Designing for Group Math Discourse

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Abstract: We are developing a socio-technical system to support group cognition among math students in the form of significant mathematical discourse about dependencies in dynamic-geometry constructions. Analysis of a pilot trial in a typical early cycle of our design-based-research approach revealed barriers to group success from both software and mathematics issues, and demonstrated that participants "cycled" between these types of issues. We are responding by developing a curriculum to address the uncovered technical and cognitive issues. We present the findings of our pilot study and the curriculum design criteria that are emerging from continuing cycles of re-design, prototyping, testing and analysis.

Introduction

We are interested in promoting group cognition (Stahl, 2006) among math students learning high-school geometry by enhancing their ability to engage in significant mathematical discourse. Recent research on math learning points to the central role of language, enabling articulation and verbal reflection about mathematical relationships (Sfard, 2008; Stahl, 2008). Because we want to exploit the computational power of computers and the advantages of networking to support online collaboration incorporating math discourse, we necessarily face the dual design constraints of technical software development and social-practice scaffolding.

Increasingly, high school students are learning online, with home schooling, resources like the Khan Academy of math YouTube videos and virtual high schools. The problem with online learning is that the current models for this are often lacking in social interaction and collaborative learning. This is, of course the motivation for CSCL research and innovation.

Discourse is fundamentally a group process, so we want to provide support for small groups of students to engage together in math discourse. This is complicated in terms of both the discourse and the technology as there are multiple facets to "significant mathematical discourse" (Stahl, 2013d). Furthermore, we are interested in taking advantage of networked computers to allow groups of students to discuss math and to work on mathematical tasks together online. We want to supply computer support for their math work and computer recordings for maintaining persistence of their discourse—which raises technological barriers to students navigating the interface.

As a research project, we approach this task with the idea of combining VMT (Stahl, 2009; Stahl, Mantoan & Weimar, 2013)—a generic computer environment for collaborative learning by "virtual math teams"—with GeoGebra (www.geogebra.org)—a popular open-source application for dynamic geometry. This involves enhancing VMT and transforming GeoGebra from a single-user application to a multi-user client integrated into VMT. When developing a socio-technical system, in addition to the technical development we need to guide the group-cognitive work by providing helpful resources and scaffolding group practices.

To get a realistic sense of how groups of students will interact within the environment we are designing, we need to conduct pilot tests throughout our design process. In order to try out our system in naturalistic settings as part of our socio-technical, design-based-research approach, as well as to provide a basis for eventual deployment, we have developed relationships with two professional education schools, where we will eventually deploy our system with practicing math teachers.

In our preliminary stage, we have run informal pilot tests with available groups. Our findings showed that these students encountered significant problems due to a lack of preparation for using the technology and for engaging in the mathematics. As a result of the analysis of these sessions—as discussed below—we realized that we would have to carefully craft a curriculum, which the teachers could follow and then adapt for their own classrooms. This curriculum would need to incorporate not only math lessons, but also tutorials about the technological environment. We started to sketch out a curriculum based on existing best practices and theories. We were fortunate that the Common Core State Standards for Mathematics (CCSSI, 2011) had recently been released and adopted by most states in the US. This provided an up-to-date, research-based outline of content for a geometry course, which was widely accepted.

As we looked at results of the initial trials analyzed below, we realized many problems needed to be addressed. These involved design issues in extending VMT, in making GeoGebra multi-user, in supporting collaboration around the activities, in teaching the deep conceptual ideas in geometry, in taking advantage of computer-supported dynamic math and in promoting significant math discourse (Stahl, 2013d). We ran several cycles of additional trials within our research group and with available college students. In each cycle, we

revised the curriculum, revised the software, ran the trial and analyzed the behaviors. Generally, there were clear lessons from each trial, which led to the next cycle.

Gradually, a set of design criteria for the curriculum was formulated and evolved. In this paper, we report findings from the early session without curriculum to identify challenges faced by technologically adept individuals when attempting to engage in significant math discourse within the GeoGebra environment. Then we review some of the lessons for the technology and some of the aspects of the discourse that we believe are important. Based on these lessons, we are now developing a curriculum based around online, small-group activities. This paper discusses the criteria for the design of that curriculum, as it is emerging from testing of trial curriculum drafts.

From a socio-technical standpoint, the curriculum is central because it mediates between the people and the technology. It tells the people what activities they should be engaging in while communicating through and working within the technology. It also models for them how to talk about math. For an online course, in which there is no teacher present to orchestrate activities and interaction, the textual curriculum provides the major scripting of collaborative sessions and the primary scaffolding of the group cognition.

Relevant Literature

Our approach to online dynamic-geometry education is based on previous research by our own team and by others in the fields of groupware design, collaborative learning and mathematics education.

Dynamic-Mathematics Software

The research on dynamic-mathematics software—such as Geometer's Sketchpad, Cabri and GeoGebra—is new and limited. Much of it merely popularizes the availability and the novelty of the approach. However, there are some important studies of aspects such as the dynamic dragging of geometric objects and the implications of dynamic visualizations for student conceptions of proof. A recent review of 37 publications on dynamic mathematics summarized the research to date (Powell & Dicker, 2012). Dynamic geometry can be effective in improving student understanding of geometry through support for visualization and exploration. There is a trade-off between having students do their own constructions versus having them manipulate prepared constructions. While the construction process may deepen understanding, it takes much longer and can be distracting from curricular goals. The ability to manipulate constructions dynamically aides students in making conjectures, exploring them and understanding their significance, but it can be seen as a substitute for deductive proof and can lower student motivation to engage in rigorous proof procedures.

The utilization of dynamic-math environments by teachers has allowed them to extend traditional materials found in textbooks, allowing for better interaction with their students in both the classroom and through technological mediation (Hohenwarter, Preiner & Yi, 2007). These dynamic-math environments have been found to make mathematical tasks more efficient and allow for more interaction and application of the theoretical knowledge associated with the mathematical task (Laborde, 2001; Öner, 2008). This success of dynamic-math environments in the classroom setting is heavily influenced by the given tasks and the interaction with the instructor who is leading the exercise (deVilliers, 2004; Mariotti, 2001; Sanchez & Sacristan, 2003). The research in dynamic math is limited to specific pedagogical approaches and needs to be developed further. In particular, previous studies focus on individual learning. This is at least in part because until now dynamic-math applications have been designed for single users. Another weakness in the literature is the lack of focus on *dependencies*, which we feel are central to understanding dynamic geometry (Stahl, 2013c).

Online Math Collaboration

The ability for students to co-construct knowledge using technology together has been studied for decades. Depending on the context, the students and teachers play different roles (Jonassen, Peck & Wilson, 1999). While technology adoption in the classroom has met with varying levels of success, using small groups for learning and co-constructing knowledge has been illustrated to be productive through all levels of education (Springer, Stanne & Donovan, 1999).

Researchers have approached studying math discourse and cognition in face-to-face media through the utilization of technology (Dion, Jank & Rutt, 2011). Research on Group Scribbles use in a primary science classroom in Singapore illustrates a transitional stage between the physical classroom and a strictly online context (Chen & Looi, 2011). The Group Scribbles environment provides similar capabilities to the VMT environment, but the interaction occurs in a classroom through tablets. The students' interactions are technologically mediated, and the teacher in the classroom provides physical mediation, allowing for technology problems to be quickly overcome so participants may focus on the problem at hand. In a series of tasks carried out using Group Scribbles, it was found that students had more agency and were given more participation opportunities compared to traditional approaches. This was found to particularly benefit passive students (Chen, Looi & Ng, 2009).

Technology has also taken the place of moderating a learning environment in an effort to facilitate more discourse and reduce direct teacher involvement in student problem solving. In an attempt to automate the support of group math cognition in the VMT environment, research has been initiated to understand how conversational agents could be used (Cui et al., 2009). These agents are used to encourage academic discourse and accountable talk (Michaels, O'Connor & Resnick, 2008), but have been met with only limited success so far (Stahl, 2013a; Stahl et al., 2010).

Understanding the technological environment of the student and how this contributes to successful or unsuccessful learning is integral to the analysis of the learning and the design of an online system (Suthers & Medina, 2010). As evidenced by prior research (Valentine, 2002) and our findings below, technology use in a learning scenario can harm the experience of students, hindering communication as much as facilitating it. Understanding the extent of barriers and modes of facilitation of math discourse in a dynamic-math environment is still limited.

Math Discourse

The theory of math learning through participation in math discourse (Sfard, 2008) specifies important mathematical discourse moves, such as encapsulation, reification, saming, routines, deeds, explorations and rituals—all defined, systematized and passed down through the community, culture, tools, procedures, language and traditions of mathematics. These interactional resources can traverse levels between individual learning, group cognition and community knowledge (Stahl, 2012; 2013b; Stahl & Öner, 2013). The theory of accountable talk (Michaels, O'Connor & Resnick, 2008; Resnick, O'Connor & Michaels, 2007) specifies discourse moves that promote accountability to the group, to standards of math reasoning and to the characteristics of the math objects. Speaking meaningfully in math discourse "implies that responses are conceptually based, conclusions are supported by a mathematical argument and explanations include reference to the quantities in the problem context as opposed to merely describing the procedures and calculations used to determine the answer" (Clark, Moore & Carlson, 2008, p. 298). Socio-mathematical norms include what counts as an acceptable, justifiable, easy, clear, different, efficient, elegant and sophisticated explanation (Yackel, 1995; Yackel & Cobb, 1996). Mathematical practices emerge from interaction, are taken up by participants and are applied repeatedly (Medina, Suthers & Vatrapu, 2009). Though Sinclair and Yurita (2008) study of how dynamic geometry changes discourse began the process, research into the nature of mathematical discourse in a collaborative dynamic-mathematics environment has yet to be conducted.

While the importance of collaborative learning for online education may be obvious to CSCL researchers (Stahl, Koschmann & Suthers, 2006) and its possible advantages have been well documented in cooperative-learning (Johnson & Johnson, 1989; Slavin, 1980) and CSCL research for decades (Sawyer, 2006), support for collaboration is still not always designed into new educational platforms. For instance, the latest hot approach to university instruction—massive open online courses or MOOCs—are generally based on the lecture paradigm, in which students passively watch talking-head videos of famous professors and are not given any sanctioned opportunities for interaction. Similarly, the acclaimed Khan Academy offers thousands of YouTube videos explaining detailed topics in school mathematics, but students have no support for interactively exploring the topics themselves or discussing them with peers. These technological opportunities are generally not designed to incorporate constructivist learning principles (Bransford, Brown & Cocking, 1999).

Method

In Fall 2011, we examined four one-hour-long chat logs from information-science graduate students taking a course on CSCL using the VMT environment. In these chats, the groups met online and attempted to solve a geometry problem within the GeoGebra environment. The students had used the VMT environment to perform collaborative writing exercises in previous weeks, but had not previously used GeoGebra. These students were enrolled in majors related to technology, suggesting that they were engaged rather than nervous about technology use. As part of the exercise, there was no explicit introduction to the GeoGebra tool or further instructions other than the assigned problem.

We were interested in analyzing these groups' interactions and their strategies for navigating a new online collaborative environment. Each log was examined independently using a thematic analysis approach that revealed themes that were typical stages of conversation. These stages include: social niceties, problem identification, technical discourse, math discourse, design suggestions and future planning. While these are separate stages of conversation, we found that each group moved back and forth between *technological* and *mathematical* discourse, behavior we termed "cycling."

We examined the logs using our initial categories as a guide to further examine this process of cycling. In our subsequent analysis, we identified the cyclical behavior triggered by individual statements distinctly indicating technical issues (involving software usage issues or software problems) versus mathematical issues and discourse (involving attempts to understand, represent and solve the geometry problem). By examining the chat logs we are able to observe phases of group interaction, how technology affects each phase and how the

technology can both facilitate and inhibit successful completion of the task in an online environment new to group members.

Findings

Analysis of the group chat logs illustrates the presence of a variety of stages of conversation by the members of the groups in the context of the problem-solving task. Each of the chats begins with an orientation, including the exchange of social niceties and resolution of unrelated issues, typically lasting two to three lines per group member.

Following the orientation stage, the groups identified the problem by either explaining it in the chat to the other group members or by referring to the posted problem in another tab. This typically involved a statement to orient the group:

> *Quick summary – we have to work thru the problem (see topic). Summarize the process in the Summary tab and post a few sentences on the wiki too. We good? (Group 1, line 16)*

This quote illustrates some of the important characteristics of this type of focusing statement, including a description of what the "speaker" is going to do with the statement, instruction and then a leading question to ensure the team is on the same page.

Table 1 illustrates the different stages identified in the chat logs of the four groups and the different places in the discussion in which math discourse began. This varied for each group, and even when groups did not start with technical issues, they arose very quickly.

Table 1. Stage identification of each of the groups

	Group 1	Group 2	Group 3	Group 4
Opening Stages	Orientation; Problem Identification			
Intermediate Stages	Role Assignment	Math Discourse	Technical Issues	Technical issues
	Technical Issues	Technical Issues	Math Discourse	Role Assignment
	Math Discourse	Design Suggestions	Math Confusion	Math Discourse
			Technical Issues	Use of Alternative Tools
			Technical Confusion	
Concluding Stages	Summarization of task/experience; social niceties; next steps			

Once Group 3 reached math discourse, they experienced confusion about the mathematical concepts, further compounded by technical issues with the tool that further confused the participants in the group and degraded the quality of math discourse:

> *I'm trying to figure out how to delete this line… I kind of messed up… do you still see a line on the screen? (Group 3, line 24-25)*

This quote highlights a number of issues that were common across multiple groups: not knowing how to delete an object (an option expected by participants), and group members being unsure that they were looking at the same objects as their fellow group members.

Both Group 1 and Group 4 achieved significant math discourse as each of the team members attempted to solve the problem, but did so with the help of outside tools. Group 1 used PowerPoint. Group 4 experienced confusion because the VMT environment did not display the same screen to all group members, so one emailed a screenshot to share the solution. This indicates that use of familiar tools or tools that work intuitively enables groups to more quickly reach effective math discourse that achieves a solution.

Overcoming Technological Barriers

While the technological tool—multi-user GeoGebra—provided many opportunities and options, it often served as a barrier for users. Barriers could be as simple as not being able to undo an action. However, even simple barriers stopped the groups from engaging in fluid math discourse and sometimes even went unresolved as the individuals found ways to work around issues. One example of this is the issue of not being able to easily rename an object. The mathematical problem these groups were attempting involved making an angle ABC. Groups began by playing with the system by adding objects to their GeoGebra screen. However, each group discovered that they were unable to simply rename the points on their screen, and the names they needed (A, B and C) were already in use by the system, though the objects they developed later in the process were better suited to solving the problem. This meant that their refined objects were confusingly named (for example, J, K and L), making math discourse about the objects in relation to the problem statement more complicated:

> *One thing we can state is how the lettering got messed up… I think that is helping to confuse us. (Group 3, lines 58-59)*

Each group experienced this issue; because of their lack of familiarity with the system, none were able to fix it.

Other barriers were easier for the groups to work around creatively. When the tool put up barriers, these groups were quick to try to work around the resulting issues, employing their understanding of other technologies to inform their decision in the VMT environment or solve the problem and move forward to math discourse. While it was unsuccessful, in the above barrier, the group attempted to rename an object when it had difficulty, which is a common solution in other tools. Additionally, many groups wished for an undo option, a common affordance in other tools:

Is there an undo function… not that I could find. That would be nice. (Group 1, lines 97-99)

Because of the nature of the work, as groups overcame tool issues and moved into math discourse, new mathematical objectives (e.g., renaming a point, adding a ray) resulted in a return to the tool and often the discovery of a new technological barrier. Even in the face of such issues with the tool, multiple groups managed to achieve effective math discourse that led to solutions. Each successive cycle of math discourse and tool use also led to difficulty with the mathematical concepts at hand, which we will now discuss.

Discourse about Math Difficulties

The goal of these chats for the students was to experience a new tool, but also to achieve math discourse around the visualization and solution of a geometry problem. Reaching math discourse proved to take some time for multiple groups, despite the fact that they were actively pursuing this goal, often within the first few lines of chat. Typically, the first approach involved developing a shared understanding of what the mathematical problem was, which we termed "problem identification" in our stage-identification process. Groups quickly entered into discussion of technological issues with the tool, but had difficulty returning to the larger goal of mathematical discourse.

Participants often employed a question structure to encourage a return to math discourse, and usually included words like "okay," "well," or "so" to bridge from the previous topic, which was typically a technological issue. In Group 4, one participant states:

Ok, we are on the same page now… we need a point in the middle. (line 179-180)

In an attempt to move past the technological barrier of not being able to effectively rename objects and establish common ground among the participants, one participant transitioned with:

Well, anyway, do we all at least see i, j, k? (Group 4, line 83)

In addition to bridging words, participants also employed explicit questions to reorient the group, for example:

Can i start by drawing two lines to create an angle? (Group 3, line 22)

These structures serve to call attention to a reorientation, and to give other participants the opportunity to request a pause in that reorientation to ensure they share understanding with the rest of the group.

Reorienting questions also served to highlight an understanding gap, pulling the group back into math discussion to provide an explanation or confirm an understanding. One example of this math-question reorientation comes from Group 2:

If you try to construct a line EF trying to connect AB and BC, wouldn't that mean A=C. (line 94)

The use of reorienting statements rotates through group members, indicating that it was not always the same participant to return the group to math discourse. Talking about technological issues could quickly grab the attention of the group, but these reorienting statements were effective at refocusing the groups' attention on the mathematical issues. When groups returned to this higher level of math discourse, there were a variety of approaches employed by individuals. Multiple participants displayed something akin to math anxiety, highlighting their lack of experience or inability:

I haven't done geometry in a long time… I'll need the hints. (Group 1, line 18)

Often, members of the group shared in their confusion, as evidenced by Group 3's experience with making the decision to look at the hints during a series of math discourse. The group looked at the hints as a whole, but each member admitted to being more confused after doing so, imagining that it could be their unfamiliarity with math causing the issue:

I'm not sure if its cause I haven't done these types of problems in a while or the hints just aren't that good. (Group 3, line 95)

However, Group 1 and Group 4 were able to achieve math discourse and a solution, notably, with the use of familiar outside technologies.

Cycles of Problems

The analysis of the pilot trial revealed cycles of problems, with the groups having to go back and forth between confronting technical problems with the software and cognitive problems with the mathematics. The cyclic nature of the alternation between technical and mathematical difficulties may have been an artifact of the task and the preliminary state of the software prototype. Though the task was to work on a geometry construction, within the online environment, software problems intervened and distracted the group. Groups tried to quickly get around the technical problems and cycle back to the math. There, they found themselves poorly prepared to tackle a geometry problem. Both the technical and the cognitive problems were consequences of the situation of

pilot-test participants in a design-based-research project. The socio-technical goal of the project was still in the distant future and the necessary supports for the participants were not yet in place. Thus, it is not a surprise that the subjects met with many serious difficulties. The point is to learn from the pilot trial: what are the most important social and technical features to be developed next?

Discussion

The experiences of the groups highlight interesting aspects of group-cognitive processes and how tool and math skills can hinder the ability to solve the problem by otherwise competent users. Clearly, while math discourse was a goal of each group, it proved difficult to achieve in the face of tool issues and feelings of math anxiety. When faced with a technical issue, the individuals blamed the tool for the inability to solve the problem, because they felt they were technically competent in general:

> *I'm an IT consultant and have to deal with various software programs meaning I'm familiar with how software should be designed and navigating my way around…this was definitely tough. (Group 4, lines 310-313)*

On the other hand, when faced with a mathematical concept that they were not familiar with, members of the groups blamed themselves for not being mathematically focused:

> *My High School Math teachers are furious with me right now I can feel it. (Group 3, line 96)*

This dichotomy between technical ability and mathematical inability was identified in each log. While this is an interesting case in our specific dataset pertaining to mathematically oriented online-learning contexts, we suspect that this phenomena may be evident in other collaborative-learning situations. Working to learn both content and the technology used to deliver that content can be overwhelming and may distract from the conceptual intent of the lesson. These difficulties are evident in our analysis as triggers of cycling and may be applicable to many technologically mediated learning situations. Because of these identified issues, it is important to build technological familiarity into any educational groupware environment to overcome technological issues early in the process. We find that in the face of tool adversity individuals defaulted to tools they were comfortable with such as PowerPoint, paper/pencil or screenshot/email. The use of familiar tools allowed the members of the groups to focus on the actual math discourse and problem solving, and isolate the effects of the tool on their productivity.

One of the most striking elements of our analysis is the concept of *cycling* in the group process between *tool* issues and *math* discourse. There was a salient presence of software functionality issues that when coupled with gaps in knowledge derailed mathematical discourse. This derailment and the students' interest in getting back on task led to cycling. Though each group experienced cycling, the groups that were most successful were able to quickly manage technological barriers and return to math discourse for the majority of their chats. We speculate that problems will exist in many groupware situations, including math learning, in which there are gaps in ability to manipulate the technology used for the learning. We believe that these findings may be transferable to other environments and contexts. As highlighted by one of the participants,

> *The issue with our first attempts was the usability of the tools – and lack of familiarity of the capabilities available within GeoGebra (Group 1, line 109)*

An increase in familiarity with the system may reduce cycling; however, further research into groups learning a system is required to determine how this might manifest under different circumstances.

In addition to our analytical findings, each of the groups had recommendations for ways to improve the technology and the process of group math problem solving in the VMT-with-GeoGebra environment. These ranged from calls for an undo option to hopes for a primer or tutorial to alleviate some technological issues.

Curriculum Design Criteria

In response to the analysis of the GeoGebra use sessions, we have been drafting a set of dynamic-geometry curricular activities, interspersed with tutorials of the technology features. Curriculum activities have been designed to promote collaborative learning, particularly as exhibited in significant mathematical discourse about geometry. Collaborative learning involves a subtle interplay of processes at the individual, small-group and classroom levels of engagement, cognition and reflection. Accordingly, the activities are structured with sections for individual work, small-group collaboration and whole-class discussion. It is hoped that this mixture will enhance motivation, extend attention and spread understanding.

The goal of our set of activities is to improve the following skills in math teachers and students:

1. To engage in significant mathematical discourse; to collaborate on and discuss mathematical activities in supportive small online groups.
2. To collaboratively explore mathematical phenomena and dependencies; to make mathematical phenomena visual in multiple representations; and to vary their parameters.
3. To construct mathematical diagrams—understanding and exploring their structural dependencies.
4. To notice, wonder about and form conjectures about mathematical relationships; to justify, explain and prove mathematical findings.

5.   To understand core concepts, relationships, theorems and constructions of basic high-school geometry.
       In other words, the activities seek a productive synthesis of the five areas of: discourse, visualization, construction and argumentation skills applied in the domain of beginning geometry. The set of activities is designed to provide an educational experience in basic geometry to math teachers and students, taking them from a possibly novice level to a more skilled level, from which they can proceed more effectively without such designed, scaffolded activities. By providing activities on different levels for each of the dimensions, we hope to help math teachers and students to increase their relevant skills – in different ways for different people.

Conclusion

Our focus has centered increasingly on facilitating and supporting lessons involving geometric dependencies (Stahl, 2013c). GeoGebra allows one to construct systems of inter-dependent geometric objects. Students have to learn how to think in terms of these dependencies. They can learn through visualizations, manipulations, constructions and verbal articulations. These can all be modeled and these skills can be developed gradually; our pilot study indicates that for successful math discourse to be achieved, supporting these skills must be an explicit priority of the socio-technical system. We are now drafting and piloting versions of curricular activities designed to develop significant mathematical discourse focused on dependencies among geometric objects (Stahl & Öner, 2013). Concomitantly, we are implementing software support for teachers and students to explore the dependencies and assembling materials for professional development to prepare teachers to enact this curriculum with their students (Stahl, 2013d).
       Our design work is guided by socio-technical implications of continuing pilot studies as the technology and pedagogy of our project co-evolve. We are countering the problems that caused negative cycling of technical and cognitive distractions by improving the software and testing the curriculum. The curriculum integrates tutorials about using the VMT and GeoGebra interfaces with carefully structured sequences of dynamic-geometry activities for virtual math teams. The activities systematically build up the background knowledge, group practices and problem-solving orientation needed for engaging in mathematical discourse.

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