Discovering Dependencies: A Case Study of Collaborative Dynamic Mathematics

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Abstract: The Virtual Math Teams (VMT) Project is exploring an approach to the teaching and learning of basic school geometry through a computer-supported collaborative-learning (CSCL) approach. As one phase of a design-based-research cycle of design/trial/analysis, two teams of three adults worked on a dynamic-geometry task in the VMT online environment. The case study presented here analyzes the progression of their computer-supported collaborative interaction, showing that each team combined in different ways (a) exploration of a complex geometric figure through dynamic dragging of points in the figure in a shared GeoGebra virtual workspace, (b) step-by-step construction of a similar figure and (c) discussion of the constraints needed to replicate the behavior of the dynamic figure. The teams thereby achieved a group-cognitive result that most of the group members would probably not have been able to achieve on their own.

Geometric Discourse

The educational research field of computer-supported collaborative learning (CSCL) arose in the late 1990s to explore the opportunities for collaborative learning introduced by the growing access to networked computational devices, like laptops linked to the Internet (Stahl, Koschmann & Suthers, 2006). The seminal theory influencing CSCL was the cognitive psychology of Vygotsky (1930/1978). He had argued several decades earlier that most cognitive skills of humans originated in collaborative-learning episodes within small groups, such as in the family, mentoring relationships, apprenticeships or interactions with peers. Skills might originate in inter-personal interactions and later evolve into self-talk mimicking of such interactions; often ultimately being conducted as silent rehearsal (inner speech, thinking) or even automatized non-reflective practices (habits). In most cases of mathematics learning, the foundational inter-personal interactions are mediated by language (including various forms of bodily gesture) (Sfard, 2008; Stahl, 2008). Frequently, the early experiences leading to new math skills are also mediated by physical artifacts or systems of symbols—more recently including computer interfaces (Çakir, Zemel & Stahl, 2009).

Based on a Vygotskian perspective, a CSCL approach to the teaching of geometry could involve collaborative learning mediated by dynamic-geometry software—such as Geometer's Sketchpad or GeoGebra—and student discourse. During the past decade, we have developed the Virtual Math Teams (VMT)

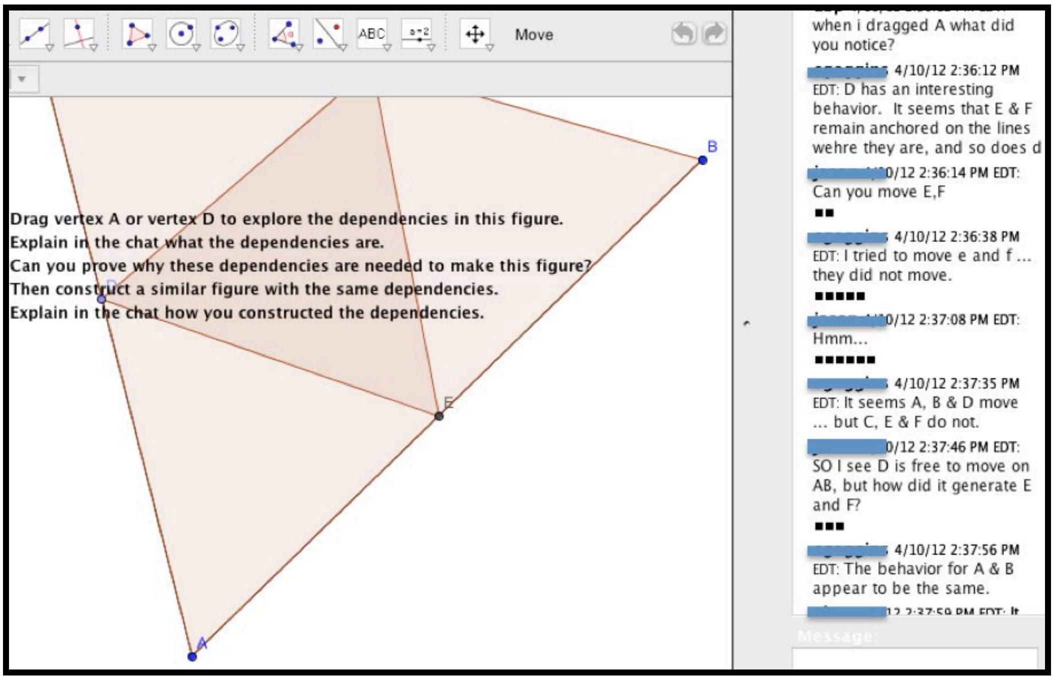

Figure 1. The dragged construction with the problem statement and some chat.

environment and are now integrating a multi-user version of GeoGebra into it (Stahl, 2009; Stahl et al., 2010). Our environment and associated pedagogy focus on supporting collaboration and fostering significant mathematical discourse. In developing this system, we have tested our prototypes with various small groups of users. Recently, two small groups worked together for about an hour on the problem given in Figure 1 (based on the construction of inscribed equilateral triangles). We will call them Group A (Jan, Sam and Abe) and Group B (Lauren, Cat and Stew). The group members are adults already familiar with GeoGebra.

The geometry problem is adapted to the VMT setting from (Öner, 2013). In her study, two co-located adults were videotaped working on one computer screen using Geometer's Sketchpad. We have "replicated" the study with teams of three adults working on separate computers with our multi-user version of GeoGebra in the VMT environment, allowing them to construct, drag, observe and chat about a shared construction. Öner chose this problem because it requires students to explore a dynamic-geometry figure to identify dependencies in it and then to construct a similar figure, building in such dependencies. We believe that the identification and construction of geometric dependencies is central to the mastery of dynamic geometry (Stahl, 2012b; 2013).

In this paper, we analyze the process through which the two groups in our study identified and constructed the dependencies involved in an equilateral triangle inscribed in another equilateral triangle. While we were able to replay the entire sessions of the groups in complete detail, observing all group interaction (text chat and dynamic-geometry actions) that the group members observed, the reader of this paper will have to imagine being able to drag points of the figure around and having the inscribed triangles remain intact due to the software maintenance of construction dependencies. Also bear in mind that the difficulty of the geometry problem comes from the geometric fact that a point is defined by the intersection of two lines and cannot be specified as the intersection of three lines.

Case Study: Group A

Group A starts by coordinating their online activity. They decide who will have initial control of the GeoGebra manipulation and they discuss in the text-chat panel the behavior they see as points of the construction are dragged (Log 1). They begin by dragging each of the points in the diagram (Figure 1).

6	14:33:56	Sam	I am good with somebody taking a stab at the dragging ...
7	14:34:10	Sam	I think maybe tell us what you intend to drag and we can discuss what we observe?
8	14:34:18	Jan	Go ahead Abe. Why don't you move the points in alphabetical order
9	14:34:36	Abe	Ok
10	14:34:43	Abe	I will try to drag point A
11	14:35:08	Jan	So the whole triangle moves... it both rotates around point B and it can dilate
12	14:35:14	Sam	So, A seems to move all the other points and scalle the whole drawing.
13	14:35:14	Jan	Which are you moving now
14	14:35:20	Sam	What are you moving now?
15	14:35:35	Abe	I first moved 1 and then D.
16	14:35:45	Sam	A, then D.
17	14:35:46	Sam	ok
18	14:36:05	Jan	so D was stuck on segment AC
19	14:36:12	Abe	when i dragged A what did you notice?
20	14:36:12	Sam	D has an interesting behavior. It seems that E & F remain anchored on the lines wehre they are, and so does d
21	14:36:14	Jan	Can you move E,F
22	14:36:38	Sam	I tried to move e and f ... they did not move.
23	14:37:08	Jan	Hmm...
24	14:37:35	Sam	It seems A, B & D move ... but C, E & F do not.
25	14:37:46	Jan	SO I see D is free to move on AB, but how did it generate E and F?
26	14:37:56	Sam	The behavior for A & B appear to be the same.
27	14:37:59	Abe	It appears that the triangles remain equilateral.

Log 1. Group A drags points in the diagram.

Note that the problem statement in Figure 1 does not explicitly state that the triangles are equilateral or inscribed. By having Abe drag points A and D, the team quickly sees that the vertices of the inner triangle always stay on the sides of the outer triangle (e.g., lines 18 and 20), indicating that the smaller triangle is inscribed in the larger one.

As Abe drags each of the vertex points, the group notices that points A, B and D are free to move, but that C, E and F are dependent points, somehow determined by A, B and/or D. Jan asks Sam to drag E and F, but Sam finds that they cannot be dragged. This sparks Jan to express wonder about how the position of point D (as it is dragged while A, B and C remain stationary) generates the positions of E and F (line 25). This is a move to consider how the diagram must be constructed in order to display the behavior it does during dragging. Meanwhile, Abe notices in line 27 that the triangles both remain equilateral during the dragging of all their vertices.

Within about three minutes of collaborative observation, the group has systematically dragged all the available points and noted the results. They have noticed that the triangles are both inscribed and equilateral. They have also wondered about the dependencies that determine the position of E and F as D is dragged. Now they start to consider how one would construct the dynamic diagram (Log 2).

47	14:45:39	Jan	What are we thinking...
48	14:46:07	Abe	okay,we have two equilateral triangles, with the inner one constrained to the sides of the outer triangle.
49	14:46:12	Sam	I think Abe summarized what is happening nicely - that both triangles remain equilateral when any of the 3 movable points are moved.
50	14:46:26	Jan	Agreed.
51	14:47:01	Jan	The thing I'm wondering about is how to generate the specific equilateral triang.e
52	14:47:02	Sam	Yes, another good bpoint - the one is contained in the other ... further, the three points of the inner triangle are constrained by the line segments that make up the outer triangle.
53	14:47:03	Abe	let's try to construct the figures?
54	14:47:20	Jan	For example, given a point on AB and a point on AC, there exists an equilateral triangle
55	14:47:38	Jan	But that's not this sketch b/c only one point is free. The rest are constrained
56	14:49:37	Jan	I'm wondering if all the three triangles that are outside the little equilateral triangle yet inside the big one are congruent.
57	14:50:39	Abe	When you say all three triangles, do you mean the three sides of the one of the triangles?

Log 2. Group A wonders about the construction.

First they all agree on the constraint that the triangles must remain inscribed and equilateral. Abe suggests that they actually try to construct the figure (line 53); through such a trial, they are likely to gain more insight into an effective construction procedure, which will reproduce the dragging behavior they have observed. Jan first notes that an equilateral triangle can be defined by the two points of its base. However, he notes that in the given figure only one of the vertices is free and it determines the other two (line 55). This leads him to wonder, "if all the three triangles that are outside the little equilateral triangle yet inside the big one are congruent." If they are congruent, then corresponding sides will all be of equal length. Abe relates the sides of the three little congruent triangles to the three sides of the interior triangle and to the three line segments on the sides of the exterior triangle. Following the excerpt in Log 2, Team A measures the three line segments AE, BF and CD, discovering that they are always equal to each other, even when their numeric length changes with the dragging of any of the free points (Log 3).

72	14:53:33	Jan	That means that CD, AE, and CF also have to be the same length, bc big triangle is equilateral
73	14:53:42	Abe	did you change what is being measured? or did you resize the figure?
74	14:53:58	Jan	I just moved point D along the side of the equilateral triangle
75	14:54:35	Abe	i c
76	14:56:16	Abe	So, shall we summarize the dependencies that we notice?
77	14:57:11	Jan	Sure who wants to start?
78	14:57:45	Sam	The inner triangle is contained by the outer triangle.
79	14:58:05	Sam	segment AC is the boundary of point D

80	14:58:14	Sam	Segment CB is the boundary of point F
81	14:58:24	Sam	Segmemt AB is the boundary of point E
82	14:58:55	Jan	So I think we may want to say F is on CB a bit differenlty.
83	14:59:10	Sam	Both triangles are equalateral no matter how the three movable points -- A, B & D -- are moved.
84	14:59:14	Jan	It is not free to move on CB. It is stuck in a particular location on CB defined by where D is on CA
85	15:00:09	Abe	The line segment CB cannot move.
86	15:00:10	Jan	So I think F is CD units away from B on BC. Its not constructed as an equilateral triangle, it happens to be an equilaterl triangle because of the construction
87	15:00:26	Jan	Agreed. I meant segment of length CB
88	15:00:38	Jan	Do you all buy that...
89	15:00:39	Jan	?
90	15:00:50	Sam	@Jan - I think that's covered by saying that both triangles are always equaleteral ... it implies both points move in conjunction with the third. (D) ... Of course, I don't teach the teachers who teach math (much), so you may have a better sense of the conventions. :D
91	15:00:59	Sam	I'll buy it.
92	15:01:04	Abe	yes, i agree!
93	15:02:28	Abe	The same can be said about E, it's constructed to be CD units from A.

Log 3. Group A identifies dependencies of the inscribed equilateral triangles.

After noting the key dependency that they discovered, AE=BF=CD (line 72), they list the other dependencies involved in constructing the figure. Line 86 provides a conjecture on how to construct the inner triangle. Namely, it is not constructed using Euclid's (300 BCE/2002) method from his Proposition 1 (the way the exterior equilateral triangle could be[1]). Rather, point F is located the same distance from B on side BC as D is from C on AC: a distance of CD. Jan asks the rest of his group if they agree (line 88). They do. Abe adds that the same goes for the third vertex: point E is located the same distance from A on side AB as D is from C on AC: a distance of CD. The work of the group on this problem is essentially done at this point. A few minutes later (line 110), Jan spells out how to assure that AE=BF=CD using GeoGebra construction tools: "Measure CD with compass. Then stick the compass at B and A."

We have seen that Group A went through a collaborative process in which they explored the given figure by varying it visually through the procedure of *dragging* various points and noticing how the figure responded. Some points could move freely; they often caused the other points to readjust. Some points were constrained and could not be moved freely. The group then wondered about the constraints underlying the behavior. They conjectured that certain relationships were maintained by built-in dependencies. Finally, the group figured out how to accomplish the *construction* of the inscribed equilateral triangles by defining the *dependencies* in GeoGebra.

Case Study: Group B

Team B goes through a similar process, with differences in the details of their observations and conjectures. Interestingly, Team B makes conjectures leading to at least three different construction approaches. First, Stew wonders if the lengths of the sides of the interior triangle are related to the lengths of a segment of the exterior triangle, like DE=DA (Log 4). The group then quickly shares with each other the set of basic constraints—inscribed and equilateral—similar to Group A's list of constraints.

15	14:37:00	Stew	and it appears that the side lengths of the inner triangle are related to the length of a portion of the orignal side
16	14:37:08	Cat	so also, there must be a constraint about the segments remaining equal, no?
17	14:37:31	Cat	@Stew, why can't they just be equal to each other?
18	14:37:47	Lauren	yes, visually it sure looks like equilateral triangles

[1] Triangle ABC can be constructed as an equilateral by locating point C at the intersection of two circles of equal radii centered on points A and B. However, the point F of triangle DEF has to be on line segment BC as well as being equidistant from D and E, which would entail constructing F at the intersection of three lines. By constructing F as the intersection of BC with a circle around B of radius equal to the length of CD, the group circumvents this problem while still imposing the necessary dependencies on point F.

19	14:37:53	Stew	yes, I think the triangles are equilateral or something like that.?
20	14:38:00	Lauren	D is free to move on AC, but E and F cant be dragged
21	14:38:58	Lauren	constructing the outer equilateral will be easy, but how do you think we should plan the construction of the inner triangle?
22	14:39:24	Stew	you can construct an equilateral but how do you make it so that its vertices are always on the outer triangle?
23	14:39:57	Lauren	Im thinking place D on AC, and construct an equilateral from there, with intersections on the sides of the outer triangle
24	14:40:12	Lauren	should we try and see what happens?
25	14:40:13	Cat	yeah, i'm not sure about making the other points stay on their respective segments
26	14:40:27	Cat	but we can maybe see the answer when we get closer
27	14:40:35	Stew	I think we'll get intpo trouble with the third side
28	14:40:38	Lauren	yeah, that will be the tricky part, but i think if we intersect they will be constrained
29	14:40:41	Stew	but, sure, let's try it
30	14:40:53	Lauren	may I start?
31	14:40:59	Cat	go for it!

Log 4. Group B noticings while dragging points in the diagram.

The group sees that the inner triangle must remain both inscribed and equilateral. This raises difficulties because the usual method of constructing an equilateral triangle would not in general locate the dependent vertex on the side of the inscribing triangle (line 22). This group, too, decides to start construction in order to learn more about the problem (line 24). They begin by constructing triangle ABC and placing point D on AC. They anticipate problems constructing triangle DEF and ensuring that both E and F remain on the sides of the inscribing triangle while also being equidistant from D. Note that the members of the team are careful to make sure that everyone is following what is going on and agrees with the approach.

45	14:44:53	Lauren	anyone have any ideas for the inner triangle?
46	14:45:37	Stew	One thing I noticed is that the sidelength of the inner triangle appears to be the distance of the longer segment on the original triangle
47	14:46:17	Cat	i wish i could copy the board :) i know that is not ideal, though
48	14:46:40	Cat	i forget what the tools do exactly, and want to just remind myself
49	14:46:57	Stew	If you made a circle that fit inside the original triangle, then its point of tangency or intersection might be useful
50	14:47:31	Stew	the trick might be to find the center of such a circle.
51	14:48:15	Stew	There are interesting centers made by things such as Cat was suggesting, the angle bisectors, or perpendicular bisectors
52	14:48:33	Lauren	yes - the center of each triagle probably is the same - do you think?
53	14:48:54	Lauren	angle bisecotrs would work
54	14:49:05	Stew	I don't think they have the same center
55	14:49:44	Lauren	maybe not....

Log 5. Group B conjectures about the construction.

In line 46 (Log 5), Stew repeats his conjecture about side DE equaling the length of "the longer segment" of AC, i.e., either CD or AD depending on which is longer at the moment. This conjecture is visibly supported by the special cases when D is at an endpoint of AC or at its midpoint: when D is at an endpoint, DE=AC or DE=AB (and AB=AC); when D is at the midpoint of AC, DE=AD=DC because the three small triangles formed between the inscribed triangles are all equilateral and congruent.

But then the group switches to discussing a quite different conjecture that Lauren had brought up earlier and that Cat is trying to work on through GeoGebra constructions. That conjecture is that it would be helpful to locate the centers of the inscribed triangles, construct a circle around the center and observe where that circle is tangent to or intersects triangle ABC. In general, triangles have different kinds of centers, formed

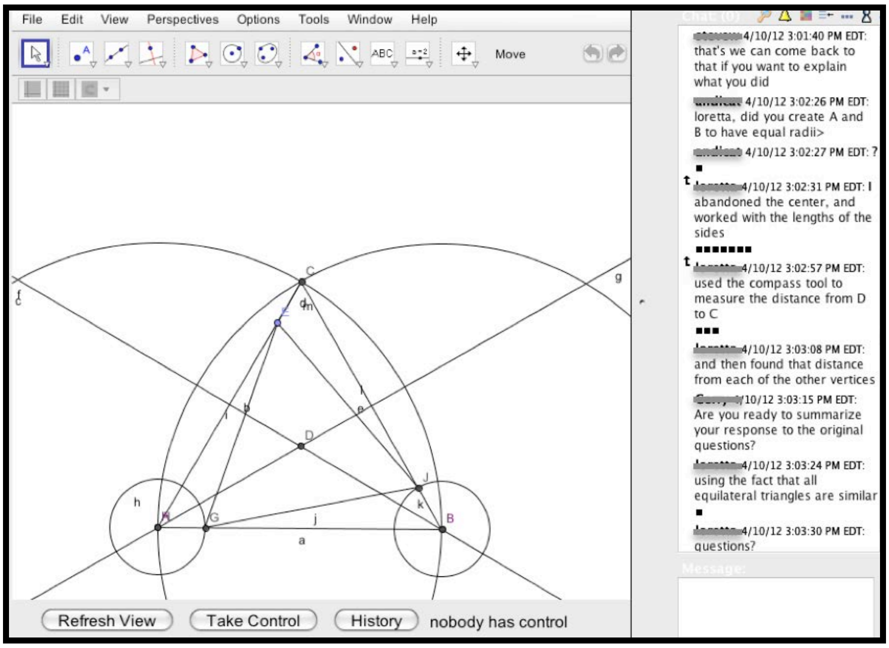

Figure 2. Finding the center and constructing equal line segments.

by constructing bisectors or the triangle's angles or by constructing perpendicular bisectors of the triangle's sides. The group discusses which to use and whether they might be the same center for both of the triangles.

Lauren does some construction (see Figure 2). She locates a point at which triangle ABC's angle bisectors meet. However, she then abandons this approach (Log 6). Instead, she pursues a new conjecture, related to Stew's earlier observations: "we know by similar triangles, that each line of the inner is the same proportion of the outer" (Lauren, line 75).

80	15:01:40	Stew	that's we can come back to that if you want to explain what you did
81	15:02:26	Cat	Lauren, did you create A and B to have equal radii>
82	15:02:27	Cat	?
83	15:02:31	Lauren	I abandoned the center, and worked with the lengths of the sides
84	15:02:57	Lauren	used the compass tool to measure the distance from D to C
85	15:03:08	Lauren	and then found that distance from each of the other vertices
87	15:03:24	Lauren	using the fact that all equilateral triangles are similar
88	15:03:30	Lauren	questions?
89	15:04:05	Lauren	is everyone convinced the inner triangle is as it should be?

Log 6. Group B constructs the dependencies of the inscribed equilateral triangles.

Lauren uses the GeoGebra compass tool with a radius of CD to construct circles around the other vertices of triangle ABC (Line 84, 85), just like Group A had done. This locates points where the circles intersect the triangle sides for placing the other vertices of the inscribed triangle with the constraint that CD=AE=BF. She then concludes by asking if the other group members agree that this constructs the figure properly.

Like Group A, Group B initiated a collaborative process of exploring the given diagram visually with the help of dragging points. They developed conjectures about the constraints in the figure and about what dependencies would have to be built into a construction that replicated the inscribed equilateral triangles. They decided to explore trial constructions as a way of better understanding the problem and the issues that would arise in different approaches. Eventually, they pursued an approach involving line segments in the three congruent smaller triangles.

It is interesting to note the role of the three small triangles formed between the two inscribed triangles. These small triangles are not immediately salient in the original diagram. Triangles ABC and DEF are shaded; the smaller triangles are simply empty spaces in between. They become focal and visible to the groups due to their relationships with the sides of the salient triangles, and particularly with the segment CD. It is the fact that these three smaller triangles are congruent that supports the insight that the necessary constraint is to make CD=AE=BF. The smaller triangles become visible through the exploratory work of dragging, conjecturing and constructing this dependency. This is precisely the kind of perception that can occur in the scaffolded inter-personal setting of collaborative dynamic geometry and then can gradually mature into increased professional vision (Goodwin, 1994) and mastery of practices of observation and discourse by the individual group members as developing students of mathematics.

Conclusion

Both Group A and Group B find a solution to the problem they address by taking advantage of the affordances of collaborative dynamic geometry. Their understanding of the problem (Zemel & Koschmann, 2013) develops gradually through dragging points, noticing how other points respond, wondering about effective constraints and conjecturing about possible dependencies to construct. Next, they begin exploratory construction. These are trial-and-error attempts in different directions. Some reach deadends or are simply put aside as more promising attempts catch the group's attention. Finally, each group agrees upon a key dependency to build into its construction. This dependency—in its connections to related geometric relationships—forms the basis for persuading the group members of a solution to the problem. This is implicitly a justification or proof of the solution. In the end, the group can construct a set of inscribed equilateral triangles, building in the dependency that CD=AE=BF. They can then prove that the triangles are inscribed and equilateral by referring to the dependency that CD=AE=BF, along with certain well-known characteristics of equilateral and congruent triangles.

Although both groups reached a similar conclusion, their paths were significantly different. First, they defined their problem differently. Group A focused on listing the constraints that they noticed from dragging points and then on proving that the given triangles were in fact equilateral. Group B, in contrast, quickly realized that it would be difficult to construct triangle DEF to be both inscribed and equilateral, since these characteristics required quite different constraints, which would be hard to impose simultaneously. Whereas Group A coordinated its work so that the members followed a single path of exploration and conjecture, Group B's members each came up with different conjectures and even engaged in some divergent explorative construction on their own before sharing their findings. Despite these differences, both groups collaborated effectively. They listened attentively and responded to each other's comments. They solicited questions and agreement. They followed a shared group approach. Together, they reached an accepted conclusion to a difficult problem, which they would not all likely have been able to solve on their own, illustrating effective group cognition (Stahl, 2006).

The case study of Groups A and B illustrates the approach of collaborative dynamic geometry. The groups took advantage of the three central dimensions of dynamic geometry—dragging, construction and dependencies—to explore the intricacies of a geometric configuration and to reach—as a group—a deep understanding of the relationships within the configuration. They figured out how to construct the diagram and they understood why the construction would work as a result of dependencies that they designed into it.

In the Virtual Math Teams Project, we are currently refining the VMT software and developing curriculum (Stahl, 2012b) to guide the use of collaborative dynamic geometry in in-service-teacher professional development and high-school geometry (Stahl, 2012a). The curriculum centers on activities like the one in Figure 1. It structures the use of dragging, constructing and dependencies, as well as effective collaborative discourse practices. The curriculum is closely aligned to the new Common Core State Standards for basic geometry and their recommended mathematical practices (CCSSI, 2011). It covers the most important propositions of Book I of Euclid's *Elements*, translating them into research-based, contemporary approaches to geometry and mathematical discourse in a computer-supported collaborative learning environment. We will continue to study the results of collaborative dynamic geometry through analysis of the discourse and geometric explorations (Stahl, 2012c).

On the basis of a continuing series of trial studies like the one just reported, we feel that the approach of collaborative dynamic geometry can translate the geometry of Euclid into an effective tool of computer-supported collaborative learning.

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