's third session, so the team has already explored similar patterns of sticks and become familiar with the features of the VMT environment during their prior online sessions. As they came to this session the team members knew that they were supposed to continue inventing and discussing new stick-patterns.

Since this dissertation is chiefly concerned with the procedures group members use to coordinate their math problem-solving activities across multiple interaction spaces, we look for perspicuous occasions of such use when we select cases for analysis. The following case study involves one particular instance of such organization. We are concerned with how the actors make sense of each other and their interaction as they proceed. The particular case we will be analyzing involved the use and coordination of actions involving both the whiteboard and the chat spaces, and so served as a useful site for seeing how actors, in this local setting, were able to engage in meaningful interaction while they were working on a math problem together.

In terms of the methodological terminology we have appropriated from Pomerantz (1990), section 4.1 will provide a characterization of an entire problem-solving session recorded in the VMT environment. More specifically, we will present how this team co-constructed a mathematical artifact they called the "hexagonal array" through a coordinated sequence of actions distributed between the chat and whiteboard spaces, and how they subsequently explored its properties by referring to and annotating shared drawings on the whiteboard. In particular, we will focus on the methodic uses of the software features enacted by the members (a) to demonstrate their math reasoning, and (b) to relate their contributions to previously performed actions across/within the dual interaction spaces of the VMT online environment. Finally, in section 4.2 we will discuss how the methods work by describing the sequential/interactional organization of the circumstances in which these methods were invoked, which will motivate the discussion of our main research questions.

4.1. Characterization of Group Interaction

Our characterization of this team's VMT session is organized in terms of a collection of excerpts that chronologically follow each other. An EM/CA analysis of each excerpt is provided as a subsection with a title that summarizes the team's collective mathematical achievement as a group in that episode. We provide transcripts that cover most of the one-hour long problem-solving session, but since our research questions are concerned with the organization of activities in which students co-construct mathematical artifacts online, we deliberately keep the focus of our analysis on the mathematical discussion.

4.1.1. Co-construction of the triangular grid

Excerpt 4.1.1

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
		19:07:52 - 19:11:00		***137 draws a hexagon shape and then splits it up into regions by adding lines. Figure 4.4.1 shows some of the key steps in 137's drawing performance.***	
694	19:11:02	19:11:16	137	Great. Can anyone m ake a diagram of a bunch of triangles?	
	19:11:17	19:11:20	137	*[137 has fully erased the chat message]*	
		19:11:21 - 19:11:38		***137 begins to delete the set of lines he has just drawn by moving them out***	
	19:11:36	19:11:42	qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
	19:11:45	19:11:46	qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
		19:11:49		***137 moved some object/s***	
695	19:11:47	19:11:51	qwertyuiop	Just a grid?	
		19:11:54 - 19:12:01		***137 moved some object/s***	
696	19:12:04	19:12:07	137	Yeah...	
		19:12:14		***137 moved some object/s***	
697	19:12:14	19:12:17	qwertyuiop	ok...	
		19:12:19		***137 moved some object/s***	
		7:12:23 - 7:14:07		***Qwertyuiop draws a grid of triangles in the space opened up by 137. Figure 4.4.2 shows some of the steps in Qwertyuiop's drawing actions.***	

Excerpt 4.1.1 is taken from the beginning of the team's third session in the Spring Fest event. The drawing actions at the beginning of this excerpt were the first moves of the session related to math problem solving. The excerpt begins with a series of drawing actions performed by 137. Figure 4.1.1 shows six snapshots corresponding to intermediary stages of 137's drawing actions: 137 initiates his drawing actions with 6 lines that form a hexagon in stage 1. Then he adds 3 diagonal lines in step 2. The 3rd snapshot shows the additional 2 lines drawn parallel to one of the diagonals. The 4th snapshot shows a similar set of 2 parallel lines added with respect to another diagonal. The 5th snapshot shows slight modifications performed on the new set of parallel lines to ensure intersections at certain places. The 6th snapshot shows the final stage of 137's drawing. Hence, the sequence of drawings and modifications altogether suggests a

particular organization of lines for constructing a hexagonal shape that encloses some triangular and diamond-shaped regions. Note that this drawing episode takes about 3 minutes and it was not interrupted by another team member, which may be taken as an indication that other members were watching 137's drawing performance.

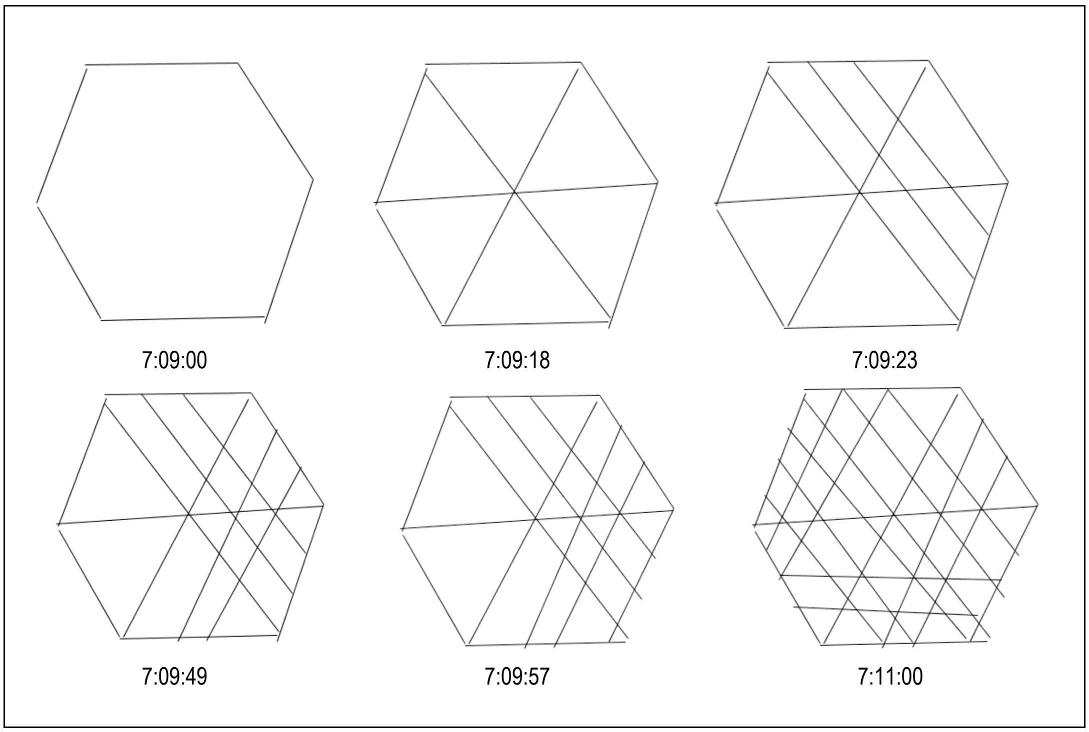


Figure 4.1.1: Six stages of 137's drawing actions obtained from the Replayer tool. The timestamp of each stage is displayed under the corresponding image. Snapshots focus on a particular region on the whiteboard where the relevant drawing activity is taking place.

137's chat posting in line 694 that follows his drawing effort suggests that he considers his illustration inadequate in some way. He makes this explicit with the ironic use of "great" at the beginning of his posting and by soliciting help from other members to produce "a diagram of a bunch of triangles" on the whiteboard. The phrase "bunch of triangles" is used as a verbal gloss to describe the desired form of the projected drawing. Then he removes the diagram he has just produced (the boxes following this posting in the chat window in Figure 4.1.3 below correspond to deletion actions on the whiteboard).

By removing his diagram, 137 makes that space available to other members for the projected drawing activity.

Qwertyuiop responds to 137's query with a request for clarification regarding the projected organization of the drawing ("just a grid?"). After 137's acknowledgement, Qwertyuiop performs a series of drawing actions that resemble the latter stages of 137's drawing actions, namely starting with the parallel lines tipped to the right first, then drawing a few parallel lines tipped to the left, and finally adding horizontal lines at the intersection points of earlier lines that are parallel to each other (see Figures 4.1.2 and 4.1.3). Having witnessed 137's earlier actions, the similarity in the organizations of both drawing actions suggest that Qwertyuiop has appropriated some key aspects of 137's drawing strategy, but modified/re-ordered the steps (e.g., he didn't start with the hexagon at the beginning) in a way that allowed him to produce a grid of triangles as a response to 137's request.

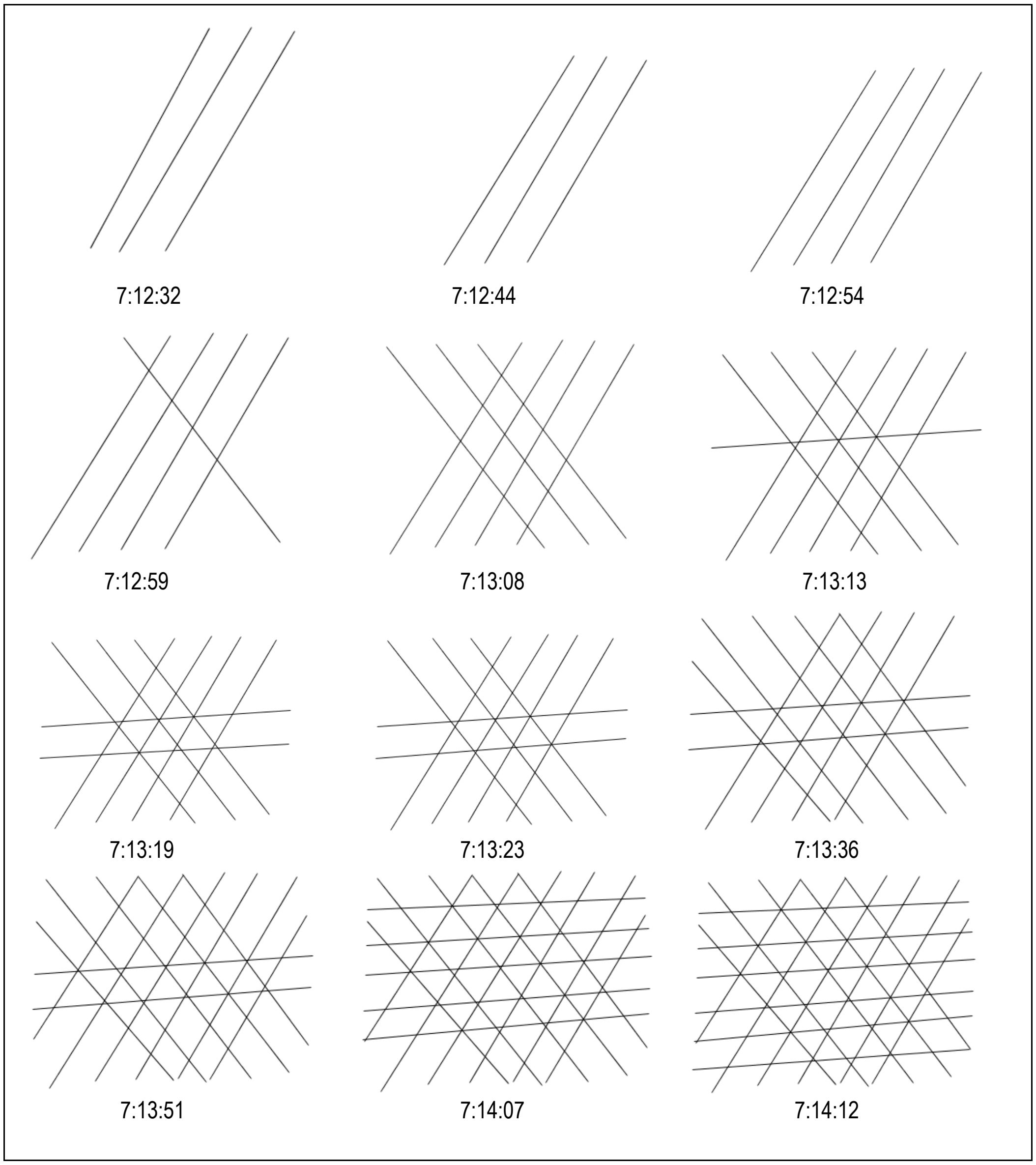


Figure 4.1.2: The evolution of Qwertyuiop's drawing in response to 137's request.

The key point we would like to make in this episode is that *the availability of the sequence of drawing actions that produces a diagram on the shared whiteboard can serve as a vital resource for collaborative sense-making*. As we have seen in Excerpt 4.1.1, 137 did not provide a specific explanation in chat about his drawing actions or about the shape he was trying to draw. Yet, as we have observed in the similarity of

Figures 4.1.1 and 4.1.2, the orderliness of 137’s actions has informed Qwertyuiop’s subsequent performance. The methodical use of intersecting parallel lines to produce triangular objects is common to both drawing performances. Moreover, Qwertyuiop does not repeat the same set of drawing actions, but selectively uses 137’s steps to produce the relevant object (i.e., a grid of triangles) on the whiteboard. Qwertyuiop does not initially constrain his representational development by constructing a hexagon first, but allows a hexagon (or other shapes made with triangles) to emerge from the collection of shapes implied by the intersecting lines. Thus, Qwertyuiop’s performance shows us that he is able to *notice a particular organization* in 137’s drawing actions, and he has *selectively appropriated and built upon* some key aspects of 137’s drawing practice.

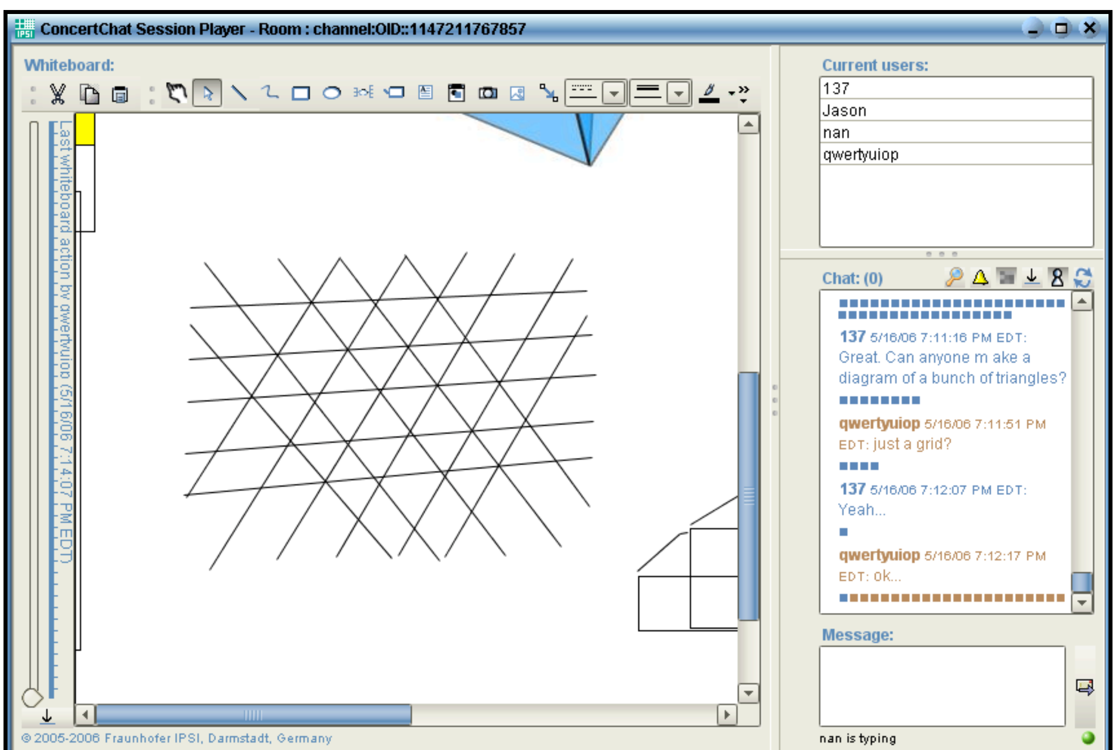


Figure 4.1.3: The state of the whiteboard when Qwertyuiop’s drawing reached its 12th stage in Figure 4.1.2

4.1.2. Introduction of the hexagonal array

Excerpt 4.1.2

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
698	19:13:40	19:14:09	nan	so what's up now? does everyone know what other people are doing?	
		19:14:12		***Qwertyuiop adds a line to the grid of triangles.***	
699	19:14:23	19:14:25	137	Yes?	
700	19:14:18	19:14:25	qwertyuiop	no-just making triangles	
		19:14:32		***Qwertyuiop adds a line to the grid of triangles.***	
701	19:14:31	19:14:33	137	I think...	Message # 699
702	19:14:32	19:14:34	Jason	Yeah	
		19:14:36		***Qwertyuiop adds a line to the grid of triangles.***	
703	19:14:44	19:14:46	nan	Good:-)	Message # 701
704	19:14:45	19:14:51	qwertyuiop	triangles are done	
705	19:14:46	19:15:08	137	So do you want to first calculate the number of triangles in a hexagonal array?	
	19:15:02	19:15:20	Nan	*[nan has fully erased the chat message]*	
706	19:15:22	19:15:45	qwertyuiop	What's the shape of the array? a hexagon?	Message # 705
		19:15:47		***137 locks the triangular grid that Qwertyuiop has just drawn.***	

Excerpt 4.1.2 shown above immediately follows Excerpt 4.1.1, where the team is oriented to the construction of a triangular grid after a failed attempt to embed a grid of triangles inside a hexagon. As Qwertyuiop is adding more lines to the grid the facilitator (Nan) posts two questions addressed to the whole team in line 698. The question not only queries about what is happening now and whether everybody knows what others are currently doing, but the placement of the question at this point in interaction also problematizes the relevance of what has been happening so far. 137's response in lines 699 and 701 treat the facilitator's question as a problematic intervention. Qwertyuiop's response indicates he is busy with making triangles and hence may not know what others

are doing. Jason acknowledges that he is following what's going on in line 702. These responses indicate that the team members have been following (perhaps better than the facilitator) what has been happening on the whiteboard so far as something relevant to the task at hand.

Following Qwertyuiop's announcement in line 704 that the drawing work is complete, 137 proposes that the team calculate "the number of triangles" in a "hexagonal array" as a possible question to be pursued next. Although a hexagon was previously produced as part of the failed drawing, this is the first time someone explicitly mentions the term "hexagonal array" in this session. What makes 137's proposal potentially intelligible to others is the availability of referable resources such as whiteboard objects, and the immediate history of the production of those objects such that the proposal can be seen to be embedded in a sequence of displayed actions. 137's use of "So" to introduce his proposal presents it as a consequence of or a making explicit of what preceded. His suggestion of it as a "first" (next) move implies that the drawings opened up multiple mathematical tasks that the group could pursue and that the proposed suggestion would be a candidate for a next move.

4.1.3. Achievement of indexical symmetry through referential work

Excerpt 4.1.3

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
704	19:14:45	19:14:51	qwertyuiop	triangles are done	
705	19:14:46	19:15:08	137	So do you want to first calculate the number of triangles in a hexagonal array?	
	19:15:02	19:15:20	nan	*[nan has fully erased the chat message]*	
706	19:15:22	19:15:45	qwertyuiop	What's the shape of the array? a hexagon?	Message # 705
		19:15:47		***137 locks the triangular grid that Qwertyuiop has just drawn.***	
	19:15:52	19:15:57	137	*[137 has fully erased the chat message]*	
	19:15:54	19:16:02	Jason	*[Jason has fully erased the chat message]*	
707	19:16:00	19:16:02	137	Ya.	Message # 706
		19:16:09		***137 created a line***	
		19:16:13		***137 created a line***	
708	19:16:13	19:16:15	qwertyuiop	ok...	
		19:16:18 – 19:16:35	137	***137 performs a few drawing actions and then erases them***	
709	19:16:20	19:16:41	Jason	wait-- can someone highlight the hexagonal array on the diagram? i don't really see what you mean...	
		19:16:45 – 19:17:28		***137 adds new lines to the grid on the whiteboard which gradually forms a contour on top of the grid. Figure 4.3.4 shows some of the steps performed by 137.***	
710	19:17:28	19:17:30	Jason	hmm.. okay	
		19:17:32		***[137 resized some objects]***	
		19:17:38		***[137 moved some object/s]***	
711	19:17:42	19:17:43	qwertyuiop	Oops	to whiteboard
712	19:17:35	19:17:44	Jason	so it has at least 6 triangles?	
		19:17:47		***[137 moved some object/s]***	
713	19:17:55	19:17:58	Jason	in this, for instance	to whiteboard
		19:18:03 - 19:18:17		***137 completes the contour by adding more lines, which forms a hexagon.***	
714	19:18:48	19:18:53	137	How do you color lines?	
		19:19:01	137	***[137 changed layout]***	
715	19:18:58	19:19:06	Jason	there's a little paintbrush icon up at the top	
		19:19:07		***[137 changed layout]***	
716	19:19:06	19:19:12	Jason	it's the fifth one from the right	
		19:19:13	137	***[137 changed layout]***	
		19:19:18	137	***[137 changed layout]***	
717	19:19:19	19:19:20	137	Thanks.	
718	19:19:18	19:19:21	Jason	there ya go :-)	
		19:19:25 – 19:19:40		***137 finishes the coloring. Now the contour is highlighted in blue (see last stage in Figure 4.3.4 below).***	

719	19:19:44	19:19:48	137	Er... That hexagon.	
	19:19:38	19:19:49	Jason	*[Jason has fully erased the chat message]*	
	19:19:51	19:19:52	Jason	*[Jason has fully erased the chat message]*	
	19:19:56	19:19:59	137	*[137 has fully erased the chat message]*	
720	19:19:52	19:20:02	Jason	so... should we try to find a formula i guess	

After a brief episode of silence for about 37 seconds, Qwertyuiop posts a message explicitly linked to 137's proposal in line 706. The referential arrow attached to the message makes it explicit that it is addressed to 137. The message is phrased as a question that calls for clarification with regards to137's use of the terms "array" and "hexagon" to describe the shape of the pattern.

Two seconds after this question, 137 anchors the triangular grid to the background. The temporal proximity of this whiteboard move to the previous question makes it difficult to see it as part of a response to the call for clarification. Nevertheless, since the anchoring move preserves the positions and the size of the selected objects and the objects affected by the move includes only the lines recently added by Qwertyuiop, 137's anchoring move seems to give a particular significance to Qwertyuiop's recent drawing. Hence 137's anchoring move can be treated as an (implicit) endorsement of Qwertyuiop's previous drawing effort. This move may have also performed in anticipation of subsequent drawing activity that will be performed on the triangular grid.

Next 137 posts an acknowledgement linked to Qwertyuiop's question in line 707. Following that he draws a line following the grid and a blue rectangle covering the grid, and then he removes the rectangle. In other words, 137 seems to be oriented to the shared drawing, but his moves do not introduce any significant change on the whiteboard yet.

Following these drawing actions Jason posts a query for clarification in line 709. The "wait" used at the beginning calls others to suspend the ongoing activity. The rest of the posting indicates that the available referential resources are still insufficient for Jason to locate what 137 is referring to with the term "hexagonal array." Moreover, the posting explicitly calls for a response to be performed on the shared diagram, i.e., in a particular field of relevance in the other interaction space. Following Jason's query, 137 begins to add a few lines that gradually begin to enclose a region on the triangular grid[12] (see Figure 4.1.4).

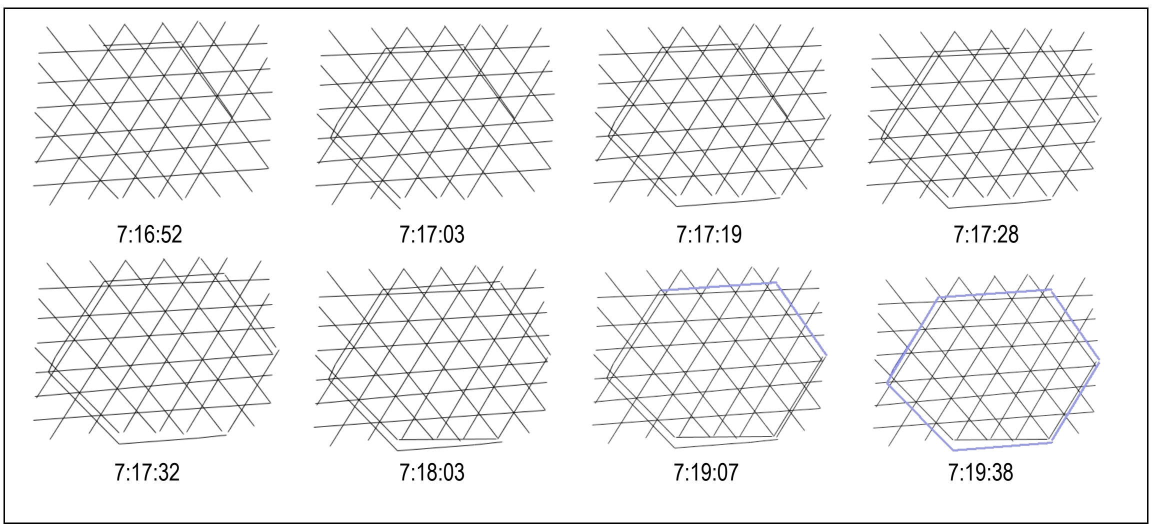


Figure 4.1.4: Snapshots from the sequence of drawing actions performed by 137

When the shared diagram reaches the stage illustrated by the 4th frame in Figure 4.1.4, Jason posts the message "hmmm... okay" in line 710. Since no chat message was posted after Jason's request in line 709, and the only shared actions were 137's work on the whiteboard, Jason's chat posting can be read as a response to the ongoing drawing activity on the whiteboard. As it is made evident in his posting, Jason is treating the

[12] In the meantime, Qwertyuiop also performs a few drawing actions near the shared drawing, but his actions do not introduce anything noticeably different since he quickly erases what he draws each time.

evolving drawing on the shared diagram as a response to his earlier query for highlighting the hexagonal array on the whiteboard: the question/answer adjacency pair is spread across the two interaction spaces in an unproblematic way.

Following provisional acknowledgement of 137's drawing actions on the whiteboard, Jason posts a so-prefaced claim in line 712. This posting is built as a declarative: "so it has at least 6 triangles" with a question mark appended to the end. The use of "so" in this posting again invites readers to treat what follows in the posting as a consequence of the prior actions of 137. In this way, Jason is (a) proposing a defeasible extension of his understanding of the sense of 137's actions and (b) inviting others to endorse or correct this provisional claim about the hexagonal array by presenting this as a query using the question mark.

In line 713 Jason provides further specificity to what he is indexing by the term "it" in line 712 by highlighting a region on the grid with the referencing tool of the VMT system. The textual part of the posting makes it evident that the highlighted region is an instance of the object mentioned in line 712. Moreover, the 6 triangles highlighted by the explicit reference recognizably make up a hexagon shape altogether. Hence Jason's explicit reference seems to be pointing to a particular stage (indexed by "at least") of the hexagonal array that the team is oriented to (see Figure 4.1.5).

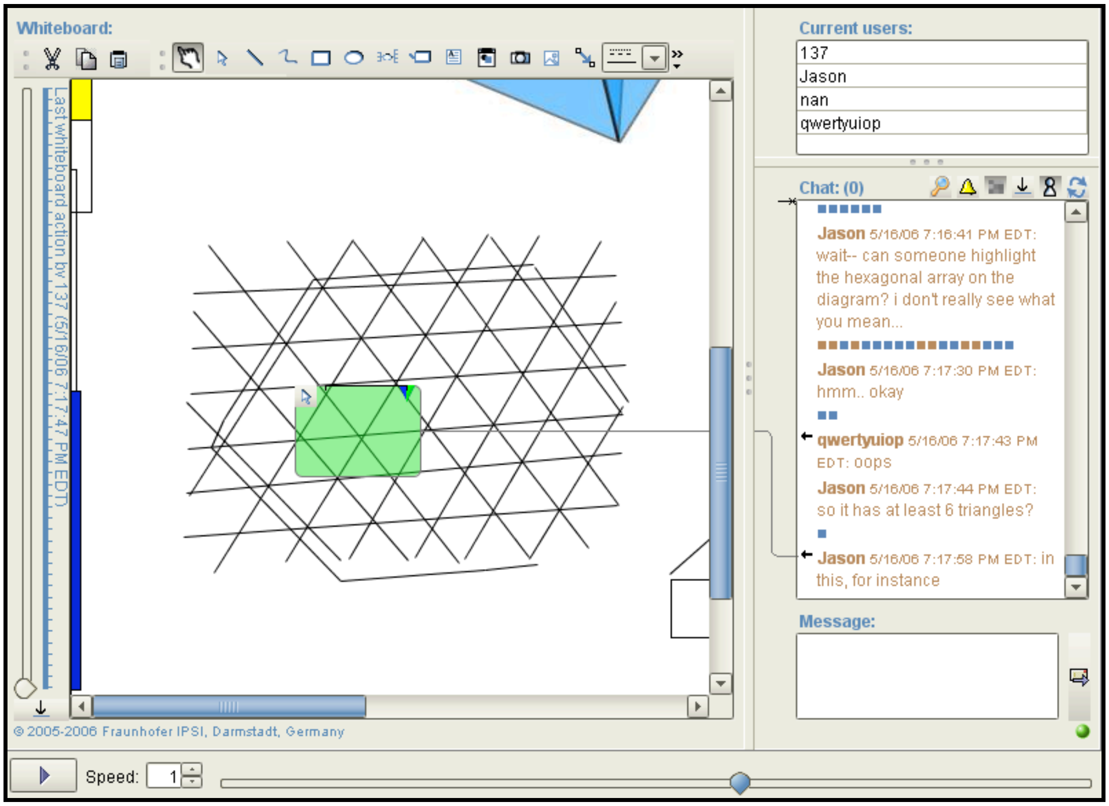


Figure 4.1.5: Jason uses the referencing tool to point to a stage of the hexagonal array.

In other words, having witnessed the production of the hexagonal shape on the whiteboard as a response to his earlier query, in lines 712 and 713 Jason displays his understanding of the hexagonal pattern implicated in 137's graphical illustration. 137's drawing actions highlight a particular stage of a growing pattern made of triangles—stage N=3, as we will see in Figure 4.1.7 below. However, recognizing the stick-pattern implicated in 137's highlighting actions requires other members to project how the displayed example can be grown and/or shrunk to produce other stages of the hexagonal array. Thus, Jason's description of the shape of the "hexagonal array" at a different stage—N=1—is a public display of his newly achieved comprehension of the significance of the math object in the whiteboard and the achievement of *indexical symmetry* among the parties involved with respect to this math object.

Although Jason explicitly endorsed 137's drawing as an adequate illustration, the small boxes in the chat stream that appear after Jason's acknowledgement in line 710 show that 137 is still oriented to and operating on the whiteboard. In line 714, 137 solicits other members' help regarding how he can change the color of an object on the whiteboard, which opens a side sequence about a specific feature of the whiteboard system. Based on the description he got, 137 finishes marking the hexagon by coloring all its edges with blue, and he posts the phrase "that hexagon" in line 719. This can be read as a reference to the shape enclosed by the blue contour, and as a response to other members' earlier requests for clarification.

This excerpt tentatively proposes a major mathematical insight. It is a visual achievement. It emerges from a visual inspection by Jason of 137's visual diagram, based on Qwertyuiop's method of visually representing hexagons as patterns of triangularly intersecting lines. By literally focusing his eyes on a smallest hexagon in the larger array and counting the number of triangles visible within a hexagonal border, Jason discovers that there are at least 6 triangles at the initial stage of a hexagon with one unit on each side. We will see how the group visualizes the generalization of this picture to other stages. But it is already interesting to note that Jason not only observes the composition of a small hexagon out of 6 triangles, but he conveys this to the rest of the group in both media: by posting chat line 712 and by referencing from chat line 713 to a visually highlighted view in the whiteboard, so that his visual understanding can be shared by the group as well as his narrative description in his claim. Having achieved a sense of indexical symmetry with respect to the hexagonal pattern implicated in the drawings, the group will orient to the task of formulating symbolic mathematical expressions to summarize the shape's pattern of growth in the next excerpts.

4.1.4. Decomposition of the hexagonal array into partitions

Excerpt 4.1.4

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
720	19:19:52	19:20:02	Jason	so... should we try to find a formula i guess	
	19:20:08	19:20:12	Jason	*[Jason has fully erased the chat message]*	
721	19:20:13	19:20:22	Jason	input: side length; output: # triangles	
	19:20:35	19:20:36	137	*[137 has fully erased the chat message]*	
722	19:20:12	19:20:39	qwertyuiop	It might be easier to see it as the 6 smaller triangles.	
723	19:20:44	19:20:48	137	Like this?	Message # 722
		19:20:53	137	[137 created a line]	
		19:20:57	137	[137 created a line]	
		19:21:00	137	[137 created a line]	
724	19:21:01	19:21:02	qwertyuiop	yes	
725	19:21:00	19:21:03	Jason	yup	
		19:21:03	137	[137 resized some objects]	
		19:21:05	137	[137 resized some objects]	
	19:21:23	19:21:23	qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
	19:21:24	19:21:26	137	*[137 has fully erased the chat message]*	
726	19:21:23	19:21:29	qwertyuiop	side length is the same...	
727	19:22:05	19:22:06	Jason	Yeah	

Excerpt 4.1.4 immediately follows Excerpt 4.1.3 presented in the previous section. Jason brings the prior activity of locating the hexagonal array on the shared drawing to a close with his so-prefaced posting in line 720 where he invokes the task of finding a formula that was mentioned by 137 earlier. Jason provides further specificity to the formula he is referring to in the next line (i.e., given the side length as input the formula should return the number of triangles as output). In line 722 Qwertyuiop takes up Jason's proposal by suggesting the team consider the hexagonal array as 6 smaller triangles to potentially simplify the task at hand. In the next line, 137 posts a question phrased as "like this?" which is addressed to Qwertyuiop's prior posting, as indicated by the use of the

referential arrow. Next we observe the appearance of three red lines on the shared diagram, which are all added by 137. Here, 137 demonstrates a particular way of splitting the hexagon into six parts: the image on the left of Figure 4.1.6 corresponds to the sequence of three whiteboard actions represented as three boxes in the chat excerpt. After 137 adds the third line whose intersection with the previously drawn red lines recognizably produces six triangular regions on the shared representation, Qwertyuiop and Jason both endorse 137's demonstration of a particular way of splitting up the hexagon shape in lines 724 and 725 respectively.

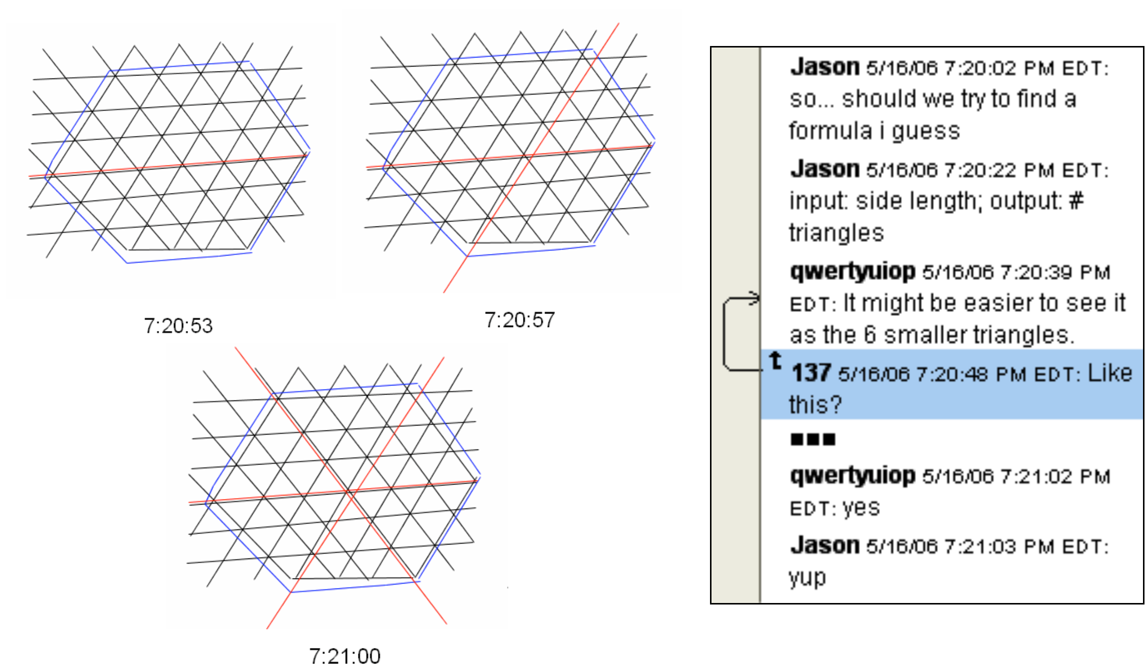


Figure 4.1.6: 137 splits the hexagon into 6 regions

One important aspect of this organization is directing other members' attention to the projected whiteboard activity as a relevant step in the sequentially unfolding exposition in chat. For instance, the deictic term "this" in 137's chat line 723 refers to something yet to be produced, and thereby projects that there is more to follow the current posting, possibly in the other interaction space. Moreover, the use of the referential link and the term "like" together inform others that what is about to be done should be read in relation to the message 137 is responding to. Finally 137's use of a different color marks the newly added lines as recognizably distinct from what is already there as the background, and hence noticeable as a demonstration of what is implicated in recent chat postings.

Again, the progress in understanding the mathematics of the problem is propelled through visual means. In response to Jason's proposal of finding a formula, Qwertyuiop suggests that "it might be easier to see it" in a certain way. Jason's proposed approach might be difficult to pursue because no one has suggested a concrete approach to constructing a formula that would meet the general criteria of producing an output result for any input variable value. By contrast, the group has been working successfully in the visual medium of the whiteboard drawing and has been able to literally "see" important characteristics of the math artifact that they have co-constructed out of intersecting lines. Jason has pointed out that at least 6 triangles are involved (in the smallest hexagon). So Qwertyuiop proposes building on this insight. 137 asks if the way to see the general case in terms of the 6 small triangles as proposed by Qwertyuiop can be visualized by intersecting the hexagon array with 3 intersecting lines to distinguish 6 regions of the array. He does this through a visual construction, simply referenced from the chat with his "Like this?" post. By staring at the final version of the array (see last frame in Figure 4.3.6 with timestamp 7:21:00), all members of the group can see the hexagon divided into 6 equal parts at each stage of the hexagonal pattern.

4.1.5. Joint discovery of a counting method

Excerpt 4.1.5

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
728	19:22:06	19:22:13	Jason	so it'll just be x6 for # triangles in the hexagon	
729	19:22:04	19:22:19	137	Each one has 1+3+5 triangles.	
730	19:22:17	19:22:23	Jason	but then we're assuming just regular hexagons	
	19:22:23	19:22:27	137	*[137 has fully erased the chat message]*	
731	19:21:53	19:22:29	qwertyuiop	the "each polygon corrisponds to 2 sides" thing we did last time doesn't work for triangles	
	19:22:28	19:22:33	137	*[137 has fully erased the chat message]*	
732	19:22:43	19:23:17	137	It equals 1+3+...+(n+n-1) because of the "rows"?	
733	19:23:43	19:24:00	qwertyuiop	yes- 1st row is 1, 2nd row is 3...	
734	19:24:22	19:24:49	137	And there are n terms so... n(2n/2)	
735	19:25:01	19:25:07	137	or n^2	Message # 734
736	19:25:17	19:25:17	Jason	Yeah	
737	19:25:18	19:25:21	Jason	then multiply by 6	
738	19:25:26	19:25:31	137	To get 6n^2	Message # 737

Immediately following the previous excerpt the team moves on to figuring out a general formula to compute the *number of triangles* in a hexagonal pattern. In line 728 of Excerpt 4.1.5, Jason relates the particular partitioning of the hexagon illustrated on the whiteboard to the problem at hand by stating that the number ("#") of triangles in the hexagon will equal 6 times ("x6") the number of triangles enclosed by each partition. In the next posting 137 seems to be indexing one of the six partitions with the phrase "each one." Hence, this posting can be read as a proposal about the number of triangles included in a partition. The sequence of numbers in the expression "1+3+5" calls others to look at a partition in a particular way. While 137 could have simply said here that there are 9 triangles in each partition, he instead organizes the numbers in summation form and offers more than an aggregated result. His expression also demonstrates a systematic method for counting the triangles. In other words, his construction is designed to highlight a particular *orderliness* in the organization of triangles that form a partition. Moreover, the sequence includes increasing consecutive odd numbers, which implicitly informs a certain progression for the growth of the shape under consideration.

About a minute after his most recent posting, 137 offers an extended version of his sequence as a query in line 732. The relationship between the sequence for the special case and this one is made explicit through the repetition of the first two terms. In the new version the "…" notation is used to substitute a series of numbers following the second term up to a generic value represented by "n+n-1" which can be recognized as a standard expression for the n^{th} odd number. Hence, this representation is designed to stand for something more general than the one derived from the specific instance illustrated on the whiteboard. 137 attributes this generalization to the concept of "rows," and solicits other members' assessment regarding the validity of his version (by ending with a question mark). 137's use of the term *rows* seems to serve as a pedagogic device that attempts to locate the numbers in the sequence on the n^{th} stage of the hexagonal pattern (see Figure 4.1.7 for an illustration of the generalized hexagonal pattern co-constructed by the group). For stages 1, 2 and 3, the hexagonal shape has 6*(1) = 6, 6*(1+3) = 24, 6*(1+3+5) = 54 triangles, respectively.

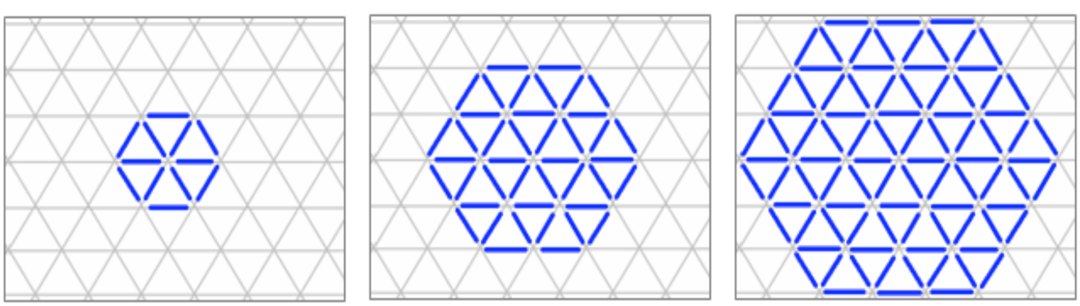


Figure 4.1.7: A reconstruction of the first three iterations of the geometric pattern.

Qwertyuiop's endorsement of 137's proposal comes in line 733. He also demonstrates a row-by-row iteration on a hexagon, where each number in the sequence corresponds to a row of triangles in a partition. In other words, Qwertyuiop elaborates on 137's statement in line 732 of the chat by displaying his understanding of the relationship between the rows and the sequence of odd numbers. Although he does not explicitly reference it here, Qwertyuiop may be viewing the figure in the whiteboard to see the successive rows. The figure is, of course, also available to 137 and Jason to help them follow Qwertyuiop's chat posting and check it.

Then 137 proposes an expression for the sum of the first n odd numbers in line 734.[13] Jason agrees with the proposed expression and suggests that it should be multiplied by 6 next. In the following line, 137 grammatically completes Jason's posting with the resulting expression. In short, by virtue of the agreements and the co-construction work of Jason and 137, the team demonstrates its endorsement of the conclusion that the number of triangles would equal $6n^2$ for a hexagonal array made of triangles. As the group collaboratively discovered, when n equals the stage number (as "input" to the formula), the number of triangles is given by the expression $6n^2$.

An important aspect of the team's achievement of a general expression in this episode is the way they transformed a particular way of counting the triangles in one of the partitions (i.e., a geometric observation) into an algebraic mode of investigation. This

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[13] 137 makes use of Gauss's method for summing this kind of series, adding the first and last term and multiplying by half of the number of terms: $(1 + n + n - 1)*n/2=2n*n/2=n^2$. Apparently, this method was understood by the students or at least not treated by them as problematic.

shift led the team members to recognize that a particular sequence of numbers can be associated with the way the partition grows in subsequent iterations. The shift to this symbolic mode of engagement, which heavily uses the shared drawing as a resource, allowed the team to go further in the task of generalizing the pattern of growth by invoking algebraic resources. In other words, the team made use of multiple realizations (graphical and linguistic) of the math artifact (the hexagonal array) distributed across the dual interaction space to co-construct a general formula for the task at hand.

4.1.6. Constitution of a new math task

Excerpt 4.1.6

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
731	19:21:53	19:22:29	Qwertyuiop	the "each polygon corrisponds to 2 sides" thing we did last time doesn't work for triangles	
	19:22:28	19:22:33	137	[137 has fully erased the chat message]	
732	19:22:43	19:23:17	137	It equals 1+3+...+(n+n-1) because of the "rows"?	
733	19:23:43	19:24:00	Qwertyuiop	yes- 1st row is 1, 2nd row is 3...	
734	19:24:22	19:24:49	137	And there are n terms so... n(2n/2)	
735	19:25:01	19:25:07	137	or n^2	Message # 734
736	19:25:17	19:25:17	Jason	Yeah	
737	19:25:18	19:25:21	Jason	then multiply by 6	
738	19:25:26	19:25:31	137	To get 6n^2	Message # 737
739	19:25:21	19:25:39	Jason	but this is only with regular hexagons... is it possible to have one definite formula for irregular hexagons as well	
740	19:24:19	19:25:46	Nan	(sorry to interrupt) jason, do you think you can ask ssjnish to check the email to see the instructions sent by VMT team, which might help?	
741	19:25:42	19:25:48	Jason	i'm not sure if its possible tho	
742	19:24:39	19:25:48	Qwertyuiop	an idea: Find the number of a certain set of colinear sides (there are 3 sets) and multiply the result by 3	
743	19:25:55	19:26:03	Jason	i did--apparently it didn't work for him	Message # 740
	19:26:09	19:26:10	Nan	*(nan has fully erased the chat message)*	
744	19:26:05	19:26:13	Jason	or his internet could be down, as he's not even on IM right now	
745	19:26:10	19:26:13	Nan	i see. thanks!	Message # 743
		19:26:23		***137 produces two green lines on the***	

		- 19:26:33		***diagonals of the hexagon and two green arrows as displayed in Figure 1***	
746	19:26:20	19:26:36	137	As in those?	Message # 742
747	19:26:46	19:27:05	Qwertyuiop	no-in one triangle. I'll draw it...	Message # 746
		19:27:10 - 19:28:08		***Qwertyuiop repositions some of the existing green lines on a particular section of the hexagon (see Figure 2 below)***	
748	19:28:09	19:28:10	Qwertyuiop	Those	
		19:28:13 - 19:28:19		***137 makes the green lines thicker (see Figure 2 below)***	
	19:28:24	19:28:25	137	*(137 has fully erased the chat message)*	
749		19:28:28	Qwertyuiop	find those, and then multiply by 3	
	19:28:27	19:28:29	137	*(137 has fully erased the chat message)*	
	19:28:30	19:28:32	137	*(137 has fully erased the chat message)*	
750	19:28:48	19:28:50	137	The rows?	
	19:28:58	19:29:00	Qwertyuiop	*(qwertyuiop has fully erased the chat message)*	
751	19:29:01	19:30:01	Qwertyuiop	The green lines are all colinear. There are 3 identical sets of colinear lines in that triangle. Find the number of sides in one set, then multiply by 3 for all the other sets.	
752	19:30:20	19:30:23	137	Ah. I see.	

The excerpt above follows (and partially overlaps with) Excerpt 4.1.5, where the team has co-constructed a formula that characterizes the number of triangles included at any given stage of the hexagonal stick pattern. Since the team had already participated in two prior sessions where they investigated similar stick patterns, they know that their shared task includes finding a similar formula to calculate the number of sticks required to produce the n^{th} stage of the hexagonal pattern. Line 731 is among the first postings where one of the team members explicitly orients to this aspect of the task at hand. By using the descriptive phrase "each polygon corresponds to two sides thing" and the temporal indexical "last time", Qwertyuiop makes a reference to a strategy the team used for calculating the number of sticks for a different stick-pattern during a prior online gathering. In the subsequent part of the posting Qwertyuiop makes an assessment of this strategy with respect to the present task at hand (where the polygons correspond to triangles) by stating that it does not quite work in this occasion. Hence, Qwertyuiop's explicit reference to a past strategy to calculate the number of sticks makes it evident that he is oriented towards this aspect of the task at hand. Finally, this can also be read as a call for a new way to approach the problem of counting the sticks in this occasion.

In the meantime, Jason makes two remarks about the shape of the pattern under consideration in lines 730, 739 and 741. Line 730 is posted when the team was about to conclude the formula for the number of triangles by multiplying the expression they got for the summation by 6 to cover the whole hexagon. The message's sequential position after this particular problem-solving step (i.e. multiply by 6) suggests that it is stated in response to the assumed symmetry in the way the pattern under consideration grows. He problematizes this aspect of the pattern at hand again in line 739. The remaining part of that message introduces the possibility of an "irregular" hexagon and proposes the task of finding a similar "definite" formula for such a case. Finally, line 741 casts some doubt regarding the possibility of finding such a formula. Although it is not immediately taken up by others and the author is interrupted by the facilitator's question, Jason's postings have mathematical significance since they introduce a new math concept related to the task at hand by problematizing an assumption that seems to be implicitly accepted so far.

Later in line 742 Qwertyuiop announces "an idea". He suggests the team find the number of a set of objects indexed by the term "collinear sides" and multiply that number by 3. The statement in parenthesis elaborates further that there are 3 such sets. The use of the term "sides" makes it evident that this statement is about the problem of finding the number of sticks to construct a given stage, as opposed to the problem of finding the number of triangles that make up a hexagon that has been recently discussed[14].

A minute after this posting 137 begins typing at 19:26:20. While the awareness marker continues to display that 137 is currently typing, he adds two green lines on the hexagon that intersect each other and two green arrows (see Figure 4.1.8 below). The green color 137 used recognizably distinguish his drawings from what is present at the background, so *color contrast* is used here again to achieve a figure/ground relationship among the objects that are layered on top of each other. Moreover, the arrows are positioned outside the hexagon and their tips are mutually pointing at each other through a projected diagonal axis.

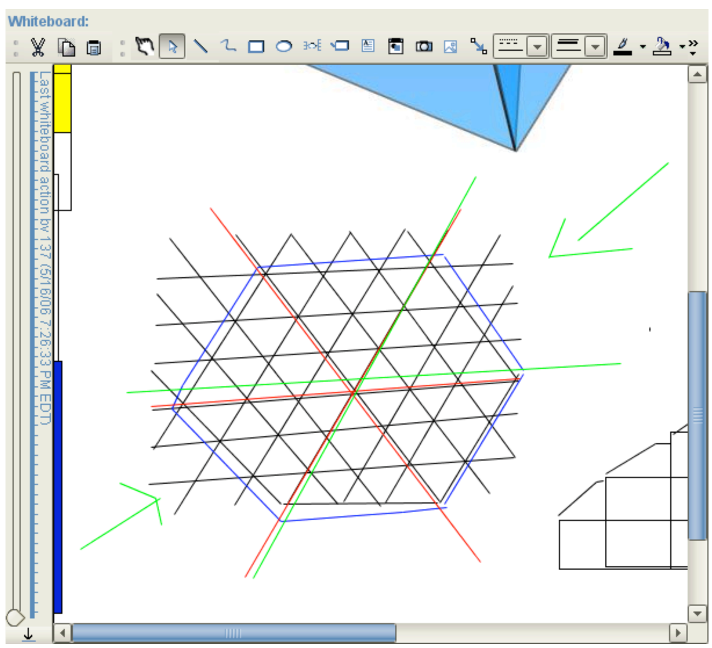

Figure 4.1.8: Green lines and arrows produced by 137

[14] There is a parallel conversation unfolding in chat at this moment between the facilitator (Nan) and Jason about an administrative matter. Lines 740, 743, 744, and 745 are omitted from the analysis to keep the focus on the math problem solving. Yet, this example illustrates that it is possible to have two conversations unfolding in parallel, which is a consequence of the persistent availability of chat messages once they are posted. As prior research concurs (O'Neil & Martin, 2003), this is an important affordance of online chat as compared to talk-in-interaction, which will be discussed further in the results section.

Shortly after his last drawing move 137 completes his typing action by posting the message "as in those?" in line 746, which is explicitly linked to Qwertyuiop's previous posting. The plural[15] deictic term "those" in this posting instructs others to attend to some objects beyond the chat statement itself, possibly located in the other interaction space. The way the drawing actions are embedded as part of the typing activity suggests that they may be designed to be seen as part of a single turn or exposition. Hence, the deictic term "those" can be read as a reference to the objects that the recently added green arrows are pointing at. Moreover, the use of the term "as" and the referential link together suggest that these drawings are related to Qwertyuiop's proposal in line 746. Therefore, based on the evidence listed above, 137 proposes a provisional *graphical representation* of what was described in narrative form by Qwertyuiop earlier and calls for an assessment of its adequacy.

In line 747 Qwertyuiop posts a message linked to 137's proposal with the referential arrow. The use of "no" at the beginning expresses disagreement and the following phrase "in one triangle" gives further specificity to where the relevant relationship should be located. The next sentence in the same posting informs everyone in the group that Qwertyuiop will continue his elaboration on the whiteboard, i.e., in the other interaction space.

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[15] 137's referential work involves multiple objects in this instance. Although the referencing tool of VMT can be used to highlight more than one area on the whiteboard, this possibility was not mentioned during the tutorial and hence was not available to the users. Although the explicit referencing tool of the system seemed to be inadequate to fulfill this complicated referential move, 137 achieves a similar referential display by temporally coordinating his moves across both interaction spaces and by using the plural deictic term "those" to index his recent moves.

Following this line, Qwertyuiop begins to *reposition* some of the green lines 137 drew earlier. He forms 3 green horizontal lines within one of the 6 triangular partitions (see the snapshot on the left in Figure 4.1.9 below). Then in line 748 he posts the deictic term “those” that can be read as a reference to the recently added lines. Immediately following Qwertyuiop’s statement, 137 modifies the recently added lines by increasing their *thickness* (see the snapshot on the right in Figure 4.1.9 below). These moves make the new lines more visible.

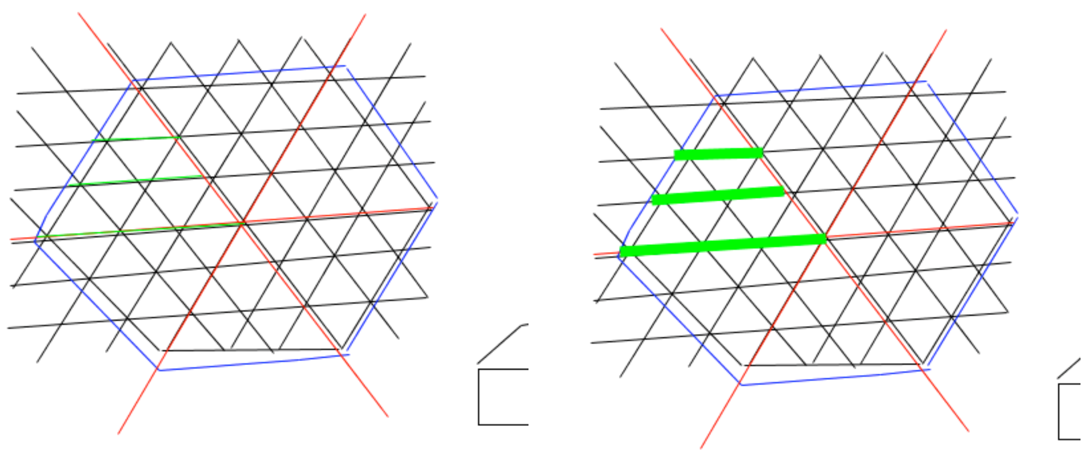


Figure 4.1.9: Qwertyuiop repositions the green lines on the left. Shortly after 137 increases their thickness.

In line 749 Qwertyuiop continues his exposition by stating that what has been marked (indexed by “those”) is what needs to be found and then multiplied by 3. 137’s posting “the rows?” follows shortly after in line 750. The term “rows” has been used to describe a method to systematically count the triangles located in one of the 6 regions of the hexagonal array earlier (see lines 732 and 733). By invoking this term here again, 137 seems to be highlighting a relationship between what is highlighted on the drawing and a term the team has previously used to articulate a method of counting. The question mark appended at the end invites others to make an assessment of the inferred relationship. A minute after 137's question Qwertyuiop posts a further elaboration. The first sentence states that the lines marked with green on the drawing are collinear to each other. The way he uses the term "collinear" here in relation to recently highlighted sticks indicates that this term is a reference to sticks that are aligned with respect to each other along a single grid line. The second sentence asserts that there are "3 identical sets of collinear lines" (presumably located within the larger triangular partition). Finally, the last sentence states that one needs to find the number of sides (i.e. sticks) in one set and multiply that number by 3 (to find the total number of sticks in one partition). Although Qwertyuiop does not explicitly state it here, the way he places the green lines indicate that he is oriented to one of the 6 larger partitions to perform the counting operation he has just described.

Following Qwertyuiop's elaboration, 137 posts "Ah. I see." in line 752. This is a token of cognitive change (Heritage, 2002) where the person who made the utterance announces that he can see something he has not been able to see earlier. Yet, it is still ambiguous what is understood or seen since no display of understanding is produced yet.

4.1.7. Co-construction of a method for counting sticks

Excerpt 4.1.7

Chat Index	**Time Start Typing**	**Time of Posting**	**Author**	**Content**	**Refers to**
752	19:30:20	19:30:23	137	Ah. I see.	
		19:30:48 - 19:30:58		***137 drew an elongated hexagon in orange***	
753	19:31:00	19:31:07	137	Wait. Wouldn't that not work for that one?	
754	19:31:11	19:31:12	Jason	Yeah	
755	19:31:12	19:31:15	Jason	beacuse that's irregular	
756	19:31:09	19:31:17	137	Or are we still only talking regular ones?	
757	19:31:20	19:31:22	137	About	
	19:31:16	19:31:24	Jason	*[Jason has fully erased the chat message]*	
758	19:30:38	19:31:24	Qwertyuiop	side length 1 = 1, side length 2 = 3, side length 3 = 6...	
	19:31:24	19:31:25	Jason	*[Jason has fully erased the chat message]*	
	19:31:27	19:31:36	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
		19:31:45 - 19:32:15		***137 removes the orange hexagon***	
	19:32:31	19:32:32	137	*[137 has fully erased the chat message]*	
759	19:32:32	19:32:50	137	Shouldn't side length 2 be fore?	Message # 758
760	19:32:52	19:32:53	137	*four	
761	19:33:06	19:33:10	Qwertyuiop	I count 3.	Message # 759
762	19:33:20	19:33:25	137	Oh. Sry.	
763	19:33:24	19:33:30	Qwertyuiop	It's this triangle.	to whiteboar d (see Figure x)
	19:33:26	19:33:30	137	*[137 has fully erased the chat message]*	
764	19:33:44	19:33:45	137	We	
	19:33:45	19:33:48	137	*[137 has fully erased the chat message]*	
	19:33:49	19:33:50	137	*[137 has fully erased the chat message]*	
765	19:33:47	19:33:54	Qwertyuiop	I don't see the pattern yet...	Message # 758
766	19:33:50	19:34:01	137	We're ignoring the bottom one?	
		19:34:10 – 19:34:18		***137 first moves the longest green line, adds an orange line segment, moves the longest line back to its original position***	
767	19:34:11	19:34:29	Qwertyuiop	no, 3 is only for side length 2.	Message # 766

About 18 seconds after 137's last posting, Qwertyuiop begins typing but he does not post anything in chat for a while. After 10 seconds elapsed since Qwertyuiop started typing, 137 begins to produce a drawing on the whiteboard. In about 10, seconds 137 produces a smaller hexagonal shape with orange color on the triangular grid. The new elongated hexagonal shape is placed on the right side of the recently added green lines, possibly to avoid overlap (see Figure 4.1.10 below). Once the hexagon is completed, 137 posts a chat message in line 753. The message starts with "wait"[16] which can be read as an attempt to suspend the ongoing activity. The remaining part of the message states that the aforementioned approach may not work for a case indexed by the deictic term "that one". Since 137 has just recently produced an addition to the shared drawing, his message can be read in reference to the orange hexagon. Moreover, since the referred case is part of a message designed to suspend ongoing activity for bringing a potential problem to others' attention, the recently produced drawing seems to be presented as a *counterexample* to the current approach for counting the sticks.

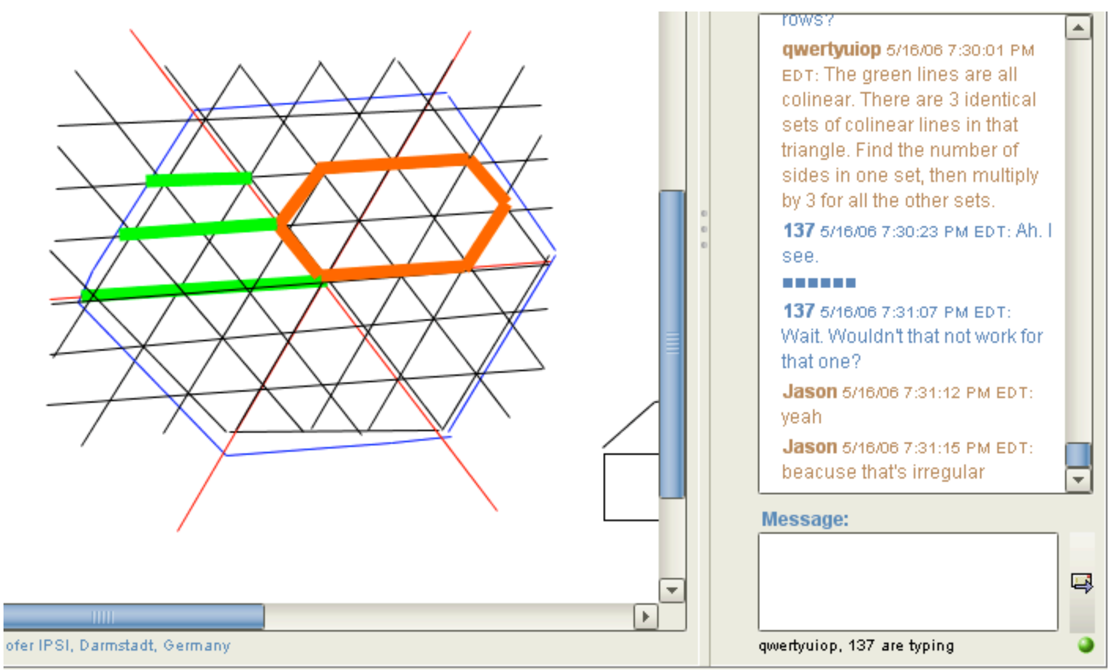


Figure 4.1.10: 137 adds an elongated hexagon in orange

[16] The token "wait" is used frequently in math problem solving chats to suspend ongoing activity of the group and solicit attention to something problematic for the participant who uttered it. This token may be used as a preface to request explanation (e.g. wait a minute, I am not following, catch me up) or to critique a result or an approach as exemplified in this excerpt.

In the next line Jason posts the affirmative token "yes". Since it follows 137's remark sequentially, the affirmation can be read as a response to 137. His immediately following posting begins with "because", which indicates that this message is designed to provide an account for the agreement. The remainder part of the message states this account/reason by associating "irregularity" with an object indexed by the deictic term "that". When these two postings are read together in response to 137's message, the deictic term can be interpreted as a reference to the recently added hexagon marked with orange. In short, Jason seems to be stating that the strategy under consideration would not work for the *orange hexagon* because it is "irregular". With this posting Jason relates the counterexample produced on the whiteboard to a point he made earlier about the possibility of considering an irregular hexagon (see line 739). Hence, Jason provides further specificity to the notion of irregularity he proposed earlier by relating it to a visual representation produced by one of his peers through his referential work.

In the meantime 137 is still typing the statement that will appear in line 756, which asks whether the hexagon under consideration is still assumed to be regular. This question mitigates the prior problematization offered by the same author since it leaves the possibility that the proposed strategy by Qwertyuiop may still work for the regular case. Moreover, with this posting 137 displays that he has paid attention to Jason's remark about the possibility of constructing irregular hexagons in this particular setting, although nobody explicitly responded to that remark when it was uttered in line 739.

In line 758, Qwertyuiop posts a chat message stating "side length 1 = 1, side length 2 = 3, side length 3 = 6...". It took about a minute for him to compose this message after he was first seen as typing at 19:30:38. The way the commas are used to separate the contents of the statement and the ellipsis placed at the end indicate that this posting should be read as an open-ended, ordered list. Within each list item the term "side length" is repeated. The notion of "side length" has been used by this team during a prior session as a way to refer to different stages of a stick-pattern. In the hexagonal case the pattern has 6 sides at its boundary and counting by side-length means figuring out how many sticks would be needed to construct a given side as the pattern grows step by step. Note that this method of indexing stages assumes a stick-pattern that grows symmetrically. So a progression indexed by side length equals 1, 2, and 3 correspond to the first, second, and third stages of the hexagonal stick pattern respectively. When the statement is read in isolation, it is not clear what the numbers on the right of the equals sign may mean, yet when this posting is read together with Qwertyuiop's previous posting where he described what needs to be found, these numbers seem to index the number of sticks within a set of collinear lines as the hexagonal array grows.

After Qwertyuiop's message 137 removes the orange lines he has drawn earlier to produce an irregular hexagon. By erasing the irregular hexagon example, 137 seems to be taking Qwertyuiop's recent posting as a response to his earlier question posted in line 756, where he asked whether they were still considering regular hexagons or not. Although Qwertyuiop did not explicitly respond to this question, his message in line 758 (especially his use of the term side length which implicitly assumes such a regularity) seems to be seen as a continuation of the line of reasoning presented in his earlier postings. In other words, Qwertyuiop's sustained orientation to the symmetric case is taken as a response to the critique raised by 137.

In line 759, 137 posts a message explicitly linked to Qwertyuiop's most recent posting. It begins with the negative token "Shouldn't", which expresses disagreement. The subsequent "side length 2" indexes the problematic item and "be fore" offers a repair for that item. Moreover, the posting is phrased as a question to solicit a response from the intended recipient. 137's next posting in line 760 repairs his own statement with a repair notation peculiar to online chat environments. The asterisk at the beginning instructs readers to attend to the posting as a correction (usually to the most recent posting of the same author). In this case, due to its syntactic similarity to the word in the repair statement, "fore" seems to be the token that is supposedly be read as "four".

In his reply in line 761, Qwertyuiop insists that his counting yields "three" for the problematized case. In the next posting 137's "oh" marks the previous response as surprising or unexpected. The subsequent "sry" can be read as "sorry", which sets an apologetic tone and can be read as backing down. In line 763, Qwertyuiop posts a message that states "it's this triangle" and explicitly points at a region on the shared drawing. The explicit reference and the deictic terms again require the interlocutors to attend to something beyond the text involved in the posting. In short, the sequential unfolding of the recent postings suggests that this posting is designed to bring the relevant triangle in which the counting operation is done for the problematic case (indexed by side length 2) to other members' attention (see Figure 4.1.11).

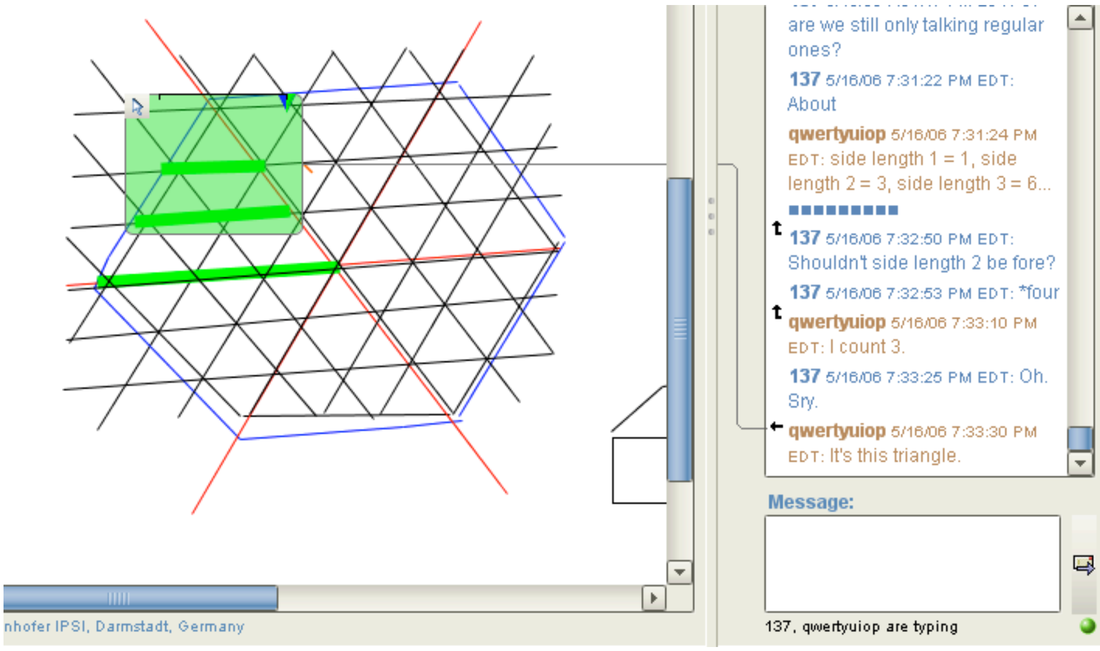


Figure 4.1.11: Qwertyuiop points to the triangle which contains the sticks to be counted for the stage indexed by sidelength=2. The green lines enclosed by the reference correspond to 1+2=3 sticks.

In line 765 Qwertyuiop posts another message explicitly pointing to his earlier proposal for the first few values he obtained through his method of counting, where he states that he has not been able to "see a pattern yet". Hence, this statement explicitly specifies "the pattern" as what is missing or needed in this circumstance. The message not only brings in a *prospective indexical* (Goodwin, 1996) "the pattern" into the ongoing discussion as a problem-solving objective, but also invites other members of the team to join the search for that pattern.

In the next line 137 posts a question that brings other members' attention to something potentially ignored so far. The term "bottom one" when used with "ignore" indexes something excluded or left out. When read as a response to Qwertyuiop's recent exposition in lines 761 and 763, the "bottom one" seems to be a reference to the part of the drawing that was not enclosed by Qwertyuiop's explicit reference. After his posting 137 performs some drawing work on the whiteboard. He moves the longest green line first, then he adds a short line segment with orange color, and then he moves the same green line back to its original location (see Figure 4.1.12). These moves make 137's orientation to a particular part of the drawing explicit. When read together with his previous question, the orange line could be seen as a marker for the problematic part previously referred as the "bottom one".

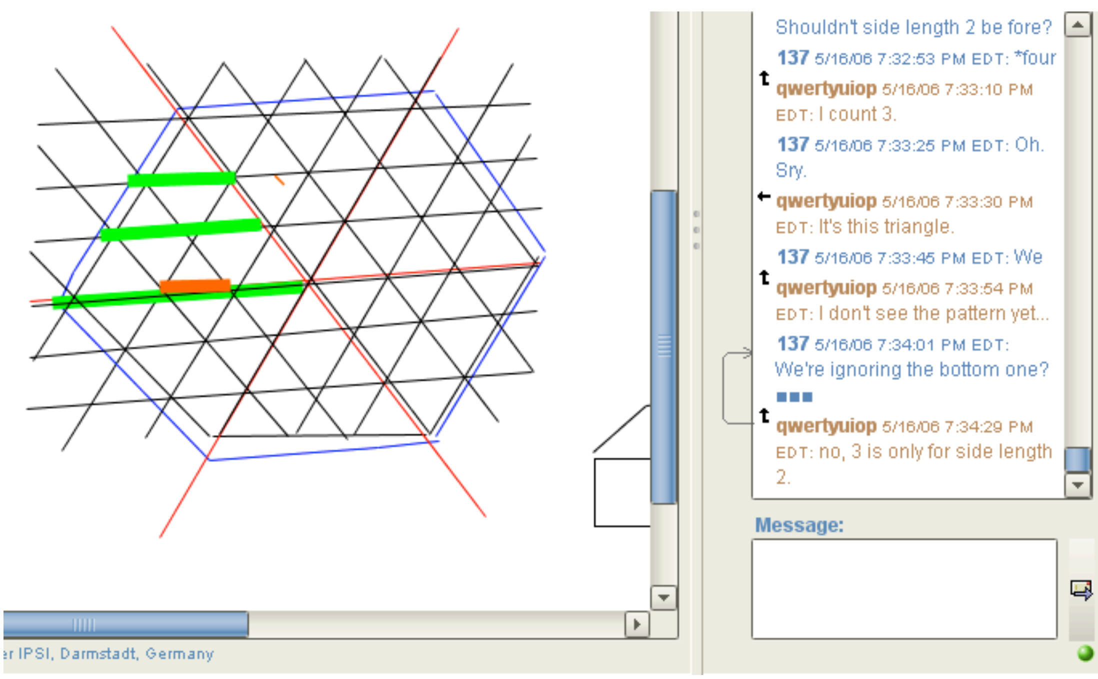


Figure 4.1.12: 137 adds an orange segment to the drawing

The next posting by Qwertyuiop, which appears in line 767, is explicitly linked to 137's question in the previous line. The message begins with "no" which marks the author's disagreement with the linked content, and the subsequent part of the message provides an account for the disagreement by stating that the value 3 is only relevant to the case indexed by "sidelength 2".

The sequence of exchanges between 137 and Qwertyuiop in this excerpt indicates that there seems to be a mis-alignment within the group about the procedure used for counting the number of sticks. This misalignment is made evident through explicit problematizations and disagreements. The way the members make use of both spaces as they interact with each other make it increasingly clear for them (a) what are the relevant pieces indexed by the terms like "collinear" and "triangle" and (b) how are they used in the counting process. Nevertheless, the misalignment between the counting procedures suggested in 137's and Qwertyuiop's contributions remains to be resolved.

4.1.8. Collective noticing of a pattern of growth

Excerpt 4.1.8

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
765	19:33:47	19:33:54	Qwertyuiop	I don't see the pattern yet...	Message #758
766	19:33:50	19:34:01	137	We're ignoring the bottom one?	
		19:34:10 - 19:34:18		***137 first moves the longest green line, adds an orange line segment, moves the longest line back to its original position***	
767	19:34:11	19:34:29	Qwertyuiop	No, 3 is only for side length 2.	Message #766
768	19:34:36	19:34:52	137	And I think the'y;re all triangular numbers.	Message #765
		19:35:03 - 19:35:16		***137's changes the color of the longest green line to red, and then to green again***	
769	19:35:06	19:35:17	Qwertyuiop	"triangular numbers"?	Message #768
		19:35:27 - 19:35:36		***137's draws a red hexagon on the diagram (Figure 6)***	
770	19:35:28	19:35:37	Jason	You mean like 1, 3, 7, ...	
771	19:35:39	19:35:39	Jason	?	
	19:35:49	19:35:50	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
772	19:35:48	19:35:59	137	Like 1,3,6,10,15,21,28.	Message #770
773	19:35:51	19:36:02	Qwertyuiop	The sequence is 1, 3, 6...	Message #770
774	19:36:02	19:36:30	137	Numbers that can be expressed as n(n+1)/2, where n is an integer.	
775	19:36:44	19:36:45	Qwertyuiop	Ah	
776	19:37:09	19:37:18	137	So are we ignoring the bottom orange line for now?	Message #766

In line 768, 137 posts a message linked to Qwertyuiop's posting in line 765. The preface "And" and the explicit reference together differentiate this contribution from the ongoing discussion about a piece that was potentially excluded from the second stage. Note that Qwertyuiop's message in line 765 refers further back to an older posting where he proposed a sequence of numbers for the first 3 stages. When 137's message is read in relation to these two prior messages, the phrase "they are all" seems to be a reference to this sequence of numbers. Hence, the message can be read as an uptake of the issue of finding a pattern that fits this sequence. Moreover, by proposing the term “triangular numbers” as a possible characterization for the sequence, 137 offers further specificity to the prospective indexical, the “pattern”, which was initially brought up by Qwertyuiop.

Following his proposal, 137 changes the color of the longest green line segment at the bottom to red and then to green again. In the meantime Qwertyuiop is typing what will appear in line 769, which can be read as a question soliciting further elaboration of the newly contributed concept “triangular numbers”. 137 continues to act on the whiteboard and he adds a red hexagon to the shared drawing (see Figure 4.1.13 below). Since the hexagon is located on the section referenced by Qwertyuiop several times earlier and shares an edge with the recently problematized orange section, this drawing action can be treated as a move related to that thread of discussion.

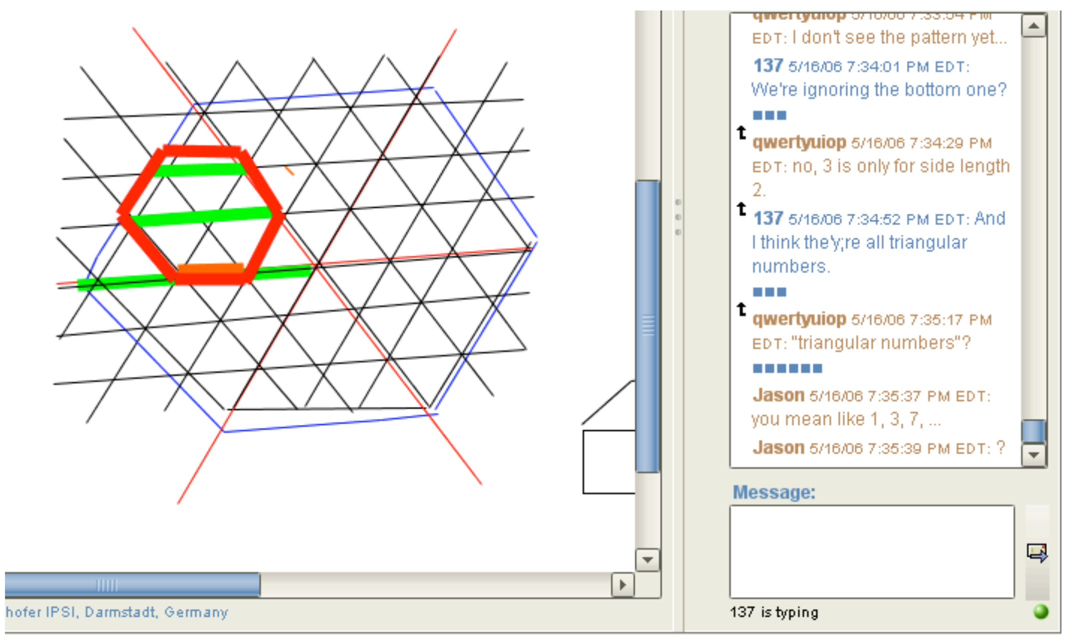


Figure 4.1.13: 137 adds a red hexagon inside the partition the team has been oriented to

Jason joins the discussion thread about triangular numbers by offering a list of numbers in line 770. The term "like" is used here again to relate a mathematical term to what it may be indexing. This posting alone can be read as an assertion, but the question mark Jason posts immediately after in the next line mitigates it to a statement soliciting others' assessment. At roughly the same time, 137 posts a substantially longer sequence of numbers, and immediately after Qwertyuiop points the difference between the sequence at hand and what Jason offered as a list of triangular numbers. In line 774, 137 elaborates his definition further by offering an algebraic characterization of triangular numbers as integers that can be expressed with the formula n(n+1)/2.

In short, the sequence resulted from Qwertyuiop's counting work based on "collinearity" has led the team to notice a relationship between that sequence and a mathematical object called "triangular numbers". The latter symbolic definition offered by 137 for triangular numbers in response to the ongoing search for a pattern has established a relationship between a geometrically motivated counting work and an algebraic/symbolic representation stated in generic form as n(n+1)/2.

4.1.9. Resolution of referential ambiguity via visual proof

Excerpt 4.1.9

Chat Index	**Time Start Typing**	**Time of Posting**	**Author**	**Content**	**Refers to**
776	19:37:09	19:37:18	137	So are we ignoring the bottom orange line for now?	Message #766
	19:37:15	19:37:32	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
777	19:37:32	19:37:36	Qwertyuiop	"green"?	Message #776
778	19:37:44	19:37:48	137	THe short orange segment.	
	19:37:56	19:37:58	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
		19:37:59 - 19:38:02		***137 changes the color of the green lines enclosed by the red hexagon to blue (see Figure 4.1.14)***	
779	19:37:49	19:38:05	137	PArallel to the blue lines.	
780	19:37:58	19:38:05	Qwertyuiop	I don't think so...	
781	19:38:20	19:38:26	137	Wait, we are counting sticks right now, right?	Message #780
782	19:38:35	19:38:48	Qwertyuiop	yes-one of the colinear ets of sticks	
783	19:38:55	19:39:08	Qwertyuiop	oops-"sets" not " ets"	
784	19:39:22	19:39:42	137	So we are trying to find the total number of sticks in a given regular hexagon?	Message #782
785	19:39:50	19:40:18	Qwertyuiop	not yet-we are finding one of the three sets, then multiplying by 3	Message #784
	19:40:22	19:40:24	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
786	19:40:25	19:40:40	Qwertyuiop	that will give the number in the whol triangle	
787	19:40:34	19:40:51	137	Then shouldn't we also count the bottom line?	Message #785
788	19:40:52	19:41:01	Jason	are you taking into account the fact that some of the sticks will overlap	Message #786
	19:41:12	19:41:15	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
789	19:41:25	19:41:41	137	Then number of sticks needed for the hexagon, right?	Message #786
790	19:41:16	19:42:22	Qwertyuiop	Yes. The blue and green/orange lines make up on of the three colinear sets of sides in the triangle. Each set is identical and doesn't overlap with the other sets.	Message #788
791	19:42:50	19:42:50	Jason	Ok	
	19:42:50	19:42:52	Jason	*[Jason has fully erased the chat message]*	
	19:42:52	19:43:01	Jason	*[Jason has fully erased the chat message]*	
792	19:43:03	19:43:11	Jason	this would be true for hexagons of any size right>	
793	19:43:09	19:43:13	Qwertyuiop	triangle, so far	Message #789
794	19:43:25	19:43:25	137	Oh.	
795	19:43:25	19:43:26	Qwertyuiop	this one	whiteboard
796	19:43:42	19:43:52	137	Yes, but they will overlap...	
797	19:43:59	19:44:13	137	Eventually when you multiply by 6 to get it for the whole figure.	
798	19:44:01	19:44:30	Qwertyuiop	no, the sets are not collinear with eachother. I'll draw it...	Message #796
		19:44:35		***Qwertyuiop moves the small hexagon in***	

		- 19:44:56		***red and blue lines out of the grid (see Figure 4.1.16)***	
799		19:44:59	137		Message #798
		19:44:59 - 19:45:17		***Qwertyuiop repositions and resizes the red lines on the grid***	
	19:45:00	19:45:17	137	*[137 has fully erased the chat message]*	
		19:45:20		***Qwertyuiop continues adjusting the red lines***	
	19:45:20	19:45:22	137	*[137 has fully erased the chat message]*	
		19:45:23 - 19:45:37		***Qwertyuiop continues adjusting the red lines***	
	19:45:36	19:45:38	137	*[137 has fully erased the chat message]*	
	19:45:39	19:45:41	137	*[137 has fully erased the chat message]*	
		19:45:41 - 19:46:16		***Qwertyuiop adds purple lines (see Figure 10)***	
	19:46:20	19:46:21	Qwertyuiop	*[qwertyuiop has fully erased the chat message]*	
800	19:46:22	19:46:34	137	Oh. I see.	
801	19:46:22	19:46:52	Qwertyuiop	Those are the 3 sets. One is red, one is green, one is purple.	
		19:47:07 - 19:47:11	137	***137 starts to make green lines thicker***	
802	19:47:04	19:47:12	Jason	wait--- i don't see the green/purple ones	
		19:47:17 - 19:47:33	137	***137 makes the purple lines thicker (see Figure 11 below)***	
803	19:47:18	19:47:40	Qwertyuiop	so we find a function for that sequence and multiply by 3	Message #774

In line 776, 137 posts a message which is explicitly linked to his prior message in line 766 where he mentioned a potentially ignored piece indexed by the phrase "the bottom one". The use of "So" at the beginning can be read as an attempt to differentiate this message from the recently unfolding discussion about triangular numbers. The subsequent part of the message brings other team members' attention a potentially ignored piece indexed by the phrase "the bottom orange line". 137 used the phrase "the bottom one" earlier, but this time he makes use of *color referencing* as an additional resource to provide further specificity to what he is referring to. At this moment a *red hexagon* and a *short orange segment* are visible on the shared drawing space, which are layered on top of the triangular grid (see Figure 4.1.13 above). The way 137 orients to the new state of the drawing indicates that his earlier drawing actions (marked in the prior excerpt) seem to be performed in preparation for this posting. Hence, this posting can be read as an attempt to *re-initiate a prior thread* about a potentially ignored piece in the counting work, which is distributed over both interaction spaces.

Qwertyuiop's message in the next line involves "green" in quotes, ends with a question mark, and is explicitly linked to 137's last message in line 776. The quotation marks seem to give significance over an object indexed by the color reference. Note that there are 3 green lines on the shared drawing at the moment (see Figure 4.1.13). The use of the color reference and the explicit link suggest that this message is posted in response to 137's question in line 776. When it is read in this way, Qwertyuiop seems to be asking if the relevant line located at the bottom should have been the green one instead.

Following this posting 137 gives further specificity to the problematized object by first stating that it is "short" in line 778. Next 137 modifies the two green lines inside the red hexagon by changing their color to blue (see Figure 4.1.14 below). Then, he posts another message in line 779 that refers to a particular location on the whiteboard that is "parallel" to the recently added "blue lines". In short, 137's recent actions suggest that the object indexed by his phrase, "short bottom orange line" segment, is the one parallel to the blue lines.

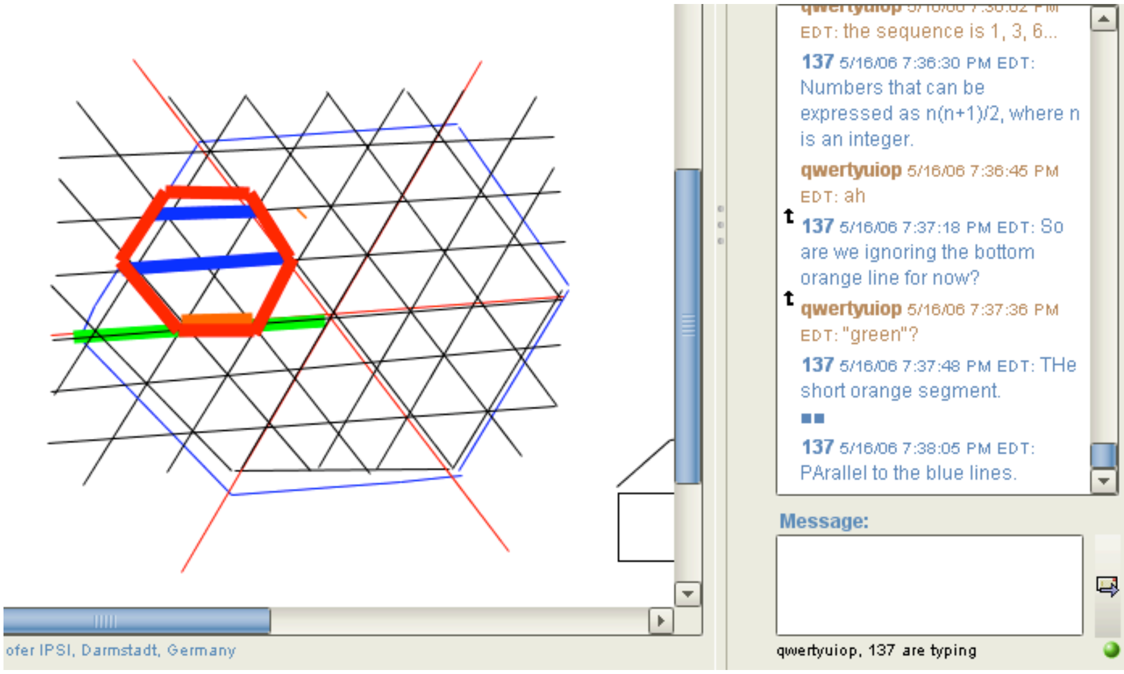


Figure 4.1.14: 137 changes the color of the green lines inside the red hexagon to blue

In line 780, Qwertyuiop states his disagreement. Since the message appears shortly after 137's point that the orange segment is left out of the computation, Qwertyuiop seems to be disagreeing with 137's remark. In the next line, 137 posts a question prefaced with "wait" that calls for suspending the ongoing activity and asks if one can still characterize what the team ("we") is currently doing as "counting the sticks". The posting is explicitly linked to Qwertyuiop's last message. By posting a meta-level question about the ongoing group process following a sustained disagreement with his peer, 137 is making it explicit that there is an asymmetry within the team with respect to the task at hand. Hence, this exchange marks a breakdown in interaction that needs to be attended to before the team can proceed any further.

In the next line, Qwertyuiop takes up this question by providing his account of the ongoing process as counting "one of the collinear sets of sticks". Next, 137 posts another question explicitly linked to Qwertyuiop's answer. The statement of the question gives further specificity to 137's earlier characterization of the counting work undertaken by the team (i.e., counting the sticks for the "whole hexagon"). Qwertyuiop's response to this question states that the focus is not on the whole hexagon yet, but on what he is referring to as "one of the three sets", which would then be followed by a multiplication by 3. In the next line Qwertyuiop continues his explanation that this will give them the number of sticks for "the whole triangle", which can be read as a reference to one of the six triangular partitions.

In line 787, 137 posts a message explicitly linked to the first part of Qwertyuiop's explanation. The posting is phrased as a question problematizing again that the bottom line should also be included in the counting operation. Next, Jason joins the discussion by posting a question linked to the latter half of Qwertyuiop's explanation in line 786, which asks him if he has taken into account "the fact that some of the sticks will overlap". The way Jason phrases his posting brings "overlap" as an issue that needs to be addressed by the counting method under discussion.

In line 789, 137 posts a chat message with a referential link to Qwertyuiop's last posting in line 786. This message seems to extend the order of computations described in Qwertyuiop's exposition by anticipating the next step of the computation, namely calculating the number of sticks needed for the hexagon once the step mentioned in 786 is achieved. In other words, 137 displays his attunement to the order of computations suggested by his peer.

As the referential arrow indicates, in line 788 Qwertyuiop responds to the overlapping sticks issue raised by Jason. He makes reference to the blue and green/orange lines to describe one of the three collinear sets of sides within the triangular partition (since the shared image has remained unchanged, this message can be read in reference to the state displayed in Figure 4.1.14 above). He further asserts that each set is identical and does not overlap. In the next line Jason concurs, and then asks if this should hold for hexagons of any size.

Following Jason's messages, Qwertyuiop posts a message linked to 137's earlier question in line 789. Qwertyuiop stresses again that the focus has been on the triangle so far. His next posting in line 795 includes a referential arrow to the shared diagram that provides further specificity about which part of the hexagon he was referring to with the indexical term "triangle" (see Figure 4.1.15).

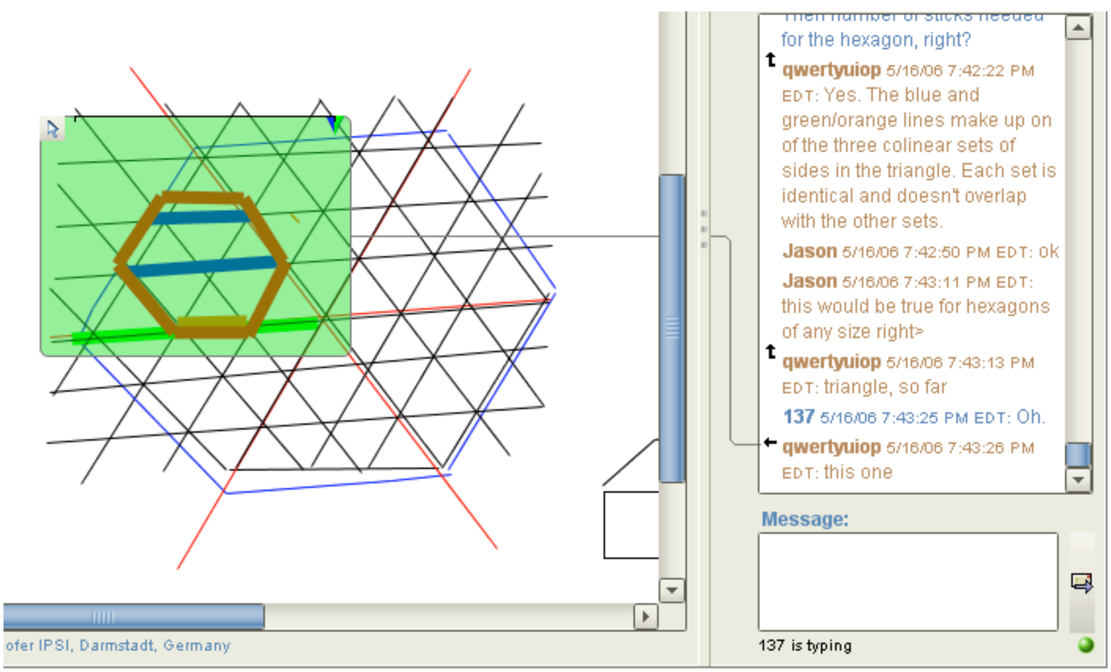


Figure 4.1.15: Qwertyuiop highlights the triangle by using the referencing tool.

In lines 796 and 797, 137 first accepts what Qwertyuiop has asserted, but points to a potential issue that will be faced when the result will be multiplied by 6 to extend the counting operation to the whole hexagon. Before 137 posts his elaboration in line 797, which states when overlap will eventually become an issue, Qwertyuiop begins typing a response to 137's first remark that appears in line 798. In that message Qwertyuiop expresses his disagreement and asserts that "the sets are not collinear with each other". Hence, this posting shows that Qwertyuiop has treated 137's use of the pronoun "they" in line 796 as a reference to the notion of collinear sets. In the latter part of his posting Qwertyuiop announces that he will draw what he is talking about, so this section of the message projects that a related drawing action will follow his statement shortly.

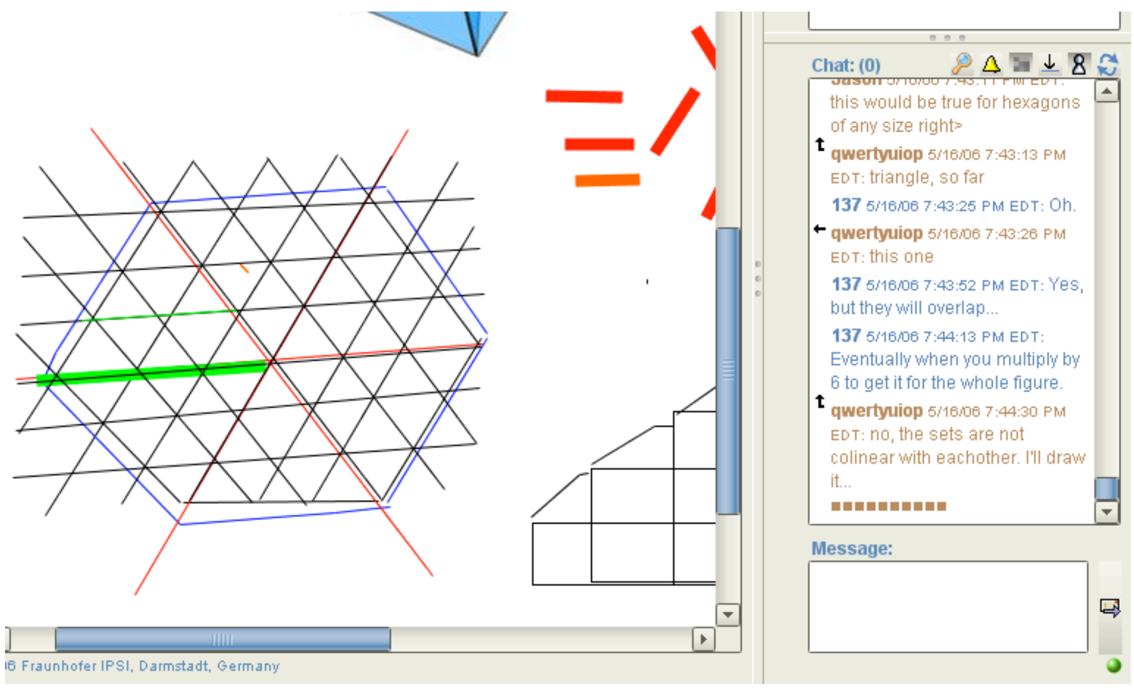


Figure 4.1.16: Qwertyuiop moves the lines added by 137 away.

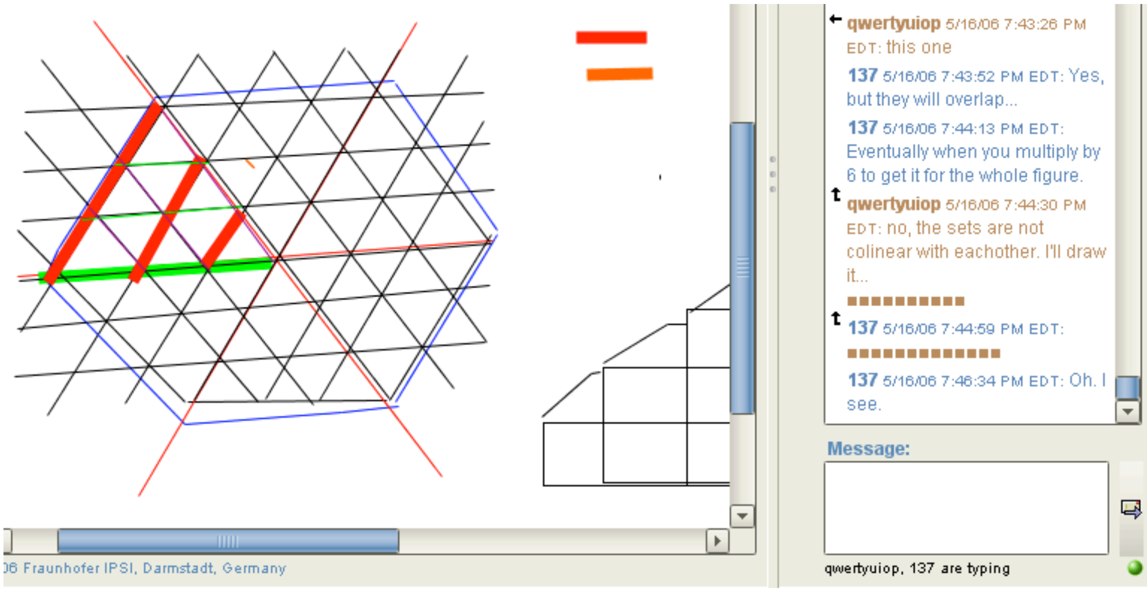


Figure 4.1.17: Qwertyuiop repositions the red lines to mark a part of the larger triangle. Then he adds two horizontal lines in green, parallel to the existing green line. Finally, he adds 3 more lines in purple. Since Qwertyuiop uses a thinner brush to draw the green and purple lines, they are difficult to see.

Figures 4.1.16 and 4.1.17 display snapshots from Qwertyuiop's drawing actions following his last posting. First he moves the red and orange lines to the side, and then he repositions the red lines to highlight 3 segments parallel to each other. Next, he adds 2 green lines parallel to the remaining green line. Finally, he adds 3 purple lines to cover the remaining sticks in that triangular section. The green and purple lines are drawn with a thin brush stroke (see Figure 4.4.10).

In line 800, 137 posts "oh I see", which can be read in response to Qwertyuiop's recent drawing work. Qwertyuiop's graphical illustration seemed to have helped 137 to notice something he had not been able to see earlier. Next, Qwertyuiop posts a message which refers to the lines he has recently drawn with the plural deictic term "those". The message provides further specificity to the math object "3 sets" by locating each set on the diagram through the use of color references “red”, “green” and “purple”. In other words, Qwertyuiop has provided a visual realization of the phrase “3 sets of collinear sides” he coined earlier, which has been treated as problematic by his teammates.

In line 802, Jason states that he cannot see the green/purple lines, which were marked with a thin brush by Qwertyuiop. In response 137 makes these new additions more visible by increasing their thickness (see Figure 4.1.18 below). The final state of the diagram presents a **visual proof** that 3 sets of collinear lines marked with green, purple, and red do not overlap with each other.

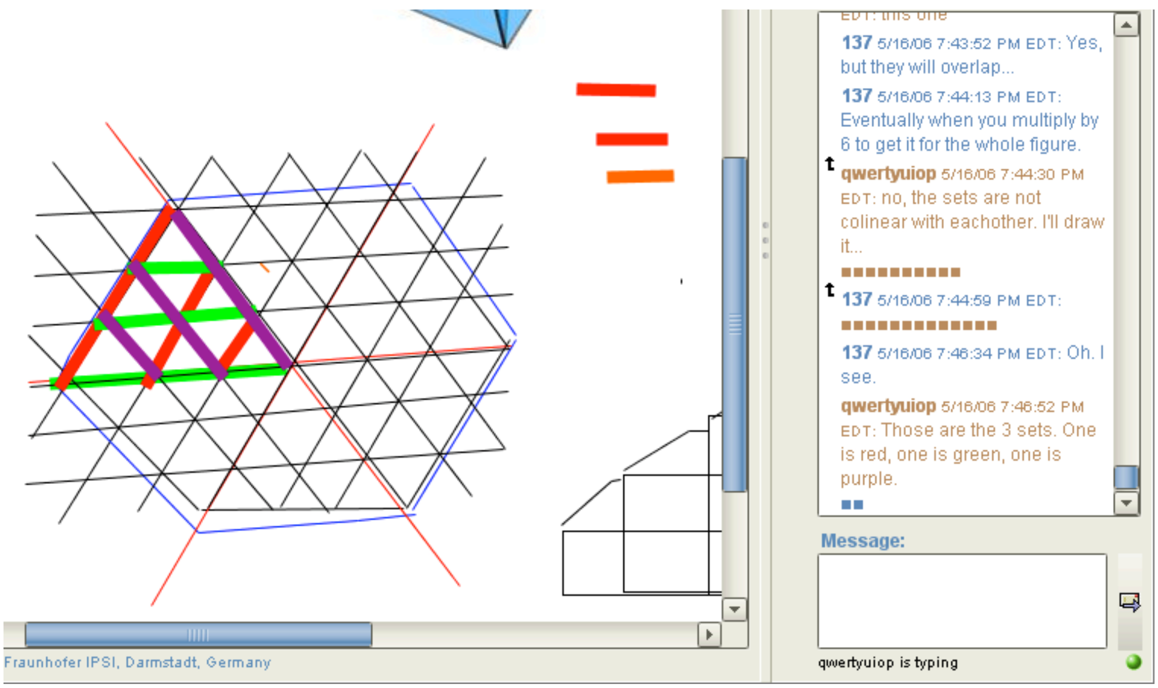


Figure 4.1.18: 137 increases the thickness of the newly added green and purple lines. The final state of the diagram presents a **visual proof** that 3 sets of collinear lines do not overlap with each other.

In line 803, Qwertyuiop provides further specificity to what needs to be found given the visual realization of the collinear sides recently produced on the whiteboard. His message is explicitly linked to a previous posting by 137 (line 774) that provides a formulaic realization for triangular numbers previously associated with the pattern of growth of collinear sides. Hence, Qwertyuiop's statement, "find a function for that sequence and multiply by 3", can be read as a proposal for a strategy to find the number of sticks required to build a triangular partition. In particular, Qwertyuiop is pointing to a candidate algebraic realization of what he has just demonstrated with visual resources on the whiteboard.

To sum up, in this episode the team has achieved a sense of *indexical symmetry* with respect to the term "set of collinear sides" and its projected application towards solving the task at hand. The challenges voiced by 137 and Jason through the course of the episode solicited further elaboration from Qwertyuiop regarding what he meant by collinear sides and how they can be used to devise a method to count the number of sticks. In particular, in this excerpt the team members worked out the overall organization of their joint problem-solving work by discussing what they are trying to find and how they should order some of the steps that have been proposed so far to arrive at a solution. For instance, Qwertyuiop's initial proposal including the term "collinear sets" focuses on one of the triangular regions. Yet, the focus on a triangular region was left implicit, which seemed to have led 137 to treat Qwertyuiop's proposal as applied to the whole hexagon. Through their discussion across both interaction spaces the team has incrementally achieved a shared understanding in terms of how a triangular region is decomposed into 3 sets of collinear, non-overlapping sides, and how that can be used to systematically count the number of sticks in that region.

4.1.10. Re-initiating the discussion of the algebraic formula

Excerpt 4.1.10

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
818	19:51:11	19:52:19	qwertyuiop	what about: f(n)=2n-1 where n is side length	Message No. 772
		19:52:28	137	***137 changes the layout of the last straight line by making it a dashed line.***	
819	19:52:55	19:53:03	137	I don't think that works.	Message No. 818
820	19:53:07	19:53:18	137	Howbout just n(n+1)/2	
821	19:53:37	19:53:41	Jason	for # sticks?	
822	19:53:38	19:53:48	qwertyuiop	that's number of sides for one set	Message No. 820
823	19:53:50	19:53:51	qwertyuiop	?	
824	19:53:57	19:53:59	Jason	oh ok nvm	
	19:54:07	19:54:09	137	*[137 has fully erased the chat message]*	
	19:54:10	19:54:16	137	*[137 has fully erased the chat message]*	
	19:54:17	19:54:19	137	*[137 has fully erased the chat message]*	
	19:54:21	19:54:25	137	*[137 has fully erased the chat message]*	
825	19:54:26	19:54:29	137	Ya.	Message No. 822
826	19:54:36	19:54:58	qwertyuiop	then x3 is 3(n(n+1)/2)	Message No. 820
827	19:55:04	19:55:07	qwertyuiop	simplified to...	Message No. 826
828	19:55:11	19:55:37	qwertyuiop	(n(n+1)1.5	
829	19:55:34	19:55:44	137	On second thought, shouldn't we use n(n-1) for these:	Message No. 826
		19:55:50 - 19:55:55	137	***137 changes the color of two dashed lines into orange (see Figure 4.4.12 below)***	
830	19:55:31	19:55:55	Nan	just a kind reminder: Jason mentioned that he needs to leave at 7p central time sharp	

A brief administrative episode including the facilitator took place between excerpts 4.1.9 and 4.1.10, which is omitted in an effort to keep the focus of our analysis on problem solving. In line 818, Qwertyuiop resumes the discussion about the shared task by proposing a formula "f(n) = 2n-1" where he declares n to be the "side length". It is not evident from the text itself what the formula is standing for. Yet, the message is explicitly linked to an older posting (line 772) where 137 posted the statement "Like 1,3,6,10,15,21,28" as part of a prior discussion on triangular numbers (see Excerpt 4.1.8

above). Hence, when this message is read in reference to line 772, it can be treated as a proposal to generalize the values derived from Qwertyuiop's geometrically informed counting method with a formula stated in symbolic form.

137 rejects Qwertyuiop's proposal in line 819 and then makes a counter proposal in the next line. As we have seen in Excerpt 4.1.8 previously, the sequence of numbers resulted from Qwertyuiop's counting method was previously associated with a math artifact called triangular numbers by 137. The counter proposal includes the same expression 137 provided earlier when he gave a definition of triangular numbers as "integers that can be represented as n(n+1)/2" (see. line 774).

Jason joins the discussion in line 821 by asking if the proposed formula is for the number ("#") of sticks. Although Jason does not specify which object (e.g. the whole hexagon) he is associating the formula with, his posting can be read as an attempt to solicit further elaboration with regards to what the recently proposed formulas are about.

Qwertyuiop's posting in the next line states that the object indexed by the deictic term "that" corresponds to the "number of sides for one set". Note that Qwertyuiop's message is explicitly linked to 137's counterproposal in line 820, so the deictic term "that" can be read as a reference to the expression "n(n+1)/2" included in 137's posting. Moreover, the message sequentially follows Jason's question. Hence, Qwertyuiop seems to be responding to Jason's query by pointing out which object the recently proposed formulas are about. The question mark Qwertyuiop posts in the next line mitigates his previous statement into a question. This can be read as a move to solicit the remaining member's (i.e. 137) assessment of the association Qwertyuiop has just offered. By making his reading of 137's formula explicit, Qwertyuiop also indicates that he concurs with the alternative expression proposed by his peer. Jason's next posting in line 824 indicates that he is now following, which comes just before 137's confirmation linked to Qwertyuiop's claim in 822. Therefore, at this point it seems to be evident for all members in the group that the algebraic expression n(n+1)/2 is associated with one of the "collinear sets of sticks" within a triangular section.

In line 826, Qwertyuiop posts a message linked back to 137's proposal in 820. The use of "then" at the beginning suggests that this message is a consequence or follow up of the message he is referring to. "x3" can be read as a reference to multiplication by 3, where the remaining part of the message provides the expression yielded by this operation. In other words, Qwertyuiop seems to be proposing the next step in the computation, given the expression for the number of sticks for a single "set". In the next two lines he further simplifies this expression by evaluating 3/2 to 1.5.

In line 829, 137 posts a message phrased as a question. The posting begins with "on second thought" which indicates that the author is about to change a position he took prior with respect to the matter at hand. The rest of the statement is phrased as a question and it is addressed to the whole team as indicated by the use of the first person plural pronoun "we". The question part associates the expression "n(n-1)" with the deictic term "these" which is yet to be specified. The posting ends with ":" which projects that more content will likely follow this message subsequently. Next, 137 begins to act on the whiteboard by changing the color of two horizontal lines from green to orange (see Figure 4.1.19 below). The temporal unfolding of these actions suggests that the sticks highlighted in orange are somehow associated with the expression n(n-1). In other words, 137's recent actions can be seen as a move for adjusting the index values in the generalized formula.

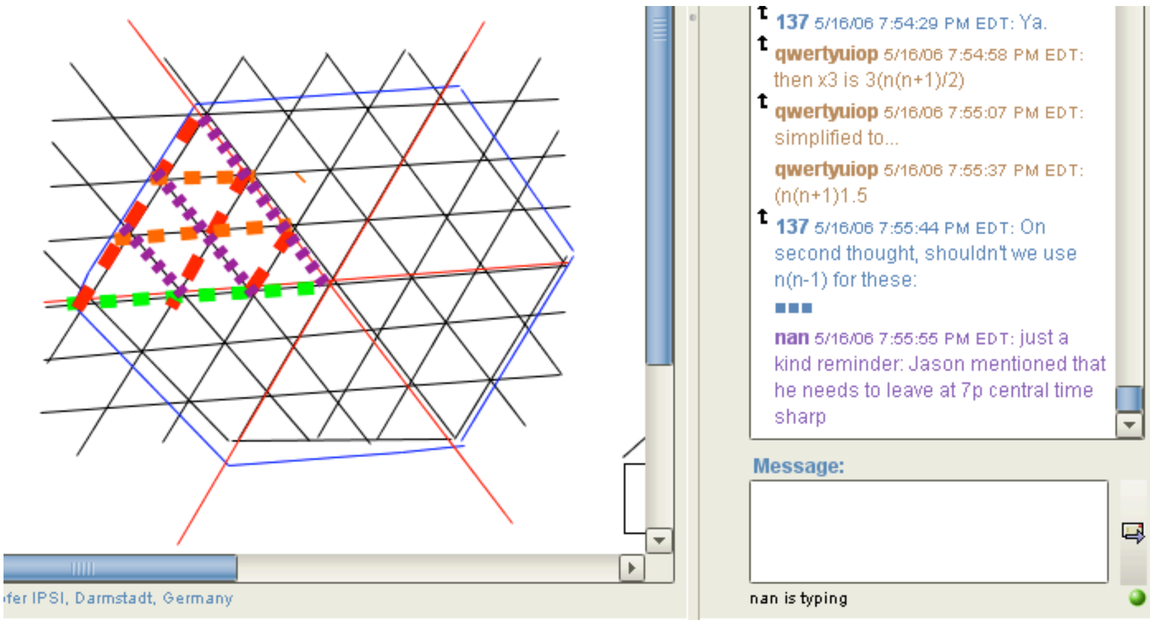


Figure 4.1.19: 137 highlights 2 horizontal lines in orange following his proposal at 7:55:44 (line 829)

In this episode, the team achieves an important transition from a geometrically motivated counting procedure applied on "one of the collinear sets" to a symbolic formula generalizing the procedure to a set of any given sidelength. The generality is achieved through one member's noticing that the sequence of numbers derived from the counting procedure corresponds to "triangular numbers", which seems to be a familiar concept at least for the member who proposed it. The formula that was provided as part of the definition of triangular numbers is then applied to the relevant portion of the pattern at hand to achieve the transition from geometric to algebraic mode of reasoning.

4.1.11. Co-reflecting on the joint achievement of the team

Excerpt 4.1.11

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
841	19:58:23	19:58:25	qwertyuiop	Back to this?	Message No. 829
	19:58:31	19:58:32	137	*[137 has fully erased the chat message]*	
842	19:58:32	19:58:34	137	Ya	
	19:58:23	19:58:35	Jason	*[Jason has fully erased the chat message]*	
	19:58:37	19:58:39	Jason	*[Jason has fully erased the chat message]*	
843	19:58:39	19:58:49	qwertyuiop	why not n(n-1)?	Message No. 829
844	19:58:39	19:58:50	Jason	you guys pretty much have the formula for this hexagon problem...	
845	19:58:57	19:59:28	qwertyuiop	We almost have it for the triangle. I don't know about the hexagon.	Message No. 844
		19:59:32	137	*[137 moved some object/s]*	
846	19:59:35	19:59:50	Jason	well that's just multiplied by a certain number for a hexagon, provided that it is regular	Message No. 845
847	19:59:58	20:00:14	qwertyuiop	but the sides of the triangles making up the hexagon overlap	Message No. 846
		20:00:17	137	*[137 moved some object/s]*	
848	19:59:52	20:00:18	Jason	well i have to leave now; sorry for not participating as much as i wanted to, it's a pretty busy night for me with school and extracurricular stuff	

At the end of excerpt 4.1.10 an administrative discussion was initiated by the facilitator about Jason's departure from the chat session[17]. Some of this exchange is left out since it involved a brief chat about the schedule of the next session. However, while Jason was saying farewell to his peers, an exchange related to the task at hand occurred which is captured in excerpt 4.1.11 above. This episode begins with Qwertyuiop's attempt to reinitiate the problem-solving work by making a reference to an older message posted in line 829 by 137. Following 137's acknowledgement in 842, Qwertyuiop posts a question linked to line 829 which indicates that he is oriented to the expression 137 proposed in that message.

[17] The session was scheduled to end at 7pm, yet the students were allowed to continue if they wished to do so. In this case Jason informed the facilitators in advance that he had to leave at 7pm central (the log is displayed in eastern time).

About a second later, Jason posts a message stating that the formula for the hexagon problem is pretty much done. Jason's use of the phrase "you guys" ascribes this achievement to the remaining members of the team. In line 845, Qwertyuiop posts a message explicitly linked to Jason's last comment. The first sentence "We almost have it for the triangle" provides an alternative account of what has been achieved so far. In his second sentence, Qwertyuiop declares that he does not know about the hexagon yet. Hence, these postings make it evident how Qwertyuiop is treating what has been accomplished so far.

In line 846, Jason posts a message linked to Qwertyuiop's latest remark. In his response Jason states that getting the formula for the hexagon requires a simple multiplicative step provided that the hexagon is regular. Qwertyuiop's response (as indicated by the referential arrow) follows next, where he brings in how the issue of overlap will play out when they move from the large triangles to the whole hexagon. This is followed by Jason's exiting remark where he apologizes for not being able to participate as much as he wanted.

In this excerpt members explicitly comment on how they should characterize their collective achievement. In other words, these postings can be read as a joint reflection on what has been done so far. Another interesting aspect of this short exchange is the apparent shift in the positions with respect to the issue of overlapping sticks in the counting procedure. Jason was the person who brought the issue of overlap for the first time in excerpt 4.1.9, yet his most recent characterization of the team's work seems to dismiss overlap as a relevant matter. Surprisingly, Qwertyuiop, who was the person previously critiqued by Jason for not taking into account the overlaps, explains now why it is still a relevant issue that needs to be attended to. Previously, Qwertyuiop argued that overlaps would not be an issue in his counting work, but that assertion applied only to the triangular section he was oriented to at that time. His most recent posting displays his awareness with regards to when the overlapping sticks will become an issue. These remarks also specify what has not been accomplished yet, and hence suggest the team find a way to address overlaps as an issue to consider next.

4.1.12. Overcoming the problem of overlapping sticks

Excerpt 4.1.12

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
853		20:01:07		*Jason leaves the room*	
854	20:01:19	20:01:31	137	Anyways, if we multiply the orange by 3, we get the:	
855	20:01:14	20:01:34	Nan	do two of you want to continue working for a bit or stop here?	
		20:01:42 - 20:01:48	137	***137 begins to add blue lines on top of the triangular grid***	
856	20:01:40	20:01:44	Nan	i guess that's the answer	Message No. 854
857	20:01:47	20:01:48	Nan	go ahead	
		20:01:49 – 20:01:53	137	***137 continues to add blue lines. The resulting shape is displayed in Figure 13***	
858	20:01:57	20:02:14	137	So then we add 12n for:	
859	20:01:28	20:02:15	qwertyuiop	actually, this doesn't complicate it that much. The overlaps can be accounted for with "-6n"	Message No. 847
		20:02:32 – 20:02:52	137	***137 adds pink contours to the shared drawing, The resulting shape is displayed in Figure 14***	
860	20:02:54	20:02:55	137	Oh.	Message No. 859
861	20:02:56	20:03:07	137	I like addition more than subtraction.	

Excerpt 4.1.12 above follows Jason's departure[18]. In line 854, 137 reinitiates the problem-solving work. In his message 137 mentions multiplying something indexed by "the orange" by 3. Figure 4.1.20 below shows the state of the shared drawing at the moment, where there are two dashed orange lines covering a portion of the hexagon. The remaining part of the message announces the outcome of the suggested operation, yet the result is not provided. Instead, the message ends with a colon ":" indicating that more content is about to follow subsequently. Next, 137 performs a series of drawing actions where he highlights a set of sticks on the triangular grid with blue lines (see Figure 4.1.21 below). These actions are done within a section of the shared drawing that has been empty. Based on the way these actions sequentially unfold and the way the drawing was set up in chat, one can read these actions as the ***visual outcome*** of the operation described in text in line 854. In short, multiplying the orange by 3 seems to yield the number of sticks highlighted in blue, which is an elaborate mathematical move spanning across textual and graphical modalities.

[18] The facilitator opens the possibility to end the session in line 855. The facilitator takes the sustained orientation of the remaining team members to the problem as an affirmative answer and lets the team continue their work.

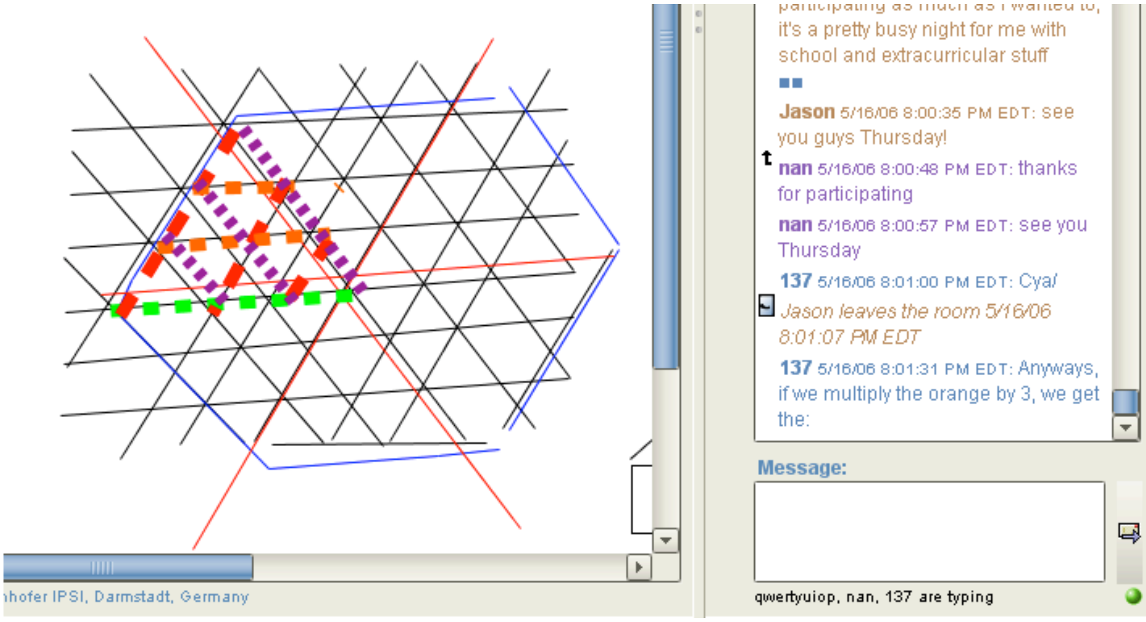


Figure 4.1.20: The state of the whiteboard when 137 began his exposition at 8:01:31 (line 854)

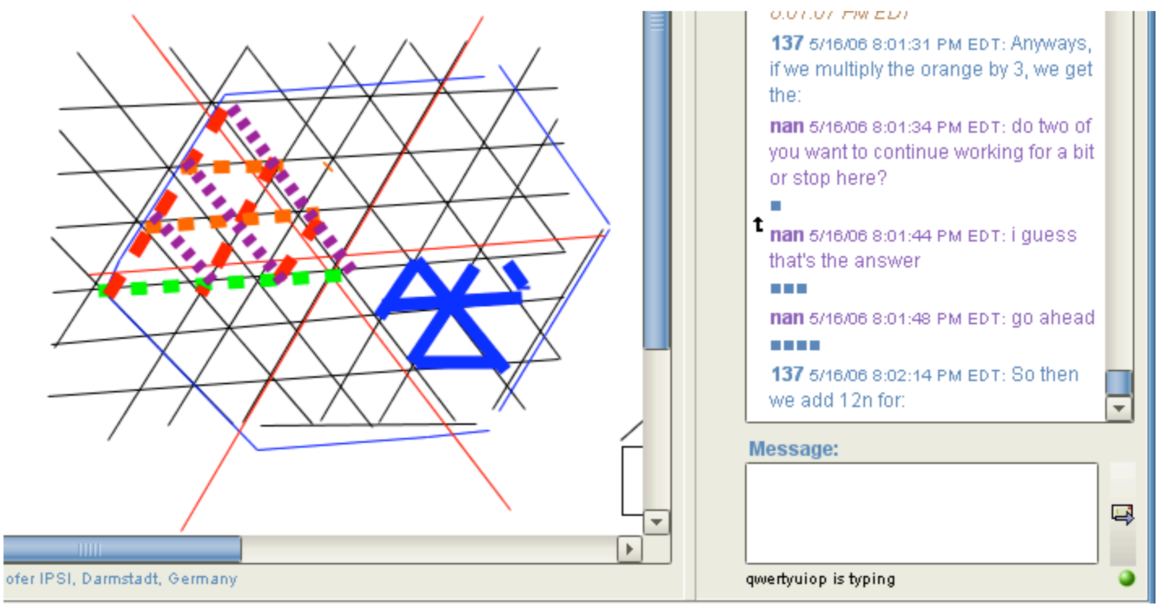


Figure 4.1.21: 137's drawing that followed his posting at 8:01:31 (i.e. line 854). The triangles added in blue follow the chat posting that proposes the multiplication of what is marked with orange by 3.

137 posts another message in line 858 which announces adding "12n" as the next step in his ongoing exposition. The message ends with "for:" which is consistent with his prior use of the colon to project that more elaboration will follow, possibly in the other

interaction space. Next, 137 begins to add pink lines to the shared drawing, which covers the boundaries and the diagonals of the hexagonal array (see Figure 4.1.22 below). The sequential continuity of 137's actions suggests that the lines marked with pink provide a geometric ***realization*** of what is indexed by the symbolic expression "12n" on the particular instance represented by the shared drawing.

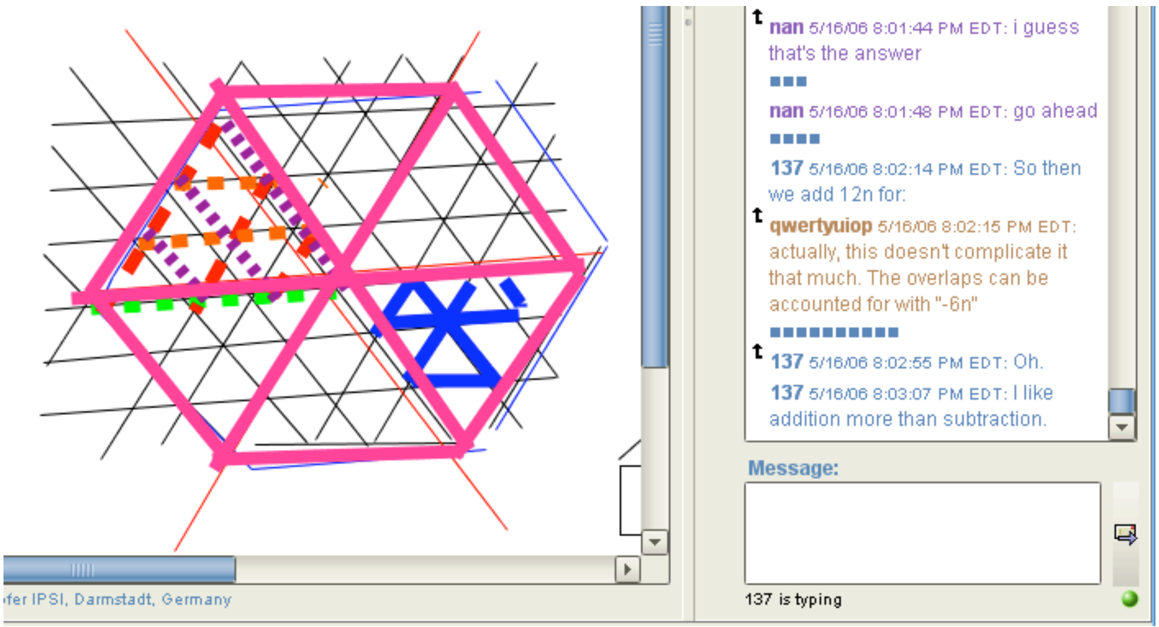


Figure 4.1.22: 137's posting "So then we add 12n for:" is followed by his drawing work where he adds the pink lines. Again the temporal continuity suggests that the pink lines show visually which sticks will be covered when the proposed computation is performed (i.e. "adding 12n")

While 137 was composing his message, Qwertyuiop was also busy typing the message that will appear in line 859. The message appears 1 second after 137's posting and just before he begins adding the pink lines. Hence, the temporal unfolding of actions suggests that these two messages were produced in parallel. In this posting Qwertyuiop makes a reference to an older message where he mentioned the problem of overlapping sticks among the 6 triangular regions. The current message announces that this may not be a big complication. The next sentence in the same post states that the overlaps can be accounted for with the expression “-6n”.

137’s response (as suggested by his use of the explicit reference) to Qwertyuiop’s proposal comes after he is done with marking the pink lines on the whiteboard. The “oh” in line 861 makes 137’s noticing of Qwertyuiop’s proposal. In his next posting, 137 states that he prefers addition rather than subtraction. The contrast made between addition and subtraction suggests that 137 is treating his and Qwertyuiop’s methods as distinct but related approaches to the task at hand.

What 137 is referring to as an “additive” approach can be observed through his prior actions distributed across both interaction spaces. 137’s approach begins with a method to cover a specific portion of one of the six partitions of the hexagon. This is referred as “multiplying the orange by three” and the outcome of this operation is marked in blue. In other words, the orange lines seem to be used as a way to index a single side of a total of 1+2 = 3 triangles (or n(n-1)/2 in general) inside one of the 6 partitions. Hence, multiplying this value by 3 covers the 3 blue triangles enclosed in a partition. Moreover, none of these triangles share a stick with the diagonals and the boundary of the hexagon, so the sticks highlighted in pink are added to cover the missing sticks. In short, the details of the additive approach is revealed through 137’s visual reasoning evidenced in his drawing actions as well as his chat postings coordinated with his drawing work.

The other approach referred as "subtraction" by 137 has been discussed by the team for a while. This approach starts with counting the sticks for one of the six partitions of the hexagon. A partition is further split into 3 "collinear sets" of sticks that does not overlap with each other. The number of sticks covered by a single set turned out to be equivalent to a triangular number. Nevertheless, since this approach covers all the sticks forming a partition and partitions share a boundary with their neighbors, when this value is multiplied by 6 to cover the whole hexagon the sticks at the boundaries (i.e. at the diagonals) would be counted twice. This is referred by the team as the overlap problem. Qwertyuiop's latest proposal provides the expression that needs to be subtracted from the general formula to make sure all sticks are counted exactly once. In contrast, the additive approach does not need subtraction since it splits the shape in such a way that each stick is counted exactly once.

The main point we would like to make about this excerpt is that 137's approach takes the previously demonstrated approaches and their critiques as resources. It brings in a new approach informed by previous discussion in an effort to address the practical issues witnessed. Hence, 137's additive approach is firmly situated within the ongoing discussion. In other words, 137's reasoning has been socially shaped; it is not a pure cognitive accomplishment of an individual mind working in isolation from others.

4.1.13. Derivation of the formula for the number of sticks

Excerpt 4.1.13

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
862	20:03:11	20:03:16	qwertyuiop	do you see why that works	Message No. 859
863	20:03:18	20:03:18	qwertyuiop	?	
864	20:03:12	20:03:29	137	So: 9n(n+1)-6n.	
865	20:03:41	20:03:45	qwertyuiop	9, not 3?	
866	20:04:13	20:04:14	137	?	Message No. 865
867	20:04:18	20:04:35	qwertyuiop	you have "9n(n..."	
868	20:04:37	20:04:47	qwertyuiop	not "3n(n..."?	
869	20:04:51	20:05:00	137	But we need to multiply by 6 then divide by 2	Message No. 868
870	20:05:10	20:05:22	qwertyuiop	x6 and /2 for what?	Message No. 869
871	20:05:44	20:05:47	137	FOr each triangle	
872	20:05:48	20:06:02	137	and /2 because it's part of the equation.	
873	20:06:03	20:06:06	137	of n(n+1)/2	
874	20:05:36	20:06:20	qwertyuiop	it's x3 for the 3 colinear sets, then x6 for 6 triangles in a hexagon... where's the 9 and 2?	
875	20:06:28	20:06:28	qwertyuiop	Oh	Message No. 872
876	20:06:35	20:06:38	137	So 18/2.	
877	20:06:42	20:06:50	137	A.K.A. 9	
	20:06:55	20:06:58	137	[137 has fully erased the chat message]	
878	20:06:48	20:07:08	qwertyuiop	(n(n+1)/2)x3x6	Message No. 873
879	20:07:14	20:07:15	137	Yeah.	
880	20:07:20	20:07:27	qwertyuiop	Which can be simplified...	
881	20:07:42	20:07:46	137	To 9n(n+1)	Message No. 880
	20:07:35	20:07:50	qwertyuiop	[qwertyuiop has fully erased the chat message]	
882	20:08:01	20:08:04	qwertyuiop	that's it?	Message No. 881
883	20:08:10	20:08:12	137	-6n.	
884	20:08:17	20:08:24	137	So 9n(n+1)-6n	
885	20:08:20	20:08:34	qwertyuiop	i'll put it with the other formulas...	

The next excerpt immediately follows the prior one. It begins with Qwertyuiop's question addressed to 137, which asks if he could see why subtracting -6n would work. In the mean time 137 seems to be busy typing the message that will appear in line 864. The use of "So" suggests that this message is stated as a consequence of what has been discussed so far. The colon is followed by the formula “9n(n+1)-6n”, which involves the term “-6n” in it. By proposing a formula making use of the term “-6n”, 137 makes his orientation to Qwertyuiop’s proposal explicit. Moreover, the sequential build up suggests that the proposed expression stands for the formula for the number of sticks for the hexagonal array.

Qwertyuiop’s next posting in line 864 seems to problematize the appearance of 9 in the proposed formula and asks if 3 should have appeared there instead. Next, 137 posts a question mark linked to Qwertyuiop’s question, which can be read as a request for more elaboration. Qwertyuiop elaborates in the next two lines by posting the part of the formula that is problematic for him and then by suggesting a repair for that part. His elaboration ends with a question mark that can be seen as an attempt to solicit his peer’s assessment. 137’s reply in line 869 states that the steps of the computation should also include multiplication by 6 and division by 2. In response Qwertyuiop asks for what part of the pattern those operations need to be done. 137’s reply spans 3 lines, where he first states “for each triangle” and then mentions that “/2” comes from the equation n(n+1)/2. Hence the sequential organization of these messages suggest that 137 associates multiplication by 6 with the triangles (i.e. the larger triangular partitions) and “/2” with the equation for triangular numbers.

In the mean time Qwertyuiop has been typing what will appear in line 874. The first sentence associates each multiplication operation with a specific section of the hexagonal pattern, namely “x3” for the 3 “collinear sets” within a triangular partition and “x6” for the

6 triangular partitions making up the hexagon. The next sentence in that posting problematizes again the appearance of 9 and 2 in the steps of the calculation. 8 seconds later Qwertyuiop posts "oh" in response to 137's remark about the equation in line 872, which indicates that the referenced message has led him to notice something new. This is followed by 137's demonstration of the derivation of 9 from the numbers previously mentioned. In the mean time Qwertyuiop is composing an expression that brings all the items they have just talked about together in symbolic form, which appears in line 878 in response to line 873 where 137 reminded him about the equation n(n+1)/2. 137 expresses his agreement in the next line. Next, they simplify the expression and subtract -6n to derive the final formula for the number of sticks.

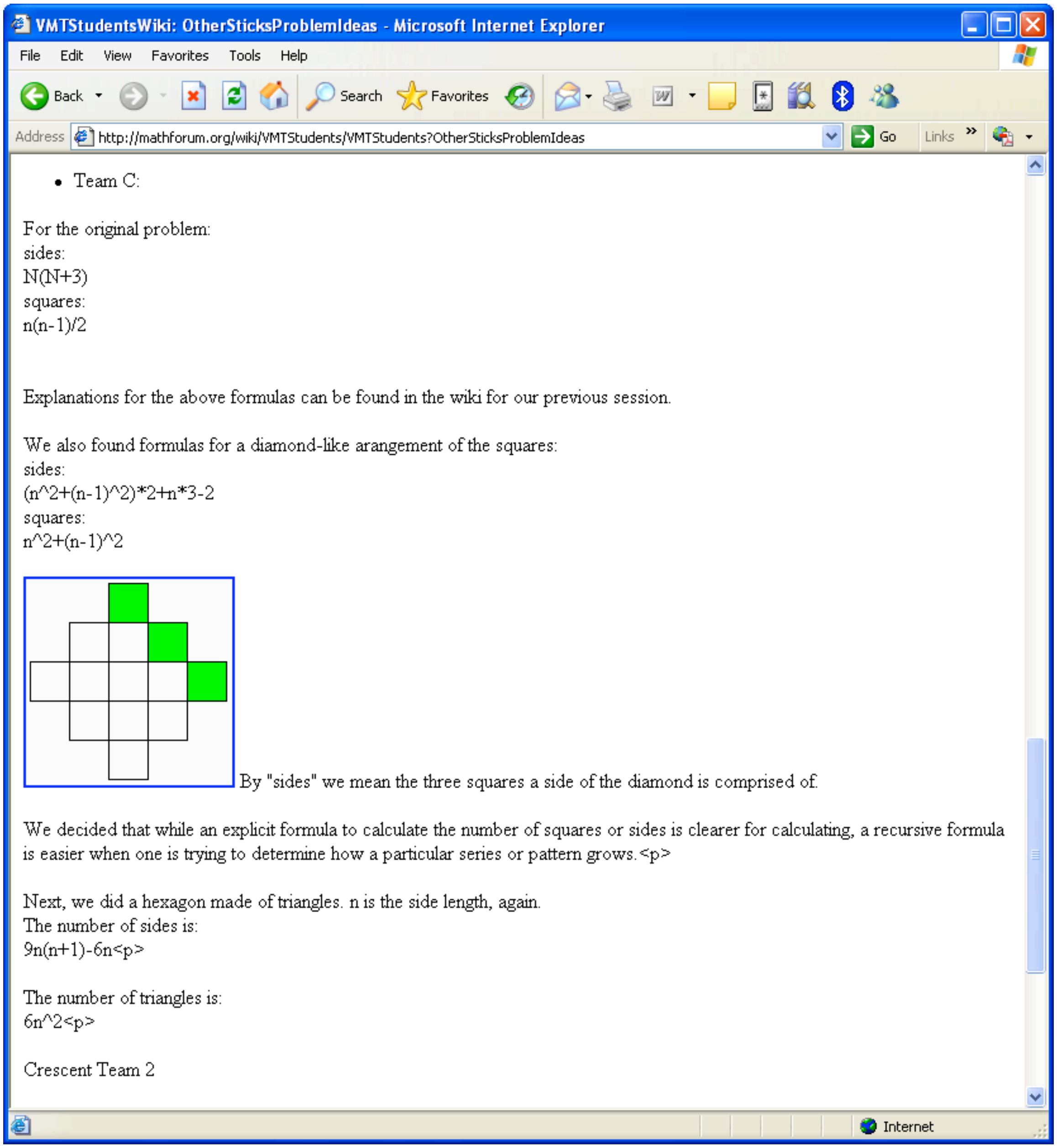
In short, the episode following 137's proposal shows that Qwertyuiop had trouble understanding how 137 derived the formula he reported in line 864. 137 seems to have gone ahead with putting together all the different pieces of the problem that have been discussed so far to produce the final formula. Note that the additive approach 137 was describing earlier included a step summarizing the pink boundary as 12n, which also includes the diagonals causing the overlap issue. The overlap between the two lines of reasoning may have informed 137's quick recognition of the algebraic implication of Qwertyuiop's subtraction move as an alternative to his own approach.

Qwertyuiop's problematizations of some of the terms that appear in the proposed formula have led 137 to reveal more details of his algebraic derivation. This exchange has revealed how each algebraic move is based on the corresponding concept the team has developed earlier (e.g., n(n+1)/2 sticks to cover a collinear set, multiply by 3 to cover 3 collinear sets making up a triangular partition, multiply by 6 to cover the hexagon, -6n to subtract those sticks counted twice). 137's contributions in this and the previous excerpts demonstrate that he can competently associate the narrative descriptions and geometric representations with symbolic formulas. Qwertyuiop's initial trouble and its resolution in the last excerpt provided us further evidence with regards to how participants made use of the narrative/geometric resources to co-construct a generalized symbolic formula addressing the problem at hand. In short, the team members complemented each others' skills as they incorporated geometric and algebraic insights proposed by different members into a solution for the task at hand during the course of their one hour long chat session.

4.1.14. The wiki summary of the team

In Chapter 3 we stated that the VMT environment includes a wiki as a third interaction space to support asynchronous interactions among virtual math teams. The instructions included in the task description (see Figure 3.5.1 above) required the teams to summarize their findings after each VMT session. Figure 4.1.23 below shows a screenshot of the wiki page where all the teams posted their summaries for the patterns they constructed. The wiki posting that corresponds to the session we analyzed above starts at the paragraph including the phrase "Next we did a hexagon…" towards the bottom[19].

[19] Note that "Crescent Team 2" following the last formula marks the beginning of a summary statement posted by another team. <p> markers seem to be html tags improperly used by the students to mark paragraphs.


VMTStudentsWiki: OtherSticksProblemIdeas - Microsoft Internet Explorer

http://mathforum.org/wiki/VMTStudents/VMTStudents?OtherSticksProblemIdeas

- Team C:

For the original problem:
sides:
N(N+3)
squares:
n(n-1)/2

Explanations for the above formulas can be found in the wiki for our previous session.

We also found formulas for a diamond-like arangement of the squares:
sides:
(n^2+(n-1)^2)*2+n*3-2
squares:
n^2+(n-1)^2

By "sides" we mean the three squares a side of the diamond is comprised of.

We decided that while an explicit formula to calculate the number of squares or sides is clearer for calculating, a recursive formula is easier when one is trying to determine how a particular series or pattern grows.<p>

Next, we did a hexagon made of triangles. n is the side length, again.
The number of sides is:
9n(n+1)-6n<p>

The number of triangles is:
6n^2<p>

Crescent Team 2

Figure 4.1.23: The team's wiki posting summarizing their work for their first three VMT sessions

The wiki posting starts with a brief textual characterization of the pattern considered by the team. It is remarked that "n is the side length", which was consistently used by this team to index stages of a pattern across sessions (see the textual description next to the diagram of the diamond pattern). Finally, the formulas for the number of sides (i.e. sticks) and the triangles for the hexagonal pattern are provided.

Although the postings for the prior sessions include relatively more content (e.g., an image displaying the diamond pattern and a sentence reflecting on the team's discussion on recursive versus explicit formulas), in general the wiki summaries of the teams do not reflect the details of the collective reasoning process that led to the co-construction of the formulas reported on the wiki page. In particular, the summaries lacked a narrative account that describes how the reported formulas were derived. This suggests that the wiki was chiefly treated by the teams as a medium to display the formulas and the patterns they investigated to other teams.

4.2. Findings

In this section, we document some of the important affordances of the VMT environment for supporting collaborative problem-solving activities online based on students' methodic uses of available software features to address their practical interactional concerns. In particular, our discussion of the mathematical affordances of the VMT environment and the coordination of actions across multiple modalities will be based on the sequential organization of the occasions in which the methods or practices are enacted.

4.2.1. Availability of the Production Process

Excerpts such as 4.1.1, 4.1.3, 4.1.4, and 4.1.9 where users display a sustained orientation to producing drawings on the whiteboard highlight a fundamental difference between the two interaction spaces: whiteboard and chat contributions differ in terms of the availability of their production process. As far as chat messages are concerned, participants can only see who is currently typing,[20] but not what is being typed until the author decides to send the message. A similar situation applies to *atomic* whiteboard actions such as drawing a line or a rectangle. Such actions make a single object appear in the shared drawing area when the user releases the left mouse button; in the case of editable objects such as textboxes, the object appears on the screens of the computers of all chat participants when the editor clicks outside the textbox to post it. However, the construction of most shared diagrams includes the production of multiple atomic shapes

[20] While a participant is typing, a social awareness message appears under the chat entry box on everyone else's screen stating that the person "is typing" (see Chapter 3). When the typist posts the message, the entire message appears suddenly as an atomic action in everyone's chat window.

(e.g., many lines), and hence the sequencing of actions that produce these diagrams is available to other members. For instance, as we observed in Excerpt 4.1.1, the availability of the drawing process can have interactionally significant consequences for math-problem-solving chats due to its instructionally informative nature. In short, the whiteboard affords an *animated evolution* of the shared space, which makes the *visual reasoning process* manifest in drawing actions *publicly available* for other members' inspection. For instance, in Figure 4.1.2 transitions from stages 1 to 2 and 7 to 8 show modifications performed to achieve a peculiar geometric organization in the shared workspace. Likewise, the sequence of drawings that leads to the drawing displayed in Figure 4.1.18 allowed team members to locate what was indexed by the term "set of 3 collinear sides". Finally, Figures 4.1.21 and 4.1.22 show cases where a textually described algebraic operation was subsequently animated on the whiteboard.

4.2.2. Mutability of Chat and Whiteboard Contents

Another interactionally significant difference between the chat and the whiteboard interaction spaces, which is evidenced in those excerpts where participants modified and annotated their shared drawings, is the difference in terms of the mutability of their contents. Once a chat posting is contributed, it cannot be changed or edited. Moreover, the sequential position of a chat posting cannot be altered later on. If the content or the sequential placement of a chat posting turns out to be interactionally problematic, then a new posting needs to be composed to repair that. On the other hand, the object-oriented design of the whiteboard allows users to re-organize its content by adding new objects and by moving, annotating, deleting, reproducing existing ones. For instance, the way 137 and Qwertyuiop repaired their drawings in Excerpt 4.1.1 by re-positioning some of the lines they drew earlier to make sure that they intersect at certain points and/or that they are parallel to the edges of the hexagon illustrates this difference. Such *demonstrable tweaks* make the mathematical details of the construction work visible and relevant to observers, and hence serve as a vital resource for joint mathematical sense making. For instance, in Excerpt 4.1.1 by seeing that Qwertyuiop successively and intentionally adjusts lines in his whiteboard drawing to appear more parallel or to intersect more precisely, the other group members take note of the significance of the arrangement of lines as parallel and intersecting in specific patterns. Likewise, Qwertyuiop's visual proof that involved repositioning of red, blue and purple lines on top of the hexagonal array in Excerpt 4.1.9 also illustrates the consequentiality of this feature of the whiteboard on joint mathematical meaning making online.

While both chat and whiteboard in VMT support persistence, visibility, and mutability, they do so in different ways. A chat posting scrolls away only slowly and one can always scroll back to it, whereas a drawing may be erased by anyone at any time. Chat conventions allow one to replace (i.e., follow) a mis-typed posting with a new one, and conversational methods allow utterances to be retracted, repaired or refined. The mechanisms of the two mediational technologies are different and the characteristics of their persistence, visibility and mutability differ accordingly. Collaborative interaction in the dual-space environment is sensitively attuned to these intricate and subtle differences.

4.2.3. Monitoring Joint Attention in an Online Environment

In Excerpt 4.1.2 we observed that the facilitator called on each participant to report on his/her understanding of the activities of other participants. Prior to the facilitator's intervention there was an extended duration in which no chat postings were published while whiteboard actions were being performed by Qwertyuiop. Because it is not possible for any participant to observe other participants, it is not possible to directly monitor a class of actions others may perform that (a) are important for how we understand ongoing action but (b) do not involve explicit manipulation of the VMT environment, actions like watching the screen, reading text, inspecting whiteboard constructs, etc. The only way to determine if those kinds of actions are occurring is to explicitly inquire about them using a chat posting.

The limited availability of the production of text contributions is consequential for the organization of interaction in computer-mediated settings. As research on Conversation Analysis (Schegloff, 2006) has shown, such resources play a fundamental role in the organization of talk-in-interaction. In particular, during face-to-face encounters, speakers routinely monitor indicators such as bodily orientation and eye gaze of their interlocutors who are co-present in the scene, and attune their ongoing performance accordingly. Likewise, listeners also monitor the ongoing speech to detect turn-transition relevant moments, display their attention to the speaker, etc. Although they are massively limited as compared to their face-to-face counterparts, awareness messages built into the VMT online environment attempt to partially address this limitation by providing users with resources/clues to monitor each others' actions, see who is present in the room, etc., to achieve a sense of co-presence and coherence among actions unfolding across dual-interaction spaces. When these awareness messages turn out to be inadequate, then participants solicit each others' attention with chat postings as demonstrated by the facilitator in Excerpt 4.1.2.

4.2.4. Methods for Referencing Relevant Artifacts in the Shared Visual Field

Bringing relevant mathematical artifacts to other members' attention requires a coordinated sequence of actions performed in both the chat and whiteboard interaction spaces. For instance, in Excerpt 4.1.3 we observed two referential methods enacted by participants to bring relevant graphical objects on the whiteboard to other group members' attention. In the first case, 137 *marked the drawing* with a different color to identify the contour of a hexagonal shape. As evidenced in other members' responses, this was designed to make the hexagonal array embedded in a grid of triangles visible to others. Jason demonstrated another method by using the explicit referencing tool to support his *textual description* of the first stage of the pattern. Likewise, in Excerpt 4.1.9 Qwertyuiop marked the shared drawing with 3 different colors to explicitly mark the 3 collinear sets he had proposed earlier. Qwertyuiop also used the explicit referencing tool in Excerpts 4.1.7 and 4.1.9 respectively to direct his teammates' attention to the relevant section of the hexagon where he was performing his counting work. In all these cases chat messages included either an explicit reference or a deictic term such as "this", "that", or "the green", which are designed to inform other members of the group that they need to attend to some features beyond the textual statement itself to make sense of the chat message.

These referential mechanisms play a key role in directing other members' attention to features of the shared *visual field* in particular ways. This kind of deictic usage isolates components of the shared drawing and constitutes them as relevant objects to be attended to for the purposes at hand. Hence, such referential work establishes a fundamental *relationship between the narrative and mathematical terminology used in text chat and the animated graphical constructions produced on the whiteboard*. The shared sense of the textual terms and the inscriptions co-evolve through the referential linkages established as the interaction sequentially unfolds in both interaction spaces.

Deictic uses of text messages and drawings presume the availability of a *shared indexical ground* where the referential action can be seen as the *figure* oriented towards some part of the shared *background*. In other words, referential moves are not performed in isolation; they rely on a part/whole relationship between the referential action (i.e., figure) and a shared visual ground. For example, the color marking of the hexagonal array in Excerpt 4.1.3 and collinear lines in Excerpt 4.1.9 worked as a referential action, because they were performed on top of an existing graphical artifact, namely the triangular grid. Even the design of the explicit referential tool, which attaches a semi-transparent green rectangle to a chat message, reflects this visual relationship between the figure (i.e., the green rectangle) and the background, which guides other members' attention to a particular location in the shared visual field. As virtual teams collaboratively explore their problem and co-construct shared artifacts, they collectively constitute a shared problem space with increasing complexity. By enacting referential practices, participants isolate features of the shared scene, assign specific terminology to them, and guide other members' perception of the ongoing activity to achieve a shared mathematical vision.

4.2.5. Coordination of Whiteboard Visualizations and Chat Narratives

The previous section focused on single actions that refer to some feature of the shared scene for its intelligibility. We argued that such actions involve a part/whole relationship that presumes the availability of a shared visual ground for their mutual intelligibility. In addition to this, such actions are also embedded within broader sequences of actions that establish their relevance. In other words, messages that establish a referential link between narrative and graphical resources routinely respond to practical matters made relevant or projected by prior actions. Thus, such actions are also tied to the context set by the sequentially unfolding discussion.

When the scope of analysis is broadened to sequence of actions that include messages with referential links, one can observe an important affordance of online environments with multiple interaction spaces: Since one can contribute to only one of the interaction spaces at a time, a participant cannot narrate his/her whiteboard actions with simultaneous chat postings, as can be done with talk in a face-to-face setting. However, as we have observed in 137's performance in Excerpts 4.1.4 and 4.1.6, participants can achieve a similar interactional organization by temporally coordinating their actions in such a way that whiteboard actions can be seen as part of an exposition performed in chat. `

For instance, in Excerpt 4.1.4, 137's drawing activity was prefaced by his chat posting "like this?" The posting was also linked to a prior message where another member suggested splitting the hexagon into 6 pieces. Hence, the deictic term "this" included in the preface was not pointing to an existing drawing or to a prior posting. Instead, it *projected* a subsequent action to be performed next by the same author. In contrast, in Excerpt 4.1.6, the drawings were performed while the author was seen as typing by others. Although the sequence of the chat and whiteboard actions are the opposite in this case (i.e. the referential move was made after the drawing was finished), 137 achieves a similar temporal organization through his use of deictic terms, referential arrows, and tokens of similarity such as "like" and "as". Therefore, these instances suggest that, although they can be ordered in different ways, the sequential organization and temporal proximity of actions are consequential for the treatment of a set of drawing actions in relation to a narrative account produced in chat.

In face-to-face settings, locational deictic terms such as "this" and "those" are used to point out contextual elements beyond the lexical content of the statement uttering them, and they are often accompanied by co-occurring pointing gestures and body movements displaying the speaker's orientation towards what is being referred to in the vicinity (Hanks, 1992; Goodwin, 2000). As demonstrated by the actual cases of use in the excerpts analyzed above, a similar organization presents an interactional challenge for the participants in an online setting with dual interaction spaces like VMT. However, as participants demonstrated in these excerpts, a functionally comparable interactional organization can be achieved online through the use of available features so that chat messages can be seen as related to shared drawings that are either on display or in production. The sequential organization of actions, explicit referencing, and the temporal proximity of actions across both spaces together guide other members' attention so that they can treat such discrete actions as a coherent whole addressed to a particular prior message or to a thread of discussion unfolding at that moment.

Another important aspect of such achievements from a math-education perspective is that it shows us how "saming" (Sfard, 2008) among narrative and graphical accounts or realizations can be done as an interactional achievement across dual-interaction spaces. This phenomenon is demonstrated in various episodes such as (a) Davidcyl's move from his drawings of the stages of the staircase pattern to a symbolic formula characterizing the pattern of growth in our first case study in Section 3.5.2, (b) Qwertyuiop's demonstration of collinear set of lines on the shared diagram in Excerpt 4.1.9, and (c) 137's exposition in Excerpt 4.1.12, where he showed the geometric implication of his proposal in narrative form by performing a drawing immediately after his chat message (see Figures 4.1.22 and 4.1.23). The referential links, the temporal proximity of actions, the awareness indicators for those actions, and the persistent availability of both prior messages and the recently added drawings are all working together as a semiotic system that allows group members to make connections among different realizations of the mathematical artifacts that they have co-constructed. Therefore, referential practices across modalities are consequential for the collective achievement of deep understanding of mathematics, which is characterized in math-education theory as establishing relationships between different realizations of mathematical ideas encapsulated in graphical, narrative, or symbolic forms.

4.2.6. Chat versus Whiteboard Contributions as Persistent Referential Resources

In all of the excerpts we have considered so far, the shared diagram has been used as a resource within a sequence of related but recognizably distinct activities. For instance, the group has oriented itself to the following activities: (1) drawing a grid of triangles, (2) formulating a problem that relates a hexagonal array to a grid of triangles, (3) highlighting a particular hexagon on the grid, (4) illustrating a particular way to split the shape into 6 smaller pieces, and (5) illustrating ways to split a triangular region into non-overlapping collections by color-coding. As the group oriented to different aspects of their shared task, the shared diagram was modified on the whiteboard and annotated in chat accordingly. Yet, although it had been modified and annotated along the way, the availability of this shared drawing on the screen and the way participants organize their discussion around it highlights its persistent characteristic as an ongoing referential resource. In contrast, none of the chat postings in prior excerpts were attributed a similar referential status by the participants. As we have seen, in each episode the postings responded or referred either to recently posted chat messages or to the visual objects in the shared space.

In short, the textual chat postings and the graphical objects produced on the whiteboard differ in terms of the way they are used as referential resources by the participants. The content of the whiteboard is persistently available for reference and manipulation, whereas the chat content is visually available for reference for a relatively shorter period of time. This is due to the linear growth of chat content, which replaces previous messages with the most recent contributions inserted at the bottom of the chat window. Although one can make explicit references to older postings by using the scroll-bar feature, the limited size of the chat window affords a referential locality between postings that are visually (and hence temporally) proximal to each other. Nevertheless, the referential arrow built into the system is occasionally used by group members to refer back to a previously stated chat message to reinitiate past discussions, which will be discussed further in the next section.

In contrast, objects drawn in the whiteboard tend to remain there for a longer period of time. They are often only erased or moved out of view when the drawing is considered inadequate for the purposes at hand and/or space is needed for new drawings related to a new topic. While they may be modified, elaborated, and moved around, whiteboard objects may remain visible for an entire hour-long session or even across sessions. Like the chat, the whiteboard has a history scrollbar, so that at least in theory any past state of the drawing can be made visible again—although in practice students rarely use this feature. Although both media technically offer a persistent record of their contents, the visual locality of the whiteboard—the fact that graphical objects tend to stay available for reference from the more fleeting chat—qualifies it as the more persistent medium as an interactional resource. This notion of persistence does not imply that the shared sense of whiteboard objects is fixed once they are registered to the shared visual field. As they continue to serve as referential resources during the course of the problem-solving effort, the sense of whiteboard objects may become increasingly evident and shared, or their role may be modified as participants make use of them for various purposes.

To use Dillenbourg & Traum's (2006) metaphor, a first glance at the chat logs might suggest that the group is narrating their problem-solving process in the chat and illustrating what they mean by "napkin" drawings in the whiteboard. However, a second look reveals that the most significant insight and sharing is occurring in the whiteboard, more along the lines of the visual "model" metaphor. Perhaps the best way to describe what is going on is to say that the group is very carefully coordinating their work in the dual space so as to achieve a shared progression of understanding of the pattern problem with an efficiency and effectiveness that could not be achieved in either a purely textual chat system or a purely graphical whiteboard. Although in this view the chat and whiteboard both function as symmetric parts of a coordinated whole in which chat references drawing and drawing illustrates chat, it is important to differentiate their affordances as well.

4.2.7. Persistence of Chat and Management of Parallel Threads

The comparison we have made between whiteboard and chat contributions in terms of their persistence does not imply that the persistence of chat messages does not carry any interactional significance. On the contrary, the persistent availability of textual messages creates the very possibility of interaction in this online environment by making the contributions available to participants for reading them in particular ways. Participants constitute a sequence of discrete/isolated sets of actions as interaction through *reading's work* (Zemel, Cakir & Stahl, 2009).

One important consequence of quasi-synchronous interactions mediated by a persistent display of text messages is that participants are not subjected to the same set of physical constraints underlying the turn-taking apparatus associated with talk in face-to-face settings. In particular, as far as talk-in-interaction is concerned, usually a single person holds the floor as the speaker due to the practical intelligibility issues involved with overlapping speech. In contrast, the persistent availability of the text messages affords simultaneous production of contributions, and hence provides more possibilities for participation. This may introduce intelligibility issues referred to as chat confusion (Fuks, Pimentel & de Lucena, 2006) or phantom adjacency pairs (Garcia & Jacobs, 1998), when simultaneously produced messages can be mistakenly treated in relation to each other. However, as we have seen in the excerpts analyzed above, participants routinely provide enough specificity to their contributions (e.g., by using the referential tool or specific tokens) and orient to the temporal/linear order in which messages appear on the screen to avoid such issues of intelligibility. When issues of intelligibility are realized during online interaction, repair mechanisms (as illustrated in Excerpt 4.1.7) have been invoked to address them. Finally, when coupled with resources such as the explicit referencing tool and repetition of specific terms (e.g., sidelength), the persistency of chat messages also allows participants to make a past point or discussion relevant to the current discussion. For instance, in line 818 in Excerpt 4.1.10, Qwertyuiop re-oriented the current discussion to the issue of devising a formula for the sequence of numbers that was stated back in line 772 by using the explicit referencing tool. Likewise, in line 841 in Excerpt 4.1.11 Qwertyuiop proposed that the team re-initiate a discussion on a point stated 13 lines above with his message “go back to this” coupled with an explicit referential link.

Another important interactional consequence of the persistent nature of text postings is that it allows participants to engage in multiple threads of discussion (O’Neil & Martin, 2003). The unique features of the VMT environment as compared to standard text-chat tools require us to expand this observation further, including the threads unfolding in the shared whiteboard. The possibility of engaging activities across multiple threads spanning both chat and whiteboard spaces is an important affordance of online environments like VMT due to the opportunities it brings in for more people to contribute to the ongoing discussion. For instance, in Excerpt 4.1.8 we have seen that 137 was engaged in two simultaneous threads where (a) he drew a line segment that was potentially ignored from the method of computation described by Qwertyuiop, and (b) he contributed to the simultaneously unfolding discussion about characterizing the pattern implicated by the numbers offered by Qwertyuiop as triangular numbers. Although the management of multiple threads across spaces can bring in confusion, the resolution of ambiguities and the intertwinement of perspectives can yield to germination/fertilization of mathematical ideas across threads. This point is well demonstrated by how the aforementioned threads led to Qwertyuiop’s proof, which (a) located visually what the term "3 sets of collinear lines" meant, (b) established that the sets do not overlap with each other, and (c) highlighted the association between a single set and a triangular number.

4.2.8. Past and Future Relevancies Implied by Shared Mathematical Artifacts

The objects on the whiteboard and their visually shared production index a horizon of past and future activities. The indexical terms in many proposals made in the analyzed excerpts (like "hexagonal array", "collinear lines", "rows") not only rely on the availability of the whiteboard objects to propose a relevant activity to pursue next, but also *reflexively* modify their sense by using linguistic and semantic resources to label or gloss the whiteboard object and its production. This allows actors to orient in particular ways to the whiteboard object and the procedures of its co-construction—providing a basis for subsequent coordinated joint activity. For example, the co-construction of the triangular grid afforded subsequent pattern constructions including regular and irregular hexagons.

This suggests that shared representations are not simply manifestations or externalizations of mental schemas as they are commonly treated in cognitive models of problem solving processes. Instead, our case studies suggest that shared representations are used as resources to interactionally organize the ways actors participate in collaborative problem solving activities. As we have seen in the case studies, once produced as shared mathematical artifacts, drawings can be mobilized and acted upon as a resource for collective reasoning as different members continue to engage with them. Shared meanings of those artifacts are contingently shaped by these engagements that are performed against the background of a shared visual space including other artifacts and prior chat messages (i.e. against a shared *indexical ground*). This does not mean that the achievement of shared understanding implies that each member has to develop and maintain mental contents that are isomorphic to each others', which is often referred as registering shared facts to a "common ground" in psycholinguistics (Clark & Brennan, 1991). Instead, shared understanding is a practical achievement of participants that is made visible through their reciprocal engagements with shared mathematical artifacts. For instance, Jason's characterization of the hexagonal array when N=1 in response to his partner's marking of the hexagonal array in the whiteboard in Excerpt 4.1.3 presents such a reciprocal display of a newly achieved shared understanding. Jason's display of his understanding did not simply include an acknowledgement of the explanation provided by his peer. Instead, he displayed his recognition of what the hexagonal array means in that situation through his actions where he extended the ongoing discussion in an interactionally relevant manner by pointing to a different stage of the pattern. In other words, Jason's actions had a dual role in the ongoing interaction, which (a) displayed his alignment with respect to the provided explanation, and (b) demonstrated an act of reasoning that draws an implication informed by that explanation. His actions *reflexively* provided further specificity to the shared artifact and *projectively* contributed to the joint effort of constituting the hexagonal pattern as a shared problem for the team. Such collective achievements open up possibilities for further coordinated (i.e., mutually intelligible) actions, and hence are consequential for the progressivity of the ongoing collaborative problem-solving activity. Achievement of progressivity (i.e., understanding as knowing "how to go on", as Wittgenstein (1953) remarked) in interaction characterizes understanding at the group level.

4.2.9. Joint Management of Narrative, Graphical and Symbolic Realizations

The excerpts studied above suggest the following in regards to how one kind of realization leads to another:

- *Graphical to Narrative*: Particular ways of drawing may inform subsequent discussion in the chat space. For instance, as we have seen in section 3.5.2, producing next steps of a pattern by copying and pasting prior stages can inform a recursive way of thinking about the pattern at hand. In this example, the noticing that a particular number of entities have to be added to go from one stage to another informed Davidcyl's subsequent symbolic characterization of the pattern of growth. Similar transitions from diagrams to narratives also played a fundamental role in the group's joint development of methods for counting the number of triangles and sticks included in the hexagonal pattern.
- *Narrative to Graphical:* When textual description turns out to be inadequate, further illustration is often provided graphically in the other interaction space. 137's illustration of splitting the hexagon into 6 in Excerpt 4.1.4 and Qwertyuip's marking of the collinear set of lines in Excerpt 4.1.10 exemplify such transitions. Such moves are carefully coordinated with chat postings that include tokens such as "like", "as" and "so", so that drawings can be seen in relation to or as a consequence of the unfolding narrative in chat.

- *Narrative to Narrative:* The narratives are chiefly co-constructed through messages posted to the chat window as short text messages. Members orient to resources such as the explicit referencing tool that connects two messages, the temporal order through which messages appear in the chat window, the specific communicative acts encoded in messages through the use of specific words, and the grammatical organization of messages to *read* this stream of texts as an unfolding online conversation.
- *Graphical to Graphical*: The availability of the details of the drawing work through which whiteboard objects are constructed can inform the production of subsequent drawings as well. The joint production of the hexagonal pattern in Excerpt 4.1.1 above shows a case where the spatial organization of one member's failed drawing attempt informed the subsequent drawing performance of another member.
- The case studies suggest that narrative and graphical realizations mutually inform each other. Hence, it is difficult to claim that one always follows or leads to the other in a unidirectional manner. However, symbolic realizations seem to be built upon graphical and/or narrative realizations. In all the cases we have studied above, associating the task at hand with symbols was achieved through a method of counting discovered by the group as part of their engagement with already produced representations. For instance, those engagements involved the annotation of images that isolate pieces to be counted and the use of verbal cues like row-by-row to direct the counting process.

The way team members oriented themselves to the shared drawing while they were exploring various properties of the hexagonal array shows that the drawings on the whiteboard have a figurative role in addition to their concrete appearance as illustrations of specific cases. In other words, the particular cases captured by concrete, tangible marks on the whiteboard are routinely used as a resource to investigate and talk about the general properties of the mathematical artifacts indexed by them. For example, the particular drawing of the hexagonal pattern in the excerpts studied above was illustrating one particular stage (i.e., N=3), yet it was treated in a generic way throughout the whole session as a resource to investigate the properties of the general pattern implied by the regularity/organization embodied in that shared artifact. Noticing of such organizational features motivated the joint development of counting practices, where relevant components of the pattern were isolated and then systematically counted.

Another important aspect of the team's achievement of general formulas, which summarize the number of sticks and triangles included in the N^{th} case respectively, is the way they transformed a particular way of counting the relevant objects in one of the partitions (i.e., a geometric observation) into an algebraic mode of investigation. For instance, in excerpts 4.1.1 through 4.1.5 where the team was oriented to the problem of finding the number of triangles required to build a hexagonal array at its n^{th} stage, this shift led the team members to recognize that a particular sequence of numbers (i.e. 1+3+5+…) can be associated with the way the partition grows in subsequent iterations. Similarly, in subsequent excerpts the team discovered that a particular alignment of sticks that they referred to as "collinear sides" corresponded to triangular numbers, which allowed them to summarize the sequence of numbers they devised into the algebraic formula $9n(n+1)-6n$. The shift to this symbolic mode of engagement, which relied on the presence of shared drawings and prior narratives as resources, allowed the team to progress further in the task of generalizing the pattern of growth by invoking algebraic methods. In other words, the team co-constructed general formulas for their shared tasks by making *coordinated* use of multiple realizations (graphical and linguistic) of the mathematical artifact (the hexagonal array) distributed across the dual interaction spaces.

CHAPTER 5. DISCUSSION

In this chapter we will discuss the findings of our case study in relation to the issues we raised as part of our review of related work in CSCL in Chapter 2. Then, we will discuss the methodological and practical implications of the case-study approach taken in this dissertation project for CSCL research and design.

5.1. Grounding through Interactional Organization

The coordination of visual and linguistic methods (across the whiteboard and chat workspaces) plays an important role in the establishment of common ground through the co-construction of references between items in the two spaces. Particularly in mathematics—with its geometric/algebraic dual nature—symbolic terms are often grounded in visual presence and associated visual practices, such as counting or collecting multiple units into a single referent (Goodwin, 1994; Healy & Hoyles, 1999; Livingston, 2006; 2008; Wittgenstein, 1944/1978). The visually present can be replaced by linguistic references to objects that are no longer in the visual field, but that can be understood based on prior experience supported by some mediating object such as a name—see the discussion of mediated memory and of the power of names in thought by Vygotsky (1930/1978; 1934/1986). Here we will elaborate on how the interactional organization that we have observed in our case study functions to ground the group's understanding of their math object (the hexagonal array) as a shared group achievement. As our literature review demonstrated, in CSCL research there has been an explicit interest in studying how affordances of online environments with multiple interaction spaces facilitate *grounding*, and how grounding processes relate to collaborative problem-solving work mediated by such online environments (Baker et al., 1999; Dillenbourg & Traum, 2006). In this section we will discuss the findings of our ethnomethodological case study in relation to the concerns and results reported in prior CSCL research on these issues.

As implied in the OCAF study (Avouris et al., 2003) discussed in Section 2.3.2, investigating grounding and problem-solving processes in online dual-interaction environments like VMT requires close attention to the relationships among actions performed in multiple interaction spaces. Our case study illustrates some of the practical challenges involved with producing mathematical models that aim to exhaustively capture such relationships. For instance, the hexagonal array that was co-constructed by the team in our case study draws upon a triangular grid that is formed by three sets of parallel lines that intersect with each other in a particular way. In other words, these objects are layered on top of each other by the participants to produce a shape recognizable as a hexagon. Despite this combinatoric challenge, a modeling approach can still attempt to capture all possible geometric relationships among these graphical objects in a bottom-up fashion. However, when all chat messages referring to the whiteboard objects are added to the mix, the resulting model may obscure rather than reveal the details of the interactional organization through which group members discuss more complicated mathematical objects by treating a collection of atomic actions as a single entity. Terminology co-constructed in the chat-and-whiteboard environment—like "hexagonal array"—can refer to complexly defined math objects.

The challenges involved with the modeling approach are not limited to finding efficient ways to capture all relationships among actions and identifying meaningful clusters of objects. The figurative uses of the graphical objects present the most daunting challenge for such an undertaking. For instance, the team members in our case study used the term "hexagonal array" to refer to a mathematical object implicated in the witnessed production of prior drawing actions. As we have seen in the way the team used this term during their session, "hexagonal array" does not simply refer to a readily available whiteboard illustration. Instead it is used as a *gloss* (Garfinkel & Sacks, 1970) to talk about an imagined pattern that grows infinitely and takes the shape illustrated on the whiteboard only at a particular stage. In the absence of a fixed set of ontological elements and constraints on types of actions a user can perform, modeling approaches that aim to capture emergent relationships among semiotic objects distributed across multiple interaction spaces need to adequately deal with the reflexive and prospective character of language in interaction. Rather than relying upon a generic approach to modeling imposed by the researchers, our ethnomethodological approach aims to discover the unique "model"—or, better, the specific meaning—that was co-constructed *by the group* in its particular situation.

In another study discussed earlier, Dillenbourg & Traum (2006) offer the napkin and mockup models in their effort to characterize the relationship between whiteboard and chat spaces. In short, these models seem to describe two use scenarios where one interaction space is *subordinated* to the other during an entire problem-solving session. The complex relationships between the actions performed across both interaction spaces in our case made it difficult for us to describe the interactions we have observed by committing to only one of these models, as Dillenbourg & Traum did in their study. Instead, we have observed that in the context of an open-ended math task groups may invoke either type of organization, depending upon the contingencies of their ongoing problem-solving work. For instance, during long episodes of drawing actions where a model of some aspect of the shared task is being co-constructed on the whiteboard (as in Excerpt 4.1.1), the chat area often serves as an auxiliary medium to coordinate the drawing actions, which seems to conform to the mockup model. In contrast, when a strategy to address the shared task is being discussed in chat (as in the excerpt where the group considered splitting the hexagon into 6 pieces in Excerpt 4.1.4), the whiteboard may be mainly used to quickly illustrate the textual descriptions with annotations or rough sketches, in accordance with the napkin model. Depending on the circumstances of ongoing interaction participants may switch from one type of organization to another from moment to moment. Therefore, instead of ascribing mockup and napkin models to entire problem-solving sessions, we argue that it would be more fruitful to use these terms as glosses or descriptive categories for types of interactional organizations group members may invoke during specific episodes of their interaction.

Another important observation made by Dillenbourg & Traum is that the whiteboard serves as a kind of shared external memory where group members keep a record of agreed upon facts. In their study the dyads were reported to post text notes on the whiteboard to keep track of the information they had discovered about a murder-mystery task. This seems to have led the authors to characterize the whiteboard as a placeholder and/or a shared working memory for the group, where agreed upon facts or "contributions" in Clark & Brennan's (1991) terms are persistently stored and spatially organized. As Dillenbourg & Traum observed, the scale of what is shared in the course of collaborative problem solving becomes an important issue when a theory operating at the utterance level like contribution theory (Clark & Marshall, 1981) is used as an analytic resource to study grounding processes that span a longer period of time. Dillenbourg & Traum seem to have used the notion of persistence to extend common ground across time to address this limitation. In particular, they argued that the whiteboard grounds the solution to the problem itself rather than the contributions made by each utterance. In other words the whiteboard is metaphorically treated as a physical manifestation of the common ground.

In our case study we have observed that the whiteboard does not simply serve as a kind of shared external memory where the group keeps a record of agreed upon facts, opinions, hypotheses or conclusions. In our sessions the whiteboard was primarily used to draw and annotate graphical illustrations of geometric shapes, although users occasionally posted textboxes on the whiteboard to note formulas they had found (see Figure 5.1.1 below). *While the whiteboard mainly supported visual reasoning and textual discussion or symbolic manipulation occurred chiefly in the chat stream, actions were carefully, systematically coordinated across the media and integrated within an interactionally* *organized group-cognitive process*. As we have illustrated in our analysis, the fact that there were inscriptions posted on the whiteboard did not necessarily mean that all members immediately shared the same sense of those graphical objects. The group members did considerable interactional work to achieve a shared sense of those objects that was adequate for the purposes at hand. For instance, the cross-hatched lines that Qwertyuiop originally drew became increasingly meaningful for the group as it was visually outlined and segmented and as it was discussed in the chat and expressed symbolically. Hence, the whiteboard objects have a different epistemic status in our case study than in Dillenbourg & Traum's experiment. Moreover, not all contents of the whiteboard were deemed relevant to the ongoing discussion by the participants. For instance, Figure 5.1.1 below shows a snapshot of the entire whiteboard as the team was discussing the hexagonal pattern problem. The figure shows that there are additional objects in the shared scene like a blue hypercube and a 3-D staircase, which are remnants of the group's prior problem-solving work. Finally, the sense of previously posted whiteboard objects may be modified or become evident as a result of current actions (Suchman, 1990). In other words, group members can not only reuse or reproduce drawings, but they can also reflexively make subsequent sense of those drawings or discard the ones that are not deemed relevant anymore. Therefore, the technologically extended notion of common ground as a placeholder for a worked-out solution suffers from the same issues stated in Koschmann & LeBaron's (2003) critique of Clark's theory. As an abstract construct transcendental to the meaning-making practices of participants, the notion of common ground obscures rather than explains the ways the whiteboard is used as a resource for collaborative problem solving.

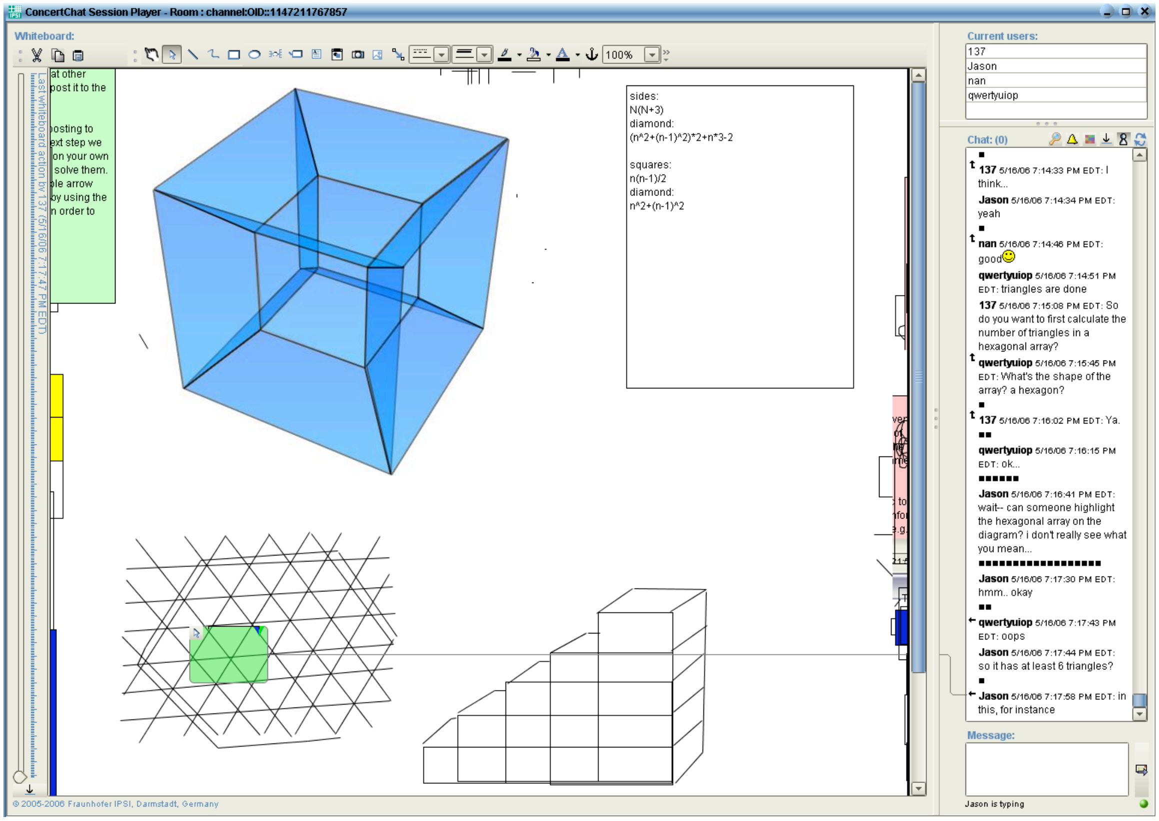


Figure 5.1.1: Snapshot of the entire whiteboard while the team was working on the hexagonal pattern

Instead of using an extended version of common ground as an analytical resource we frame our analysis using the notion of "indexical ground of deictic reference," which is a term we appropriated from linguistic anthropology (Hanks, 1992). In face-to-face interaction, human action is built through the sequential organization of not only talk but also coordinated use of the features of the local scene that are made relevant via bodily orientations, gesture, eye gaze, etc. In other words, "…human action is built through simultaneous deployment of a range of quite different kinds of semiotic resources" (Goodwin, 2000, p. 1489). Indexical terms and referential deixis play a fundamental role in the way these semiotic resources are interwoven in interaction into a coherent whole.

Indexical terms are generally defined as expressions whose interpretation requires identification of some element of the context in which it was uttered, such as who made the utterance, to whom it was addressed, when and where the utterance was made (Levinson, 1983). Since the sense of indexical terms depends on the context in which they are uttered, indexicality is necessarily a relational phenomenon. Indexical references facilitate the mutually constitutive relationship between language and context (Hanks, 1996). The basic communicative function of indexical-referentials is “to individuate or single out objects of reference or address in terms of their relation to the current interactive context in which the utterance occurs” (Hanks, 1992, p. 47).

The specific sense of referential terms such as *this*, *that*, *now*, *here* is defined locally by interlocutors against a shared indexical ground. Conversely, the linguistic labels assigned to highlighted features of the local scene reflexively shape the indexical ground. Hence, the indexical ground is not an abstract placeholder for a fixed set of registered contributions. Rather, it signifies an emergently coherent field of action that encodes an interactionally achieved set of background understandings, orientations and perspectives that make indexical expressions like “hexagonal array” intelligible to interlocutors (Zemel et al., 2008).

Despite the limitations of online environments for supporting multimodality of embodied interaction, participants make substantial use of their everyday interactional competencies as they appropriate the features of such environments to engage with other users. For instance, Suthers et al.’s (2003) study reports that deictic uses of representational proxies play an important role in the interactional organization of online problem-solving sessions mediated by the Belvedere system. The authors report that participants in the online case devised mechanisms that compensate for the lack of gestural deixis with alternative means, such as using verbal deixis to refer to the most recently added text nodes and visual manipulation of nodes to direct their partner's attention to a particular node in the shared argument map.

In contrast to the Belvedere system, VMT offers participants additional resources such as an explicit referencing mechanism, a more generic workspace that allows producing and annotating drawings, and an awareness feature that produces a sense of sequentiality by embedding indicators for drawing actions in the sequence of chat postings. Our case study shows that despite the online situation's lack of the familiar resources of embodied interaction, team members can still achieve a sense of shared access to the meaningful objects displayed in the dual-interaction spaces of the VMT environment. Our analysis indicates that coherence among multiple modalities of an online environment like VMT is achieved through members' methodical uses of the features of the system to coordinate their actions in the interface. In particular, we observed that the witnessable details of the orderly construction of shared inscriptions (e.g., the way objects are spatially arranged in relation to each other through sequences of actions) and the deictic references that link chat messages to features of those inscriptions and to prior chat content are instrumental in the achievement of indexical symmetry (intersubjectivity) with respect to the semiotic objects relevant to the task at hand.

Through coordinated use of indexical-referential terms and highlighting actions, team members help each other to literally "see" the relevant objects implicated in the shared visual field and to encode them with locally specified terminology for subsequent use. Moreover, the integration of both modalities in this manner also facilitates joint problem solving by allowing group members to invoke and operate with multiple realizations—graphical, narrative and symbolic—of their mathematical task. We have seen that such coordinated work across modalities can be a powerful problem-solving resource since it allows participants to invoke various mathematical practices relevant to the task at hand and to make use of them in mutually elaborating ways.

5.2. Implications for CSCL Design and Pedagogy

In this section we will reflect on some of the practical implications of the findings of our case study and the discussion above for (a) incorporating curricular activities facilitated by a CSCL system like VMT to promote math learning with understanding at schools, and (b) suggesting design improvements for the next iteration of the VMT environment along the line of the Design-based Research framework discussed in Chapter 3.

The collective achievements of the virtual math team documented in our case study suggest that in an online environment tailored to an institutional context like the Math Forum that promotes inquiry, creativity, and exploration in mathematics, small groups of students can *co-construct their own mathematics* by collectively making use of what they know about mathematics while they are collaboratively working on open-ended math problems. Although the complexity of the mathematics students engage with and the rigor of arguments they produce are understandably different as compared to problems and arguments formulated by professional mathematicians, the interactional organization of the *discovery work* (Garfinkel, Lynch & Livingston, 1981) that lead to mathematical innovations and results in both instances have important commonalities. In particular, in both instances the work of discovery involves "…noticing, of directing partners' attention, and of seeking, negotiating, and securing ratification of an understanding…" (Koschmann & Zemel, 2006, p. 356) through various forms of mathematical artifacts. Thus, when combined with tasks specifically designed to entice reasoning and creativity, an online environment like VMT can support the kinds of collaborative problem-solving activities that stimulate learning math with understanding as recommended by many calls for reform in math education (e.g., NCTM, 2000).

One practical implication of the discussion above is that teachers can observe how students make use of the math concepts presented in the classroom by incorporating online collaborative learning activities in their curriculum. In our case studies of VMT sessions we observed that asking students to come up with their own math problems promoted ownership of the task by the students, opened up room for negotiation regarding what would be a mathematically interesting problem for the students, stimulated discussion on what math concepts/tools would be relevant for the task at hand, and triggered the invention of a new math terminology in reference to the observations and discoveries they made. The teams that participated in our case studies accomplished all of this by participating in a collaborative math discourse in a less formal, peer-group setting outside their classrooms.

As our case study of a team of three secondary students demonstrated, the *persistent records* of these interactions can serve as a vital tool for reflection not only for students but also for the teachers. In particular, teachers can observe their students in action, monitor how they engage with mathematical practice, and adjust their lesson plans based on their reviews of chat activities. Through the replayer feature of VMT, teachers can observe in those logs how their students co-construct their own concepts and what difficulties they are experiencing with the subject matter. Teachers can use their students' ways of symbolizing and languaging that are made visible in those logs as a resource to attune the classroom discussion of formal math concepts covered by the curriculum in terms of the diagrams and the terminology invented by the students. In short, as compared to other assessment devices like exams or homeworks, the persistent records of chat logs can help teachers to better grasp their students' mathematical skills in a less formal peer-group setting, and open the possibility of developing instructional strategies better tailored to students' needs.

Since the spring fest activities sponsored by the VMT project spanned a relatively shorter duration of time (e.g. about 2-3 weeks), we could not investigate how the potential uses of collaborative activities outlined above may influence math learning at the classroom level. The impact of pedagogical practices that may have been enacted with and around a CSCL environment like VMT needs to be empirically investigated. Thus, one way to accomplish that could be to extend our work with a longitudinal study that incorporates online collaborative problem-solving sessions in a semester-long math class, which focuses on the mutually informative relationships emerging between the students' mathematical practices and the teacher's instructional practices at the classroom level.

Although the EM/CA approach we took for the analysis of the affordances of the VMT environment is descriptive rather than prescriptive, an improved understanding of the organization of online interactions uncovered by this methodology can inform the design and development of similar CSCL systems that incorporate multiple interaction spaces (Crabtree, 2003; Dourish & Button, 1998). In particular, our analysis highlighted the importance of *representational practices* that make the details of the construction process of mathematical artifacts visible to everyone and *referential practices* that coordinate multiple realizations of those artifacts in graphical, narrative, and symbolic forms. In the next paragraphs, we will discuss possible ways to improve the VMT system in the light of these practices that emerged as findings of our case studies.

One can consider various ways to improve the representational affordances of the VMT environment by implementing more advanced mathematical features similar to the ones provided by existing math packages. Some of the possibilities include the addition of (a) *virtual manipulatives* similar to those provided by GeoGebra or Geometer's Sketchpad that allow precise geometric constructions/manipulations, (b) a *spreadsheet* application and a *graphic plotter* that allow the exploration of functions and correlations, and (c) *simulations* that can be used for conducting probability experiments or exploring the motion of physical objects.

All of these additions will potentially enrich the collaborative problem-solving activities online by offering new ways to engage with mathematical artifacts. Nevertheless, since existing versions of these systems are chiefly designed for single users, they may not be directly emulated in multi-user CSCL environments. One important consideration suggested by our results is that the construction of digital artifacts in these systems should be transparent, so that the process that leads to their production can be (at least partially) observed by other users.

In addition to this, providing resources to help users coordinate their actions across new modalities and artifacts needs to be carefully considered due to its fundamental importance for the achievement of joint meaning-making online. As it was demonstrated in our case study, participants enacted referential practices to establish a sequential organization among turns taken across modalities, which helped them coordinate multiple realizations and achieve a shared understanding of their ongoing activity. The practical realization of potential improvements suggested by the addition of new functionalities depends on the extent groups of users can manipulate and monitor the manipulation of the artifacts constructed with those features.

Lastly, we will conclude this section with a few observations on the integration of the chat and the wiki spaces in the VMT environment. As we have pointed out in Section

4.1.14, the wiki component was added to the VMT system to support asynchronous interactions among virtual math teams. We aimed to provide support for teams to publish their results online through co-authored wiki pages, in resemblance to professional mathematicians who share their findings with the rest of the math community by publishing papers in journals or posting informal contributions to systems like arxiv.org. Therefore, the wiki component will likely have a significant impact on supporting longitudinal classroom studies like the one we described above, as well as scenarios of use that scale up to larger collectivities such as the Math Forum online community.

During the VMT Spring Fest event at least in one instance the wiki facilitated interaction between two teams; one of the teams considered a pattern contributed by another during their chat session and pointed out an error in one of the reported formulas. Nevertheless, our analysis of the wiki postings (e.g. see section 4.1.14) suggested that the wiki summaries lacked the details and the rationale that was present in the chat discussion. This suggests that the weak connection between the chat and the wiki applications had been consequential for the ways wiki summaries were composed and organized by the students. In particular, the weakness of the connection forced users to reproduce what they had found during their chat sessions in the wiki, without being able to reuse the drawings and narratives they had previously co-constructed in chat.

The wiki application offered basic functionality for composing simple web pages, which includes composing, editing, and formatting text as well as displaying image files uploaded to the system. Since the wiki does not support the production of images, students had to use a third party drawing tool such as Microsoft Paint to reproduce their drawings if they chose to include them as part of their summaries. Although the task included creation of new geometric patterns that are very difficult to describe in text, only one team out of the five that participated in the Spring Fest event uploaded an image file along with their summary. In retrospect, this observation suggests that even a simple whiteboard function that would allow a team member to save a selected portion of their whiteboard as an image file could have helped the teams to enrich their wiki summaries.

Finally, from a methodological perspective the wiki environment made it difficult to trace the development of the wiki postings. The teams occasionally discussed what they should post to the wiki towards the end of their chat sessions, but this process was largely not available to researchers, because the wiki version used did not keep track of who made the changes to its pages. We were only able to view the historical evolution of the changes made to a wiki page. Therefore, the wiki had limited affordances in terms of revealing the process through which summaries were produced, and hence made it difficult for us to study students' summarization practices.

All the reasons listed above highlight the need for improving the integration between the chat and the wiki applications. We took some initial steps towards this goal by adding a tabbed interface to the chat application, which allocates a dedicated whiteboard to compose a summary and a shared browser to monitor wiki pages and other web pages (see Figure 5.2.1 below). Now students can reuse the content they have already co-constructed in the VMT environment while they compose their summaries. More importantly, groups can do the summarization collaboratively in the same online environment, which will allow us to investigate summarization practices enacted by students. Nevertheless, the chat application still does not support automatically transferring the contents of the summary tab to the corresponding wiki page. This step is crucial for helping teams to mobilize their collective findings in terms of co-authored wiki pages that adequately reflect what they have accomplished together. Therefore, further improving the chat/wiki integration will be an important improvement to support larger scale collaborations at the classroom and community levels.

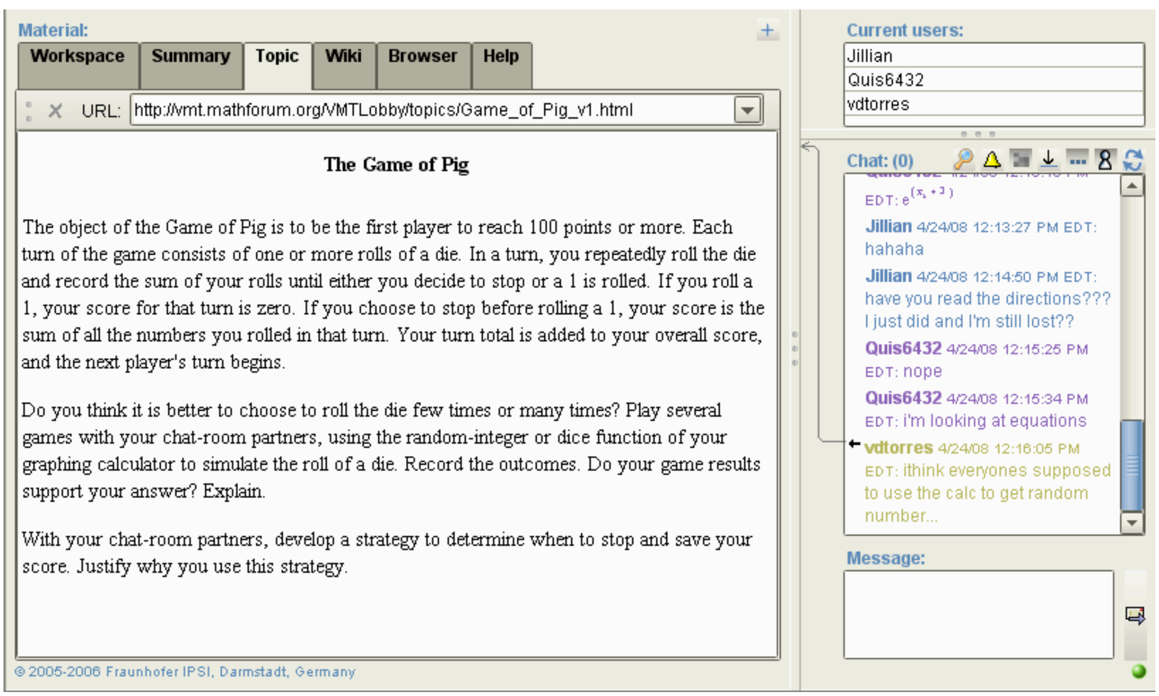


Figure 5.2.1: A snapshot of the new VMT Chat application with additional tabs

CHAPTER 6. CONCLUSION

In this chapter we will conclude the dissertation by summarizing the findings of our case study in relation to the research questions stated in Chapter 1. Overall, this dissertation investigated how online small groups of students co-construct mathematical artifacts by using graphical, textual, and symbolic resources provided by a CSCL system with multiple interaction spaces, and how they achieve a shared sense of their joint problem solving activity. We organized our main research question around three more specific questions each focusing on a distinct theme:

> RQ1 (Mathematical Affordances): What are the similarities and differences of the different media in VMT (text chat, whiteboard, and wiki) for the exploration and use of **mathematical artifacts**?

> RQ2 (Coordination Methods): How do groups in VMT **coordinate** their *problem-solving actions* across different interaction spaces as they co-construct and manage a shared space of mathematical artifacts?

> RQ3 (Group Understanding): How can collaborating students **build shared mathematical understanding** in online environments? How do they create math artifacts that incorporate multiple realizations?

We employed a practice-oriented case study approach motivated by insights from ethnomethodology and conversation analysis to identify the methods/practices students enacted in the VMT online environment to make sense of each other's actions. In our case study we investigated how a group of three upper-middle-school students put the features of an online environment with dual-interaction spaces into use as they collaboratively worked on a math problem they themselves came up with.

Overall, our ethnomethodological analysis of excerpts suggests that the users of CSCL environments like VMT need to address the following set of recurrent practical concerns for participating in collaborative math problem solving activities online: (a) identify and construct relevant mathematical artifacts to constitute a common math problem, (b) refer to those artifacts and their relevant features to achieve a mutual orientation (i.e., a sense of reciprocity among perspectives) towards the artifacts deployed in the shared visual field, and (c) manipulate and assess the manipulation of those shared artifacts with respect to mathematical norms known by the participants towards a solution to the task at hand.

Our analysis focused on how the participants enacted the affordances of the online environment as they methodically addressed the practical concerns listed above in interaction. We observed that while participants were acting in accordance with each other in this online environment, they gave a form to their understandings through their actions, and gradually *reified* those understandings into shared artifacts. Therefore, focusing on the *interactional organization* of online chats allowed us to empirically investigate (a) the affordances of the environment for producing mathematical actions and (b) the mathematical reasoning and understandings evidenced in those actions.

As far as the first research question is concerned, our analysis of the *representational practices* enacted by the students in the VMT environment has revealed important insights regarding the *mathematical and communicative affordances* of whiteboard and chat spaces:

- We observed that the whiteboard can make visible to everyone the animated evolution of a geometric construction, displaying the *visual reasoning* embodied in the sequential and spatial organization of drawing actions.
- Whiteboard and chat contents differ in terms of *mutability* of their contents, due to the object-oriented design of the whiteboard that allows modification and annotation of past contributions. Since the modifications made to the whiteboard objects are visible to everyone, participants can witness the visual reasoning embodied in the updated spatial organization of objects.
- Both media offer a persistent record of their contents, which creates the very possibility of interaction in this online environment by allowing participants to read the contributions in particular ways. Participants explicitly oriented to this particular affordance of the environment when they respond to each other's actions, make a past point or discussion relevant to the current discussion, and participate in multiple threads of activities simultaneously unfolding in chat and whiteboard spaces.

- The media differ in terms of the *persistence* of their contents: whiteboard objects remain in the shared visual field until they are removed, whereas chat content gradually scrolls off as new postings are produced. Although contents of both spaces are persistently available for reference, due to linear progression of the chat window, chat postings are likely to refer to visually (and hence temporally) proximal chat messages and to graphical whiteboard objects.
- Due to the more persistent nature of whiteboard objects, the shared drawings on the whiteboard index a horizon of past and future activities while they continue to be used as a relevant resource by the team across different problem solving episodes.

In an effort to address the second research question, we focused on the *referential practices* enacted by the team members to coordinate their actions across the dual interaction spaces of the VMT environment:

- Participants enact referential practices to bring the relevant math artifacts indexed by locally devised terms such as "hexagonal array" to other members' attention.
- Participants use *explicit* (e.g. the referential arrow provided by the system) and *verbal* references to guide each other about how a new contribution should be read in relation to prior chat messages and whiteboard objects.
- Temporal proximity of actions across dual-spaces serve as another important resource for the participants to read the narrative unfolding in chat in reference to ongoing drawing activity in the whiteboard.

- Whiteboard drawings and chat messages differ in terms of the illocutionary acts they support. Diagrams alone cannot fulfill the same kind of interactional functions that can be achieved by text postings such as "asking a question" or "expressing agreement". Hence, whiteboard objects are made interactionally relevant through chat messages that either (a) project their production as a next action, or (b) refer to already produced objects.
- The previous point can also be seen as members' orientation to a limitation of this environment as a communication platform; one can act only in one space at a given time in this online environment, so it is not possible to perform a simultaneous narration of a drawing as one can do in a face-to-face setting. Through temporal coordination of actions, participants can achieve a similar interactional organization online.
- Indexical terms stated in chat referring to the witnessable production of shared inscriptions facilitate the reification of those terms as meaningful mathematical artifacts for the participants. Indexical terms referring to co-constructed artifacts are used as a resource to index/encode complicated mathematical concepts and procedures in the process of co-constructing new ones.
- Different representational affordances of the dual-interaction spaces allow groups to develop multiple realizations of the math artifacts to which they are oriented. Shared graphical inscriptions and chat postings are used together as semiotic resources in mutually elaborating ways. Methods of coordinating group interaction across the media spaces also interrelate the mathematical significances of the multiple realizations.

Finally, we investigated how groups achieve a shared understanding of the mathematical artifacts they have co-constructed in the VMT environment. From an ethnomethodological perspective, the achievement of shared understanding or joint sense-making is synonymous to investigating how members achieve order among their actions (Garfinkel, 1967). Since the participants act by exchanging textual and graphical artifacts in the VMT environment, we argue that the groups achieve shared mathematical understanding by establishing a reciprocity of perspectives towards the shared math artifacts they have co-constructed, which entails the sequential organization of the representational and referential practices we listed above. In particular, we observed that actions performed in both interaction spaces constitute an evolving historical and indexical ground for the joint work of the group. What gets done now informs the relevant actions to be performed next, and what was done previously can be reproduced/modified depending on the circumstances of the ongoing activity. As the interaction unfolds sequentially, the sense of previously posted whiteboard objects and chat statements may become evident and/or modified as the new actions reflexively specify prior actions.

Through the sequential coordination of chat postings and whiteboard inscriptions, the virtual math team whose work was analyzed in detail in our case study, successfully solved their mathematical challenge, to find formulas for the number of small triangles and sticks in a hexagonal array of any given side-length. Their interaction was guided by a sequence of proposals and responses carried out textually in the chat medium. However, the sense of the terms and relationships narrated in the chat were largely instantiated, shared, and investigated through observation of visible features of graphical inscriptions in the whiteboard medium. The mathematical object that was visually co-constructed in the whiteboard was named and described in words within the chat. Finally, two symbolic expressions were developed by the group, grounded in the graphical artifact that evolved in the whiteboard and discussed in the terminology that emerged in the chat. The symbolic mathematical results were then posted to the wiki, a third medium within the VMT environment. The wiki is intended for sharing group findings with other groups as part of a permanent archive of work by virtual math teams.

Our case study demonstrates that it is possible to analyze how math problem solving—and presumably other cognitive achievements—can be carried out by small groups of students. The students can define and refine their own problems to pursue; they can invent their own methods of organizing their work; they can use unrestricted vocabulary; they can coordinate work in multiple media, taking advantage of different affordances. Careful attention to the sequentiality of references and responses is necessary to reveal *how* the group coordinated its work and how that work was driven by the reactions of the group members' actions to each other. Only by focusing on the sequentiality of the actions can one see how the visual, narrative and symbolic build on each other as well as how the actions of the individual students respond to each other to co-construct math artifacts, personal understanding, group agreement and mathematical results that cannot be attributed to any one individual, but which emerge from the interaction as complexly sequenced. This analysis illustrates a promising approach for CSCL research to investigate aspects of group cognition that are beyond the reach of quantitative methods that ignore the full sequentiality of their data.

To sum up, the focus of our ethnomethodological inquiry is directed towards documenting how a virtual team achieved a sense of reciprocity and coherence among their actions in an online CSCL environment with multiple interaction spaces. We looked at the moment-to-moment details of the practices through which participants organize their chat utterances and whiteboard actions as a coherent whole in interaction—a process that is lost in statistical analyses of multiple cases, where categorization and aggregation miss the rich and vital relationships of indexicality and sequentiality. We observed that referential practices enacted by the users are essential in the coordinated use of multimodalities afforded by such environments. The referential uses of available features are instrumental not only in allocating other members' attention to specific parts of the interface where relevant actions are being performed, but also in the achievement of reciprocity (intersubjectivity, common ground, shared understanding, group cognition) among actions in the multiple interaction spaces, and hence a sense of sequential organization across the spaces.

In our case study, we have seen the establishment of an indexical ground of deictic references co-constructed by the group members as an underlying support for the creation and maintenance of their joint problem space. We have seen that nexus of references created interactionally as group members propose, question, repair, respond, illustrate, make visible, symbolize, name, etc. In the VMT dual-media environment, the differential persistence, visibility and mutability of the media is consequential for the interaction. Group members develop methods of coordinating chat and drawing activities to combine visual and conceptual reasoning by the group and to co-construct and maintain an evolving shared indexical ground of their discourse.

In this study, we have transformed the problem of common ground from an issue of sharing mental representations to a practical matter of being able to relate semiotic objects to their indexed referents. The references do not reside in the minds of particular actors or in a platonic/idealized realm transcendental to the activity, but have been crafted into the presentation of the chat postings and drawing actions as shared math artifacts through the details of wording and sequential presentation. The references are present in the data as affordances for *understanding* by group participants as well as by analysts. The *meaning* is there in the presentation of the communication objects and in the network of interrelated references, rather than in re-presentations of them. The understanding of the references is a matter of normally tacit social practice, rather than of rationalist explicit deduction. The references can be explicated by analysis, but only if the structure of sequentiality and indexicality is preserved in the data analysis and only if the skill of situated human understanding is applied.

In our case study of an hour long session, three students construct a diagram of lines, triangles and hexagons, propose a math pattern problem, analyze the structure of their diagram, devise a method of systematic counting, make and justify claims about patterns observed and derive algebraic formulas to solve their problem. They do this by coordinating their whiteboard and chat activities in a synchronous online environment. Their accomplishment is precisely the kind of educational math experience recommended by mathematicians (Livingston, 2006; Lockhart, 2009; Moss & Beatty, 2006). It was not a mental achievement of an individual, but a group accomplishment carried out in computer-supported discourse. By analyzing the sequentiality and indexicality of their interactions we explicated several mechanisms of this group cognition by which the students coordinated the meaning of their discourse and maintained adequate reciprocity of understanding.

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VITA

Education

Ph.D. in Information Science & Technology 2009
College of Information Science & Technology, Drexel University

M.S. and M.C.I.T. in Computer & Information Science 2003
School of Engineering and Applied Sciences, University of Pennsylvania

B.S. in Mathematics 2001
School of Arts & Sciences, Middle East Technical University

Selected Publications

Cakir, M. P., Zemel, A., & Stahl, G. (2009). *The joint organization of interaction within a multimodal CSCL medium.* International Journal of Computer-Supported Collaborative Learning, 4(2), 155-190.

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