Translating Euclid:
Liberating the Cognitive Potential of
Collaborative Dynamic Geometry

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Abstract. How should one translate the classic-education approach of Euclid's geometry into the contemporary vernacular of computer visualization, social networking and discourse-centered pedagogy? The birth of geometry in ancient Greece and its systematization by Euclid played an important role in the development of deductive reasoning and science. As it was translated and refined over the centuries, however, geometry lost some of its cognitive power and its very nature became obscured. Recently, computer-supported versions of dynamic geometry have been developed, which afford visualization, manipulation, exploration, conjectures about constraints and construction of dependencies. Particularly within a context of computer-supported collaborative learning, a dynamic-geometry environment can facilitate the gaining of mathematical insight and understanding that is the hallmark of geometry. A case study involving an intriguing geometric configuration illustrates this within a trial test of the Virtual Math Teams collaboration environment with a multi-user version of the GeoGebra dynamic-mathematics application. This indicates how significant mathematical discourse about dynamic geometry within such environments can make salient the construction of dependencies as central to deductive cognition.

Translating from Classic to Post-Modern Philosophy

A small community of geometers in ancient Greece established a set of discourse practices and inscription methods that defined subsequent literate rational thought in the West. A cognitive history of this accomplishment is documented in Netz (1999). Latour (2008) reviewed Netz' analysis and suggested some of its significance. Since the golden age of Greece, geometry and rationality have gone through many transformations. To approach the question—What form should mathematics education take in the 21st Century?—it is helpful to first review the historic development of geometry.

The inscriptions. Using very primitive technology—ephemeral sketches in the sand and more persistent and portable diagrams on papyrus, wood, wax or clay tablets—Euclid's predecessors constructed intricate diagrams using just straightedge and compass. The diagrams consisted of points, line segments and arcs or circles. Construction sequences were used to establish dependencies among the components of constructed diagrams. Importantly, components were labeled with letters. The labels allowed accompanying texts to reference specific components, thus providing a clear visible connection between the elements of the inscription and specific statements in the text.

The texts. The textual discourse of the early geometers consisted of a highly stylized, formulaic language. The language of geometry was derived from everyday written Greek, but required specialized training to be used. The language was geared to stating parts of propositions and proofs, such as the statement of given conditions, or well-known propositions that contributed to the proof, or steps of construction and of proof conclusions. Presentations of geometric propositions consisted primarily of proofs with their accompanying labeled diagrams.

The propositions. In addition to mastering the inscription and discourse practices, a mathematician had to be very familiar with a corpus of established propositions. The knowledge of these propositions was

probably passed down through apprenticeship in small, distributed communities of geometers. Euclid compiled the theorems systematically, providing a persistent and literate basis for this knowledge, which spread around the world for thousands of years.

The content of geometry—definitions of basic geometric objects, common notions, logical equivalencies, postulates and previous propositions—was assumed in the presentation of proofs. Also implicit in the geometric texts was a practice of rational thought, which made the proof persuasive as necessarily true. That is, for instance, that the truth of a theorem was not dependent upon the particularities of the diagram, the construction process, the set of referenced theorems or the text of the proof. Rather, the diagram, construction, propositions and argumentation were merely means for bringing the reader to a transcendent mathematical truth. Geometry invented the sense of apodictic or deductive truth: a form of truth that was evidenced by the procedure of the geometric proof.

Euclid's *Elements* (300 BCE/2002) was a compilation of geometric proofs. By reading a sequence of these proofs, one could gain proficiency in the geometric practices. Therefore, the ancient proofs were carefully compiled, categorized and formalized in various ways over the centuries—both before and after Euclid, but famously by him. Reading an edition of Euclid's *Elements* was considered a cornerstone of a classic education until the era of public education. Contemporary geometry textbooks for high school can still be seen as a variation on this.

During the intervening 2,500 years of codification, the practices of geometric discourse, inscription, labeling, construction and proof have lost much of their cognitive power. Not all students of geometry still experience the sense of rational necessity as an exciting discovery (Lockhart, 2009).

Perhaps it is time to re-invent the practices of geometry in the age of computer-supported collaborative learning. This would involve reformulating each of the practices of discourse, inscription, labeling, construction and proof. This would not be the first time that the presentation of geometry has been reinterpreted, but could be a decisive opportunity for rejuvenating it. Let us review an outline of how geometry has developed up to now.

The Origin of Geometry

Folk Geometry

Ever since people stopped wandering and settled down on patches of land, they have probably had ways to measure out the land ("geo-metric"), build structures in various shapes and conceive of various visual forms. Look at the intricate patterns woven into fabrics or carved into rocks, pottery and jewelry in pre-literate cultures. Here, the designed objects carried aesthetic and social values. They had not yet been quantified and made comparable based on a universal system of equivalences (see the literature of ethno-mathematics).

Throughout history, there have always been developments in practical mathematics, which interact with the pure or academic mathematics of professional mathematicians. The practical approach to geometry as techniques for dividing plots of land or calculating distances dominated textbooks in the Middle Ages, following the lead of Fibonacci's *Practica Geometriae* published in 1220. For instance, the practical navigational needs of ship captains in the era of global exploration, colonization and world trade drove the invention of complex algorithms, detailed numeric tables and computational instrumentation (Hutchins, 1996). As an
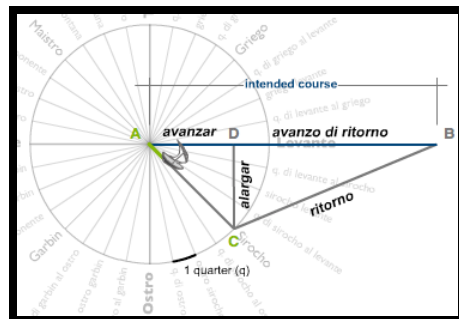
Figure 1. Course correction via the *marteloio* method. Reproduced from http://brunelleschi.imss.fi.it/michaelofrhodes/navigate_toolkit_basics.html.

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example, a 15th Century Venetian method for correcting a ship's bearings after being blown off course by the wind provided tables based on a drawing (see Figure 1) and trigonometric computations (Long, McGee & Stahl, 2009). This, in turn pushed the development of logarithms in mathematics and even the design of early computers (Gleick, 2011).

Formal, systematic geometry first emerged from common practice in the pre-Socratic days of Greece, from which few artifacts survive to tell the story. It developed the method of deductive reasoning and helped to transform the nature of literacy, science and human cognition (Husserl, 1936/1989; Netz, 1999).

The First Geometers

In the 5th and 4th centuries BCE, a small, distributed network of members of the Greek upper class developed a highly formalized version of geometry. Theirs was one of the first specialized applications of writing using an alphabet. They combined a formalized subset of written Greek with related line drawings. Significantly, the endpoints and intersections of the lines and arcs of the drawings were labeled with letters, which were used to reference them in the text. They created a genre combining text and diagram that spanned oral and literate worlds—incorporating the urge to persuade using words while pointing to objects—with the tools of the literate minority.

We barely know a few names of these early geometers; most surviving copies of their work are reproductions, translations or interpretations from hundreds of years later. Although their work was not particularly highly valued in the mainstream Greek culture, the "hobby" of doing geometry employed impressive intellectual skill. The tightly argued texts—circulated around the Mediterranean on parchment scrolls and clay tablets—were written in a minimalist style that was hard to follow. The newly invented discourse of proofs relied on an abstraction of geometric configurations to formal abstractions, such as that "a line is breathless length"—i.e., a line has no thickness or any other characteristics other than its measurable length. To follow the argument of a proof—let alone to formulate a new proof—one had to have memorized and understood an extensive corpus of previous definitions, postulates and propositions.

In order to effectively structure their proofs as self-contained and incontestable arguments, geometers had to reduce their subject matter to purely formal aspects, such as the length of lines. In addition, they laid out the proofs themselves in a clearly structured order, which made explicit the goal of the argument and the fact that the goal was achieved in the end. Each proof consisted of several discrete steps—sometimes as many as 40. The steps of a proof were restricted to formal relationships, such as that one line or angle was equal in measure to another. The argument uniting the steps to arrive at the stated goal unfolded through reliance upon a small set of transitive connections, such as that if A=B and B=C then A=C and A, B and C are all equal. These connections were accepted from the start as part of the geometry enterprise. The standardization of the minimal language of geometry made it clear that only these established connections were being used to make the deduction. Their transitive nature ensured that a proof that followed the conventionalized rules would be a valid, convincing deduction.

The history of mathematics can be viewed as an on-going process of defining math objects and rules in ways that produce elegant, consistent, rigorous proofs (Lakatos, 1976); Greek geometry is a prime example of this. The definitions of abstract points, lines and circles allowed one point to stand for any point and one line to be equivalent to any other, except for length and the points that it passed through. Also, the rules of deduction were simple and easily combined to build up more complicated deductions without introducing problems. As long as one restricted one's discourse to this small, carefully crafted, well-defined and orderly domain of geometric objects, a controlled vocabulary and transitive rules, one's proofs could be unassailable and universally persuasive.

The early Greek geometers proved propositions about geometric objects that go far beyond today's high-school geometry in insight and complexity. This would surely have been impossible without the use of diagrams. Even the simplest geometric arguments are difficult to follow without studying diagrams. The human mind is severely limited in its ability to handle long sequences of utterances and to keep track of

many inter-related objects within short-term memory. The diagrams allow people to take advantage of their powerful visual analytic skills. The lettered labels on the objects in the diagram provide deictic references to the objects intended by specific written phrases, effectively integrating the visual situation and the linguistic deduction. Through the coordination of formal proof discourse with labeled diagrams, the Greeks could prove and communicate rather involved propositions.

Plato's Academy

The cognitive importance of geometry was well recognized from the beginning. Plato (428 BCE to 348 BCE) certainly felt that the study of mathematics was good training for philosophy. Above the entrance to Plato's Academy was inscribed the phrase "Let none but geometers enter here." Plato's mentor, Socrates, is shown in one of Plato's early dialogs demonstrating a geometric proof to a servant named Meno. Plato's successor, Aristotle, made original contributions to geometry, as well as conceptualizing deductive logic.

While Plato did not engage directly in the practice of geometry in his surviving writings, there seems to be a complex interaction between his philosophy and the nature of Greek geometry. Latour (2008) argues that Plato wanted to use the deductive power of geometry to support his philosophic claims. Plato was in intellectual competition with the Sophists, who used rhetoric to convince their audience, and with the political leadership, who called upon established authority and the gods. Plato questioned authority, brought his audience to a sense of *aporia* (awe, based on puzzlement in the face of an impasse in the usual approach to a topic) and then tried to convince through logical argument, modeled to some extent on the new deductive style of the geometers. However, Latour claims that Plato could not succeed at adopting the geometry model because the success of geometry's deductive power flows from its formalism, its rejection of all content, whereas Plato needed to retain the content because he was interested in content-full topics like the Good, the True and the Beautiful. These topics are based on the richness of everyday language and cannot be reduced to well-defined meanings, relations of equivalence and limited language.

Perhaps Plato was pushed in the direction of his doctrine of Forms or Ideas by the model of geometry. If he could say, as he certainly did, that he was not talking about a specific just act, but about the concept of justice itself, which applies to all just acts without having any of the specifics of any one such act, then perhaps he could formalize his concepts so that his arguments about them would have the deductive power of geometry: the characteristic that they cannot be doubted and are self-evidently true. Unfortunately for Plato, he was determined to discuss broad, complex topics based on vague terms of everyday language, whereas the success of geometry relied upon radically restricting its discourse.

As Heidegger puts it, the philosophic experience that follows awe is intended to change one's view:

> In philosophy propositions never get firmed up into a proof. This is the case, not only because there are no top propositions from which others could be deduced, but because here what is "true" is not a "proposition" at all and also not simply that about which a proposition makes a statement. All "proof" presupposes that one who understands—as he comes, via representations, before the content of a proposition—remains unchanged as he enacts the interconnection of representations for the sake of proof. And only the "result" of the deduced proof can demand a changed way of representing, or rather a representing of what was unnoticed up until now. (Heidegger, 1938/1999, p.10)

Following a proof step by step involves the manipulations of formal components, representing things in terms of abstract symbols, illustrative diagram elements, standardized terminology and transitive comparisons. But a philosophic argument cannot reduce its topic to a representation of the topic, like a geometer can reduce a line to a labeled diagram of a line. Even the idea, form or concept of justice is not a representation of justice, but a rich understanding of what all just acts are about. Further, the point of a

philosophic argument is not simply to deduce a truth, but to persuade the audience about how they should live a good, just and beautiful life by gradually transforming their thinking.

Euclid's Elements

Not much is known of Euclid as a person. It is assumed that he lived c. 323 to 283 BCE. It is possible that Euclid studied in Plato's Academy in Athens and it is likely that mathematics was studied in the Academy. In one of the few surviving references to Euclid, it is noted that Apollonius (developer of the theory of conics and irrational numbers) "spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought." By "scientific" we can assume Apollonius primarily meant systematic.

It is not known if Euclid actually proved any new propositions or if he just compiled well-known proofs, working in the great library of Alexandria, an early gathering place of the world's knowledge. There had been some previous attempts to compile the propositions of geometry, but none were considered of comparable power to Euclid's. Euclid published 13 volumes of geometry, in which the propositions were not only organized based on their subject matter, but built on each other systematically. They all followed a similar template from statement of goal to declaration of conclusion and they were apparently all accompanied by clear, labeled diagrams corresponding to the steps of the proof. Although the inter-relationships among the propositions were implicit in their individual original proofs, it must have taken a deep understanding and overview to put together all the propositions so systematically and to preface them with a clear statement of the assumed definitions, postulates and common notions. Euclid's presentation of geometry has stood up to scrutiny for 23 centuries and has inspired and influenced scientific and mathematical thought in the Western world more than any other text.

Roman and English Translations

Unfortunately, we have no extant copies of the *Elements* as written by Euclid. We must rely on translations of translations and copies of copies of those. Each translation is necessarily an interpretation and many copyists try to "improve" the presentation. The standard English version now is a recent republication (Euclid, 300 BCE/2002) of Heath's 1908 translation of Heiberg's 1883 scholarly Greek version. The earliest printed Greek version is from 1533, predated by a printed Latin version from 1482.

Each edition made different changes: eliminating whole sections from each proof to avoid redundancies, adding clarifying phrases, etc. For instance, the introductory list of definitions, postulates and common notions was not originally broken down into a numbered list (as it is now), or even into separate sentences. It provided a general introduction to the terminology, rather than a set of axioms that could be referenced in the proof (as they are now). In addition, translations—most significantly from the original Greek to the Roman way of thinking, which strongly influenced Western scientific thinking when Latin was the lingua franca—transformed syntax and changed tenses and ways of referencing. It is particularly unfortunate that we have no exemplars of ancient diagrams, only medieval and modern versions.

Organization and systematization seem to be inherent in the practice of geometry. The notion of rigor in proof entails meticulous step-by-step procedures, precisely formulated and carefully built upon one another. In classical education, training in geometry was considered a means of disciplining unruly minds. And the birth of geometry may have contributed significantly to the rationalization of the Western mentality. However, the historical development of geometry following its birth further refined its systematic nature. Where Socrates was a free spirit intellectually and Plato sought after the essences, their follower, Aristotle was more of a systematizer, initiating a tendency that led to the great system builders in philosophy and the hierarchical thinking of the Neo-Platonist church, which dominated the medieval mentality. The library of Alexandria, where Euclid presumably assembled the *Elements* of geometry, was an historic effort to compile and categorize written knowledge. Such efforts were part of the strivings of secular and religious state leadership (like Alexander the Great) to establish, manage and control

increasingly large and complex civilizations. The formalization, systematization and bureaucratization of knowledge paralleled that of the military, politics, faith and the economy. Geometry provided a model, and it was itself in turn transformed in that direction.

Pappus of Alexandria (340, Book VII) was an organizer of mathematical knowledge, like Euclid but 600 years later. Perhaps the last major classical Greek mathematician, Pappus drew the important distinction between analysis and synthesis. Analysis is Euclid's method. It starts from "what is sought as if it were existent and true" and works back to the given conditions and previous propositions. It then reverses the sequence to present a deductive proof derivation. Synthesis is a form of exploration that begins from the given conditions and previous propositions and investigates their implications. As (Livingston, 1999) argues, the process of proving is a winding synthetic discovery process, later disguised in a linear analytic presentation. The nature of work in dynamic geometry is more naturally a synthetic approach as contrasted with a classical Euclidean paradigm.

Axiomatic Geometry

People—including many mathematicians—tend to think of mathematical objects as some kind of other-worldly abstractions, as mental constructs that have no physical characteristics but obey logical rules (axioms and their corollaries). This view may be indirectly derived from Plato's doctrine of Ideas as a realm of essences divorced from the physical world—a view furthered in philosophy by Descartes and perhaps motivating the formal axiomatization of mathematics. As mentioned previously, Plato may have been influenced by early geometry; now the influence is fed back. One consequence is that the geometric diagram is now viewed as a rather arbitrary and secondary illustration of the abstract ideas discussed in a proof. This may be an unfortunate distortion of the central role of the diagram in the work of the first geometers. It may also obscure the important role of diagrams in the learning, exploration and understanding of geometry in schools today (Livingston, 1999).

By the twentieth century, mathematics was viewed as an axiomatic system. The vision of Leibniz was worked out by Frege and other logicians, culminating in Russell and Whitehead's detailed system. Although Gödel's and Turing's work established surprising limits to this vision, the influence on geometry was significant. Euclid's proofs are now read as axiomatic procedures. Over the centuries, the prevailing paradigms of hierarchy, logic and axiomatization have ineluctably continued the interpretive transformations of Euclid's texts.

The historical development of *reason*, in which geometry has played a key role, can be considered from many perspectives. In terms of individual personal development, Piaget (1990) identified the child's transition from concrete to abstract thinking as pivotal. The educational role of geometry (and algebra) has always been seen as an important means for the training of abstract thinking. On a societal level, the movement from orality to literacy (Ong, 1998) can be seen as a primary watershed in human cognition. As discussed, the origin of geometry was an integral part of the emergence of literacy, including practices in visual representation, mathematics and deduction.

The rise of rationality has brought problems as well as progress. Philosophic analyses as different as those of Adorno and Horkheimer (1945) and Heidegger (1979) trace the origin of totalitarian fascism in the Second World War all the way back to the early Greeks. The tendency to reduce the richness of nature and interpersonal living to quantifiable representations is not only empowering, but also distorting of healthy human relationships. This historic tendency includes the emphasis on quantification and calculation in the rise of capitalism and bourgeois organizational management, rational planning and the exploitation of nature or human labor as disposable resources (Swetz, 1987). The emergence and development of geometry has been an integral element of the historical development of rational reason— although it has not often been analyzed in this context. The view dominating contemporary thought—for instance in cognitive science and artificial intelligence—has been attributed by Hutchins (1996, p.370) to

"a nearly religious belief in the Platonic status of mathematics and formal systems as eternal verities rather than as historical products of human activity."

Dynamic Geometry

In recent decades, the teaching of geometry in public schools has moved away from the presentation of proofs in bureaucratized Euclidean style, in an attempt to make the basic concepts of the field more accessible. But the underlying mathematics has changed very little. The development of dynamic geometry, by contrast, introduced a rather different geometric paradigm. The significance of the difference was not entirely intentional and has not yet been well described or documented.

Dynamic geometry emerged from the potential of personal computers to provide interactive diagramming tools with embedded computational support. The core technology actually considerably predates personal computers with Sutherland's (1963) SketchPad software, which provided a graphical user interface with an object-oriented draw program before there were GUIs, object-oriented programming or draw programs. Video games developed the technology further—and actually largely drove the personal computer market from its start. The next section describes the innovation of dynamic geometry.

From Euclidean Geometry to Dynamic Geometry

In the late 1980s, Nicholas Jackiw, the designer and programmer of Geometer's Sketchpad, began working with Eugene Klotz on one of the first instances of a dynamic-geometry program at the Visual Geometry Project, a forerunner of the Math Forum (Scher, 2000). At about the same time, Jean-Marie Laborde began Cabri, in France. The developers of Geometer's Sketchpad and Cabri shared ideas in the mid 1990s. In 2002, Markus Hohenwarter launched GeoGebra as an open-source dynamic-mathematics environment. These programs have subsequently become popular around the world. Although each of these programs has subtle differences in their geometric-construction paradigms and somewhat different functionality, they are fundamentally similar in their affordances for students of geometry. They make geometry dynamic by allowing a person using the system to construct a geometric diagram with labels and then to move the interconnected geometric objects by dragging their points around. As objects are moved, they maintain dependencies that were part of the construction process. This should be clear in the following example.

An Example of Dynamic Geometry Construction

In Figure 2, we see a construction in which equilateral triangle DEF has been inscribed in equilateral triangle ABC[1]. Figure 3 shows the same construction after point D has been dragged upwards. A user can move Point D by placing the cursor on point D and dragging the point in the construction. However, the movement of point D is constrained by the construction to always remain on line segment AC and to not go past its endpoints. This is characteristic of dynamic geometry.

Notice that in addition to point D moving, points E and F also moved when point D was moved. The line segments connecting these points and forming triangle DEF have moved with their endpoints, effectively rotating triangle DEF. This is because of how the inscribed triangles were constructed. They were constructed in a special way in order to preserve the equilateral characteristic of triangle DEF. The larger triangle ABC can also be rotated by dragging one of its vertex points, such as point A. No matter how any of the points in the construction are dragged, the other points will move in ways that maintain the equilateral character and inscribed relationship of the triangles.

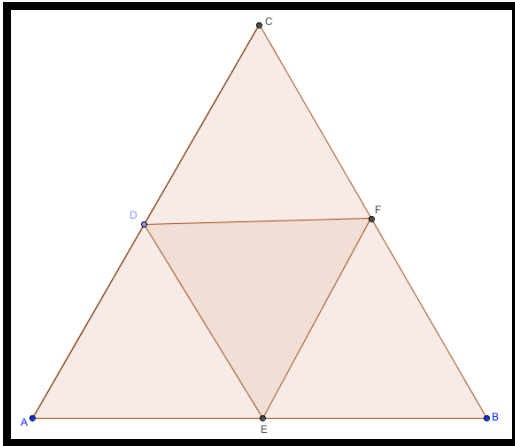[1] This construction was suggested by (Oner, 2013).
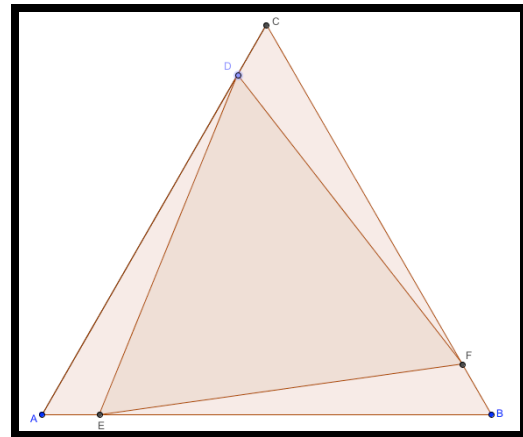
Figure 2. Inscribed equilateral triangles.

Figure 3. Inscribed equilateral triangles after dragging point D.

By studying Figures 2 and 3, it may be possible to figure out how to construct the triangles so that they will maintain their equilateral character dynamically. First, we can construct triangle ABC to be equilateral by following Euclid's first proposition. Starting from an arbitrary line segment AB, we construct a circle centered on point A and going through point B. Then we construct a second circle centered on point B and going through point A. These two circles intersect above and below AB and we mark one of the intersections as point C (see Figure 4). We then construct triangle ABC, connecting the points. We know that triangle ABC is equilateral because (as Euclid argued) its three sides are equal in length to line segment AB because they are radii of the same circles. If one subsequently drags point A or B, changing the length of AB, then the circles with radius AB will both change, moving point C in precisely the right way to keep ABC always equilateral. We can say that the position of point C is "dependent" upon the length of AB, and consequently triangle ABC is defined by this dependency. Constructing dependencies is fundamental to dynamic geometry. As in the example we just went through, these dependencies are *implicit* in Euclidean geometry, but become *visible* in the construction and manipulation of dynamic geometry.
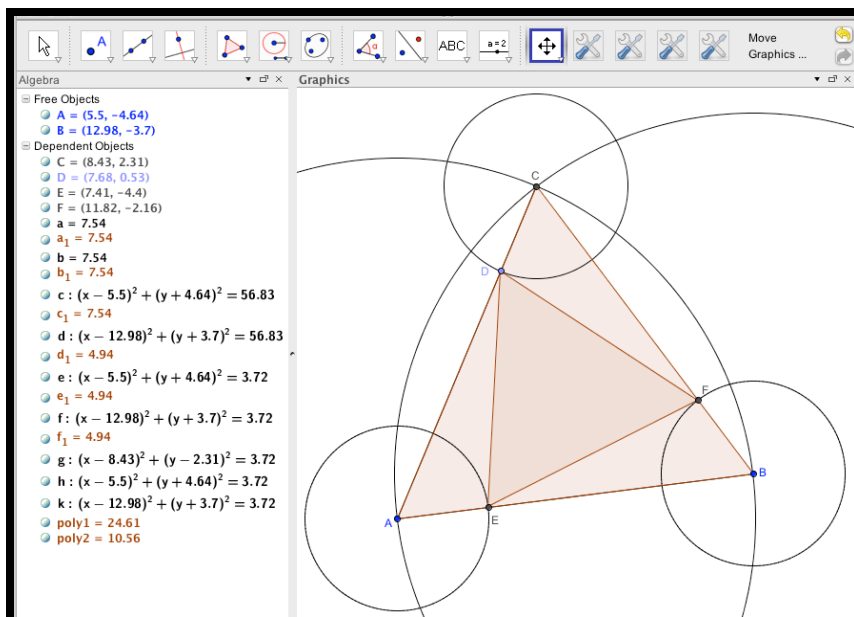

Figure 4. Constructing the dependencies for inscribed equilateral triangles

Figure 4. Constructing the dependencies for inscribed equilateral

Figure 4. Constructing the dependencies for inscribed equilateral

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Having constructed a dynamically equilateral triangle ABC, how do we construct an inscribed dynamically equilateral triangle DEF? We can place points D, E and F on the three sides of ABC, but they will not be constrained to stay at equal distances from each other. If we try to use Euclid's Proposition 1 again we run into problems. Say we construct line segment DE and then construct circles of radius DE around D and E. The intersection will not fall along line BC. Even if it did happen to fall there, we could not locate point F at the intersection of three lines because that would be over-constrained.

We need a different approach. By dragging the triangles in Figure 2, we might notice that the distance of the vertices of the smaller triangle are always at equal distances from the corresponding vertices of the larger triangle. In other words (or symbols): AE=BF=CD (see Figure 4). In fact, if you specify that these three line segments are equal, it is easy to prove by Euclidean methods that the three triangles formed between the two equilateral triangles are all congruent. This ensures that the sides of the inner triangle are equal if the sides of the outer triangle are equal. Thus, if we can impose the constraint that AE=BF=CD, then we can construct a dynamically equilateral triangle DEF inscribed inside of an equilateral triangle ABC.

Figure 4 shows how the inscribed triangles were constructed within GeoGebra. Point D was placed on line AC. A circle was constructed with GeoGebra's compass tool, with center at C and going through D. The circle was moved to be centered on point A and also moved to be centered on point B. Points E and F were constructed where these circles intersected the sides of triangle ABC, establishing the dependency in the construction that AE=BF=CD. Triangle DEF was then constructed as a dynamically equilateral triangle.

The point is how visualization (drawing) and conceptualization (proof) are so intermingled in this example. By dragging the construction, you discover how to construct it—and you can then prove why that works. GeoGebra provides tools to explore, to construct and to impose dependencies on the construction.

Defining Custom Tools

The toolkit of GeoGebra reflects some of the refinements (reinterpretations) of Euclidean geometry in recent math pedagogy. For instance, it distinguishes as different kinds of objects: "lines" (infinite straight lines passing through two defining points), "segments" (finite line segments terminated at the two endpoints that define them) and "rays" (infinite lines starting at one endpoint and passing through a second defining point). Euclid called these all "straight lines." GeoGebra also provides a "compass" tool (used in Figure 4). This is based on the theory that Euclid used a straightedge and compass for his drawing and that one can fix the opening of a compass and draw circles with the same radius by locating the compass at different centers. However, Euclid does not do this in his proofs. After showing how to make equal-length line segments using a circle to construct an equilateral triangle in his first proposition, Euclid dedicates his second proposition to demonstrating how to copy a length from one line to another given point. Figure 5 (left) shows how to do this using GeoGebra, following Euclid's procedure.

In Figure 5, the length of segment AB is copied to ray CD using the dozen steps of Euclid's Proposition 2 (detailed below). Then a custom tool is created to automate this process, much as the compass tool does. Also, a custom tool is created to automate the construction of an equilateral triangle (Figure 5 upper right). The construction in figure 4 is then recreated using the new custom tools: the base of the larger triangle is defined by two points, L and M, to which the new custom triangle tool adds point N. An arbitrary point O is next provided on LN as a vertex of the inner triangle. Using the new custom copy tool, the length of NO is copied onto LM, defining P, and the length of LP is copied onto MN, defining Q (Figure 5 lower right). Triangle OPQ, inscribed in LMN, is equilateral and can be dragged without losing its equilateral character.
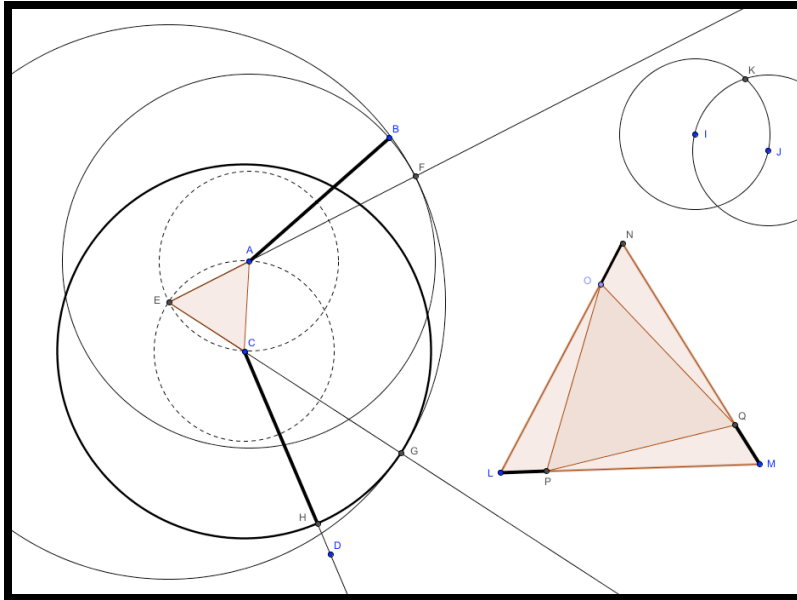
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Figure 5. Copying a length, AB, from one line to another, CH on CD. (left)

Identifying the third vertex for an equilateral triangle IJK. (upper right)

Duplicating Figure 2 with the use of custom tools. (lower right)

In case you are interested in the details, here is how segment AB was originally copied onto ray CD, using the procedure in Euclid's second proposition plus an extra circle to align the length along the ray:

> The goal is to place at a given point along a given line a straight line equal to a given straight line.
> Let C be the given point on ray CD, and AB the given straight line (Figure 5 left). Thus it is required to place on CD starting at C a segment equal to segment AB. Let the equilateral triangle ACE be constructed on AC (using the construction procedure of Proposition 1). Let ray EA and ray EC be produced, extending out from the triangle. Let a circle centered on A and through B be produced, with point F at the intersection with ray EA. And again, let a circle centered on E and through F be produced, with point G at the intersection with ray EC. As Proposition 2 argues, EF=EG and EA=EC, so CG=AF; but AB=AF, so CG=AB and the length AB has been copied to point C. Now let a circle be produced with center C going through G and let point H be at the intersection of this circle and ray CD.
> Then, CH=CG, so also CH=AB and the length AB had been copied to segment CH along ray CD.

This illustrates how the tools of construction in dynamic geometry are intimately related to the procedures in Euclid's proofs. Once a valid construction procedure has been proven, one can define a tool to encompass that procedure, such as GeoGebra's compass tool for copying a line length in accordance with Proposition 2. Gradually, one can expand the construction toolkit with new custom tools—paralleling the way propositions build on one another systematically in Euclid's *Elements*. GeoGebra provides a toolkit of dozens of tools, which can be derived from straightedge and compass constructions in accordance with Euclid's propositions. Users can define their own versions of these or build further upon them.

Dynamic geometry differs from previous presentations of geometry in at least three significant features: dynamic dragging, dynamic construction and dynamic dependencies.

Dynamic Dragging

The ability to drag points is the most striking characteristic of dynamic geometry. Most research on dynamic geometry has focused on this feature. Most classroom usage of dynamic geometry also centers

on this feature, providing students with dynamic diagrams and encouraging them to explore the diagrams by dragging points.

Previous media for diagrams have not allowed one to vary the figures except in imagination. Papyrus, clay tablets, parchment, books, pencil on paper or chalk on blackboard were not interactive media. The most one could do was to stare at the fixed diagram and imagine moving points or lines to vary the configuration. This meant that one rough graphical representation might have to illustrate an infinity of possible variations. For instance a proof concerning equilateral triangles might apply to equilateral triangles of all sizes, rotated at all possible angles, while the diagram had a fixed size and inclination.

In the preceding example (Figure 2), for instance, it would have been hard to know important features of the diagram without being able to drag the vertices of the two triangles. Through dragging point A, one can easily and naturally discover that the two triangles remain equilateral and inscribed as the size and orientation of the larger triangle is varied over arbitrary ranges. Through dragging point D, one discovers as well that the inner triangle can remain equilateral and inscribed with point D anywhere along AC, including at the endpoints. One may also notice that the area of triangle DEF varies continuously from a minimum when point D is centered on AC to an area equal to that of ABC when point D is at an endpoint of AC. Significantly for the example, one may notice while dragging point D that AE=BF=CD remains true.

With fixed diagrams, it took a certain "professional vision" (Goodwin, 1994) of mathematicians to see important mathematical relationships in diagrams. Certain features of geometric configurations are visible even in fixed diagrams. For instance, the fact that in the construction of ABC in Figure 4 the circles centered on A and B of radius AB actually do intersect (at some point C) is visible; whether this remains true for any configuration of A and B may be ascertained by staring at the diagram and imagining different locations for A and B. Dragging makes this easier to see—providing a way to train students to see like mathematicians.

Another traditional skill of mathematicians is to design a diagram to be effective for a given proof. For instance, Figure 2 illustrates a proposition: that an equilateral triangle can be inscribed in another equilateral triangle. However, it illustrates a special case, which might not generalize: the base AB is roughly horizontal and the point D is roughly at the mid point of AC. Figure 3 may represent the general case better. It also makes more salient the fact that AE=BF=CD. The ability to drag offsets the need for the traditional skill. It makes it easy for students to drag a diagram to explore what special cases exist and what relationships seem to persist in general as specifics change. It is no longer crucial to select a "representative" case since an arbitrary view can be dragged through whole ranges of possible variations.

Of course, it is possible that students will not drag a construction into every possible case or even that a given dynamic construction cannot be dragged into every case covered by a specific proposition. But dragging can provide the mediated experience of apprenticeship in geometry that can lead to the ability to conduct what Husserl (1929/1960) called "eidetic variation" in one's imagination to reveal constants under change.

Dragging also gives students a hands-on, visceral sense of the constraints and characteristics of a geometric diagram. It enhances the bodily involvement of the interaction between person and diagram in which "creative discovery" can take place (Merleau-Ponty, 1955). Perhaps this will be even further heightened when tablet computers fill the role that clay tablets originally played. Exploration through dragging is intertwined with the possibilities of construction. As a student learns to initiate various kinds of constructions, she starts to see new possibilities for dragging, for seeing constraints and patterns and possibilities. As her body is extended by the computer interface into the digital world, she gains a sense of how to move within that world, to live and perceive in a dynamic-geometry world (Merleau-Ponty, 1961/1964). Then, as the student starts to look at the dynamic environment through the eyes of a designer of structural dependencies, she can see constraining relationships at work, as well as interesting potential

transformations of them. Such embedded, skilled vision produces targets or hunches (informal conjectures) to explore through purposeful dragging.

Dynamic Construction

Previous media for diagrams not only limited variation, they required the diagrams to be completely constructed prior to the construction or presentation of the proof. Thus, where Euclid's Proposition 2 begins in the English translation, "Let A be the given point" (Euclid, 300 BCE/2002, p.3), this is actually stated in the original Greek in the perfect imperative (as already having been done): "Let the point A have been taken" (Netz, 1999, p. 25). The proof then proceeds to point to the existing diagram and describe the relationships within it. The text often relies on the pre-existence of the diagram for its sense. As an example, after Euclid specifies line AB in Proposition 1, he says, "With centre A and distance AB let the circle BCD be described" (Euclid, 300 BCE/2002, p.3). Point B has been defined as an endpoint of line AB and point D is simply an unspecified point anywhere on the new circle. But point C cannot be defined in the text until another circle is described, which intersects the first circle at point C, thereby specifying point C. It is only because the whole diagram already exists and includes point C that the text of the proof can use the label C as part of the designation of the first circle.

As Netz (1999) documents with numerous examples, Euclid's texts often rely upon the pre-existing diagrams for their sense. On the other hand, the diagrams rely on the texts for their interpretation. Text and diagram are mutually determinative, with the labels of points relating the two. Dynamic geometry overcomes the necessity of completing one before the other. The diagram can be constructed in parallel with the unfolding of the textual argument—at least in a live presentation. (In this printed document, I am limited to placing screenshots interspersed in my text.)

Livingston (1999) argues that there is a significant difference between how a conjecture is explored and how its proof is presented. One must first discover an interesting relationship and then piece together an argument. This usually proceeds through exploration, with its trials, deadends and backtracking. The final presentation is then orchestrated as a logical deduction, straight from givens to conclusion with the minimum necessary steps. The conclusion is presented as though it necessarily always existed—rather like the refined diagram that pre-existed the proof.

The past perfect tense has always characterized mathematics. Even as new objects were created in history—conics, irrational numbers, logarithms, infinitesimals, imaginary numbers, hyper-spheres—they were always taken as having always already existed. They were not treated as newly *created* human artifacts, designed for their interesting properties, but as *discovered* ideal objects in an otherworldly realm of mathematical objects (Lakoff & Núñez, 2000). Whether or not this view motivated Plato's theory of Forms, subsequent mathematics generally adopted a Neo-Platonic attitude, obscuring the important role of exploration and invention in the practice of mathematics.

Dynamic geometry can reverse this obfuscation. Students can now construct diagrams as an integral part of their exploration of geometric relationships. Using the construction tools of dynamic geometry, students can explore mathematical conjectures through trial constructions. The bureaucratic format that Euclidean proofs have evolved into can be replaced with active exploration, which does not assume the diagram is complete beforehand, treats mathematical objects as human artifacts designed to have interesting features, leads to moments of *aporia* and breakthroughs of insight as deduction unfolds as a creative form of discovery.

The ability to construct dynamic diagrams that present intriguing puzzles—which support exploring conjectures and which illustrate proofs—is itself a subtle skill, a skill that must be learned through instruction, apprenticeship and practice. This skill can be developed as part of the process of learning geometry content. For instance the geometric objects like points, lines, circles and triangles are also graphical objects in dynamic geometry environments; students can learn the characteristics of the objects by constructing and dragging the graphics. In fact, much of Euclid's *Elements* can be read as instruction

in construction: Proposition 1, how to construct an equilateral triangle; Proposition 2, how to copy a line segment of a given length to another position; Proposition 3, to measure off the length of a shorter segment along a longer one; Proposition 9, to bisect an angle; Proposition 10, to bisect a line segment; Proposition 11, to construct a perpendicular to a line at a point on it; Proposition 12, to construct a perpendicular to a line from a point not on it; etc. The art of construction has always been central to geometry, although it has not always been stressed as a creative skill.

Dynamic Dependencies

The key to constructing for exploration is the construction of dependencies. This is another potential implicit in Euclid, but not adequately recognized, acknowledged or researched.

As we have seen, the construction in Proposition 1 demonstrates how to build in the dependency that defines an equilateral triangle: that its three sides must be of equal length. Given an initial side AB, Euclid adds circles centered on points A and B, each of radius AB. He then labels an intersection of the two circles as point C, the third vertex of equilateral triangle ABC. The lengths of the new sides AC and BC are dependent upon the length of side AB because they are radii of circles of which AB is a radius—and all radii of a given circle are the same length by definition of a circle. In a fixed drawing, we might just say that AC and BC have been constructed to be the same numeric length as AB. But in a dynamic construction, one can drag point A or point B and change the length of AB. If the dependency has been properly constructed, then point C will move in response to the movement of point A or point B precisely the right way to maintain the equality of all three sides.

The dynamic diagram in Figure 4 was constructed in such a way that it remained a diagram of two inscribed equilateral triangles no matter how any of its points were dragged. The dependencies included that points D, E and F remain on segments AC, AB and BC, respectively, as well as that AE=BF=CD. The defined dependencies ensure that the two triangles remain inscribed and both equilateral under any change in size or rotation of either triangle or the dragging of any point.

It is possible to have a computer whiteboard similar to Sutherland's (1963) original SketchPad in which lines and points can be drawn as movable objects. One can draw an equilateral triangle on such a whiteboard by placing three lines of equal length meeting at their endpoints. However, if one then drags a vertex, the triangle either falls apart entirely, loses its equilateral characteristic or fails to re-size.

A dynamic geometry environment must implement computational mechanisms behind the scenes to both maintain the desired dependencies and at the same time allow permitted manipulations. In fact, the first thing the environment does is to keep track of which objects are independent and can be dragged freely (like A and B) and which are dependent and can only be dragged in limited ways (like D) or cannot be dragged directly at all but just move in response to other points on which they are dependent (like C, E and F). It requires very special software to support dynamic geometry. The software we have today—such as Geometer's Sketchpad, Cabri or GeoGebra—was carefully designed to maintain arbitrarily complex geometric dependencies while making the user experience seem extremely natural. This is the hidden power of dynamic geometry. Once one gets used to the paradigm of dynamic geometry and thinks in terms of constructing dependencies, everything works the way that one would expect it to. The user does not have to worry about the secret software mechanisms.

Dependencies lie at the heart of Euclid's geometry, but they have been largely buried in the traditional understanding of geometry. This is a philosophic issue. Heidegger would say that the Being of the geometric objects was concealed through the Greek and then the Roman and then the English way of caring for and speaking about the objects and their dependencies.

The traditional understanding of geometry that has been passed down from the Greeks through its subsequent translations, reinterpretations and refinements over the centuries confuses the causality of dependency and proof. Consider the diagram of inscribed equilateral triangles (Figure 2). One could start with equilateral triangle ABC in the completed diagram and specify that AE=BF=CD. Then one could

prove that triangle DEF must be equilateral through a logical deduction, perhaps including an argument about the three small triangles outside of DEF being congruent. This would establish the truth of the equilateral nature of DEF. That is the traditional perspective. Euclid's proofs are commonly conceived of as such discoveries of existing truths in the realm of Platonic ideas. Euclid's *Elements* are read as building up a system for proving these truths.

But this reverses the causality. For, if we have constructed DEF by using the constraint that AE=BF=CD, then we already know that we imposed this constraint in order to construct an inscribed triangle that would be equilateral. It is not a matter of discovering some mysterious otherworldly truth; it is a matter of having intentionally built in the character of equal side lengths into our construction of DEF. It is not a matter of a formal logical deduction, which unfolds with necessary truth. The drawing of equal circles at A, B and C is not originally a means for proving that the sides of an equilateral triangle have always already been equal; the circles are part of the construction of the dependency that itself ensures that the sides will be and will remain equal. The truth of the proof was built in by the construction. It was hard to see this the way that geometry has been conceptualized throughout history, but easy to see in dynamic geometry if one focuses on the construction of the dependencies as an active, creative, inquiry process. We built in the dependency as we constructed the diagram: that is why the triangles are equilateral! Taking the diagram as already given before the proof is presented, combined with the traditional assumptions about the nature of geometric objects as divorced from human activity, hides what has transpired. Experience with dynamic geometry—including dragging, constructing and designing dependencies—exposes the *creation* of objects, diagrams and relationships by people.

Shockingly, the mathematics and education literature on dynamic math has not discussed this central role of designed dependencies—and of the ability to construct dependencies in dynamic geometry. Although dependencies lie at the heart of proof and although dynamic geometry software is explicitly built on the maintenance of dependencies, I have not found a single publication discussing the role of dependencies in dynamic geometry. More generally, I have not found any in-depth discussion of the relationship of dependencies to proof. This is a symptom of the extent to which the hidden nature of geometry has been obscured.

In a typical geometry proof, recognition of the central underlying dependency is the key potential insight into why the proof works—the door to the "aha moment." The diagram illustrates some relationship not because of a mysterious otherworldly truth, but because the diagram was constructed with dependencies that built in that relationship. If students learn to think in terms of dependencies and to construct diagrams around dependencies and to search for dependencies, then geometry might be a lot more exciting and meaningful and the students might consider themselves more successful as mathematical thinkers.

While most classroom use of dynamic geometry today merely uses it as a visualization tool, to allow students to drag existing diagrams around, the technology has a greater power: to empower students to construct their own diagrams, to build their own dependencies into the objects and even to fashion their own dynamic construction tools. Then they can read Euclid's *Elements* as a guide to designing and constructing interesting objects and tools, rather than as an old-fashioned compendium of irrelevant truths to be memorized. Geometry can become an exciting design challenge, in which one creates innovative mathematical objects and imposes interesting dependencies through thoughtful construction.

The Case of the Mysterious Incenter of a Triangle

Perhaps it is clear in the case of Euclid's first proposition that one is imposing constraints through the construction that account for the result of the sides being equal length. But aren't there some geometrical relationships that just inhere to certain shapes and are not built into figures by the dependencies of our constructions? What about the surprising fact that the three bisectors of the angles of any triangle all meet at one point? It does not seem like we have built this property in through some construction constraints—
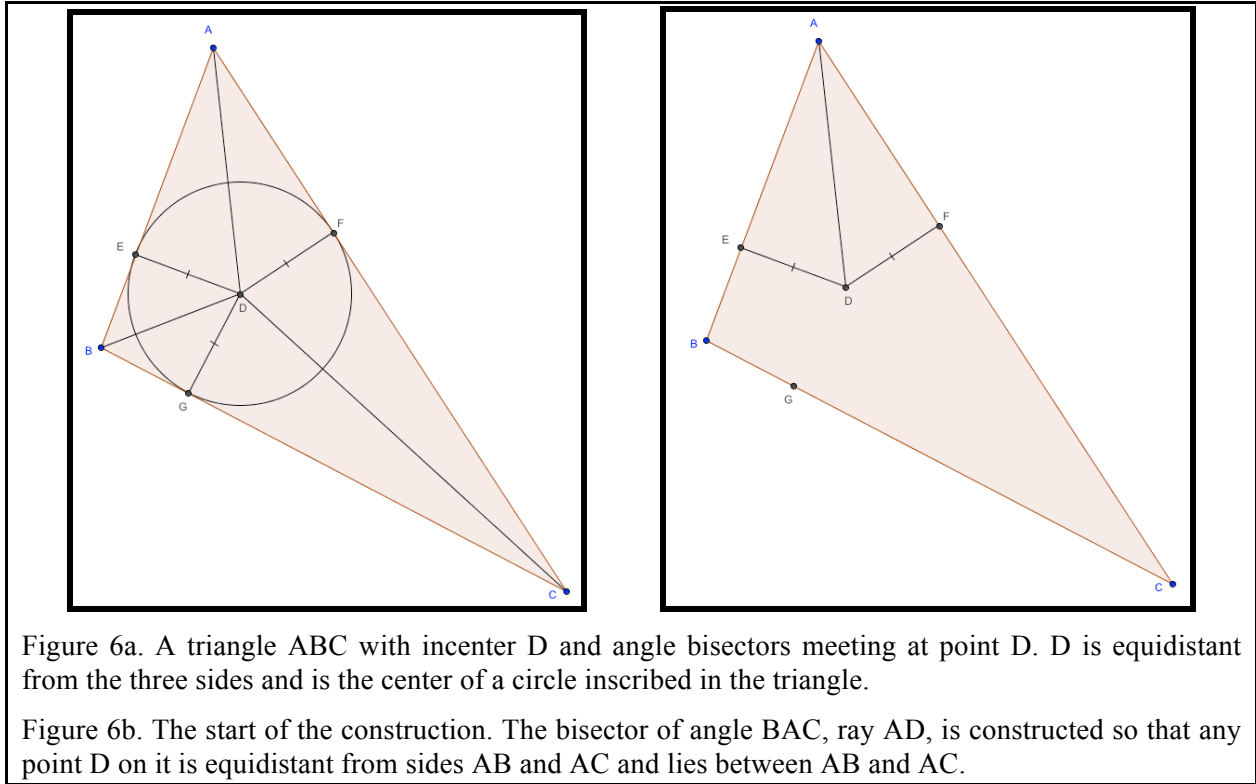
Figure 6a. A triangle ABC with incenter D and angle bisectors meeting at point D. D is equidistant from the three sides and is the center of a circle inscribed in the triangle.

Figure 6b. The start of the construction. The bisector of angle BAC, ray AD, is constructed so that any point D on it is equidistant from sides AB and AC and lies between AB and AC.

it is simply a property of any plain triangle (see Figure 6a). Furthermore, the point of concurrency of the angle bisectors—named the "incenter" of the triangle—happens to be exactly equidistant from the three sides of the triangle. It turns out interestingly that the incenter is always inside the triangle, for any kind of triangle (unlike some other special points of triangles). Moreover, if one constructs a circle inscribed in the triangle, it will happen that the center of the triangle is precisely at the incenter. These all seem to be mysterious properties of the ideal geometric object, triangle; it is assumed that they must be deductively proven from axioms and other propositions to convince us of the generality of these relationships, their Platonic truth.

Let us investigate—using dynamic geometry—the standard belief that the relationships associated with the incenter are inherent characteristics of triangles that are not imposed by constraints designed into the construction, but are properties of triangles to be discovered, whose validity is to be deductively proven. Rather than starting from the completed figure, let us instead proceed through the construction step by step. As our first step, we construct one of the angle bisectors (see Figure 6b). We actually construct the angle bisector by constructing a ray AD that goes from point A through some point D that lies between sides AB and AC and is equidistant from these sides. It seems quite natural that if one point D is equidistant from the two sides of the angle then every point on ray AD will similarly be equidistant from the two sides—as can formally be shown using similar right triangles. It also seems quite natural that if every point of ray AD is equidistant from the sides then AD is the bisector of angle BAC—as can be formally shown by demonstrating that triangles ADE and ADF are congruent. (DE=DF by construction; side AD is shared; by Pythagoras' Theorem, the third sides must be equal; so the triangles are congruent by SSS and therefore angle DAE and angle DAF are equal, so they bisect angle EAF.) Note that we have constructed the dependency of ray AD on the sides AB and AC with the constraint that DE=DF.

The second step is to construct the bisector of angle ABC. We construct a ray BH that goes from point B through some point H that lies between sides AB and AC and is equidistant from these sides. Ray BH will intersect ray AD at some point inside the triangle because both rays proceed between their two sides until

they pass through the side opposite their angle. We will now look at that point of intersection as point D; everything we said about our construction of point D for ray AD and BD will still hold for this specific point D. Also, everything we said about our construction of point H will hold for this specific point D on ray BH. Notice that the construction of ray BD has imposed the dependency that DE=DG. Combining this with the dependency that DE=DF from the construction of ray AD, we have DE=DF=DG.

The third step is to construct ray CD from the vertex C to the intersection of AD and BD at point D. We have imposed the dependencies that DE=DF=DG, so we know that DF=DG and therefore the triangles CDF and CDG are congruent. This implies that ray CD is in fact the bisector of angle ACB. Therefore, the fact that the bisectors of the three angles of a triangle are all concurrent is not a mysterious surprise, but a direct consequence of the dependencies we imposed when constructing the bisectors.

The other properties associated with the incenter follow naturally from this construction. We have seen that DE=DF=DG, which means that a circle centered on point D with a radius equal to the length of DE will pass through points E, F and G—with line segments DE, DF and DG being radii of the circle. These radii are perpendicular to their respective triangle sides, thanks to our construction of them, making the circle tangent to the triangle sides at points E, F and G—so that the circle is inscribed in the triangle.

The incenter of a triangle is not some mysterious property of the triangle, to be discovered by deductive proof from a given figure like Figure 6a, but a consequence of the dependencies constructed into the figure, such as the constraints imposed by constructing the bisector of the angle in Figure 6b.

If we had "constructed" ray AD as the bisector of angle A in GeoGebra by simply using the built-in angle-bisector tool, we would not have noticed that we were thereby imposing the constraint that DE=DF. It was only by going back to first principles, such as the angle-bisection construction procedure in Euclid's 9th proposition that we can see this. The packaging of the detailed construction process in a new tool obscured the imposition of dependencies. This is the useful process of "abstraction" in mathematics: While it allows one to build quickly upon past accomplishments (like Euclid's 9th proposition), it has the consequence of hiding what is taking place in terms of imposing dependencies. This is the "dialectic of enlightenment" (Adorno & Horkheimer, 1945): progress in rational thinking brings with it the danger that important phenomena become obscured, misunderstood, forgotten, repressed. Rationalization of society can lead to fascism, totalitarianism or mindless bureaucracy. In mathematics, it can lead to deadening memorization in place of insight—in the name of efficient training.

While dragging figures that have already been constructed and even constructing with a large palette of construction tools can be extremely helpful to students for exploring geometric relationships and coming up with conjectures to investigate, such an approach can give the misimpression that the relationships are abstract truths to be accepted on authority and validated through routinized deduction. It is also important for students—at least for those students who want a deeper understanding of what is going on—to be able to construct figures for themselves, using the basic tools of straightedge (line) and compass (circle). They should understand how other tools are built up from the elementary construction methods and should know how to create their own custom tools, for which they understand the incorporated procedures.

Of course, simply constructing figures is not enough. One must be able to reflect upon what is being accomplished in the construction and what one is trying to accomplish—and that involves discourse. Within a social setting of collaboration, students will want to share their ideas, questions, conjectures and discoveries with their friends, generating occasions for geometric discourse and collaborative learning. They can exchange constructions and custom tools, which incorporate and preserve their creative insights. In a multi-user environment, small groups of students can explore dynamic drawings together and discuss the construction process as they work on it.

Geometric Discourse

The educational research field of computer-supported collaborative learning (CSCL) arose in the late 1990s to explore the opportunities for collaborative learning introduced by the growing access to networked computational devices, like laptops linked to the Internet (Stahl, Koschmann & Suthers, 2006). The seminal theory influencing CSCL was the cognitive psychology of Vygotsky (1930/1978). He had argued several decades earlier that most cognitive skills of humans originated in collaborative-learning episodes within small groups, such as in the family, mentoring relationships, apprenticeships or interactions with peers. Skills might originate in inter-personal interactions and later evolve into self-talk mimicking of such interactions; often ultimately being conducted as silent rehearsal (thinking) or even automatized non-reflective practices (habits). In most cases of mathematics learning, the foundational inter-personal interactions are mediated by language (including various forms of bodily gesture) (Sfard, 2008; Stahl, 2008). Frequently, the early experiences leading to new math skills are also mediated by physical artifacts or systems of symbols—more recently including computer interfaces.

Based on a Vygotskian perspective, a CSCL approach to the teaching of geometry would involve collaborative learning mediated by dynamic-geometry software and student discourse. In the past decade, we have developed the Virtual Math Teams (VMT) environment and are integrating a multi-user version of GeoGebra into it (Stahl, 2009; Stahl et al., 2010). In developing this system, we have tested our prototypes with various small groups of users. Recently, two small groups worked on the problem given in Figure 6 (based on the construction of inscribed equilateral triangles). We will call them Group A (Jan, Sam and Abe) and Group B (Lauren, Cat and Stew). The group members are adults already familiar with GeoGebra.

Case study: Group A

Group A starts by coordinating their online activity. They decide who will have initial control of the GeoGebra manipulation and they discuss in the text-chat panel the behavior they see as points of the construction are dragged (Log 1). They begin by dragging each of the points in the diagram (Figure 7).

Figure 7. The dragged construction with the problem statement and some chat.

6	14:33:56	Sam	I am good with somebody taking a stab at the dragging ...
7	14:34:10	Sam	I think maybe tell us what you intend to drag and we can discuss what we observe?
8	14:34:18	Jan	Go ahead Abe. Why don't you move the points in alphabetical order
9	14:34:36	Abe	Ok
10	14:34:43	Abe	I will try to drag point A
11	14:35:08	Jan	So the whole triangle moves... it both rotates around point B and it can dilate
12	14:35:14	Sam	So, A seems to move all the other points and scalle the whole drawing.
13	14:35:14	Jan	Which are you moving now
14	14:35:20	Sam	What are you moving now?
15	14:35:35	Abe	I first moved 1 and then D.
16	14:35:45	Sam	A, then D.
17	14:35:46	Sam	ok
18	14:36:05	Jan	so D was stuck on segment AC
19	14:36:12	Abe	when i dragged A what did you notice?
20	14:36:12	Sam	D has an interesting behavior. It seems that E & F remain anchored on the lines wehre they are, and so does d
21	14:36:14	Jan	Can you move E,F
22	14:36:38	Sam	I tried to move e and f ... they did not move.
23	14:37:08	Jan	Hmm...
24	14:37:35	Sam	It seems A, B & D move ... but C, E & F do not.
25	14:37:46	Jan	SO I see D is free to move on AB, but how did it generate E and F?
26	14:37:56	Sam	The behavior for A & B appear to be the same.
27	14:37:59	Abe	It appears that the triangles remain equilateral.

Log 1. Group A drags points in the diagram.

Note that the problem statement in Figure 6 does not say that the triangles are equilateral or inscribed. By having Abe drag points A and D, the group quickly sees that the vertices of the inner triangle always stay on the sides of the outer triangle (e.g., lines 18 and 20), indicating that the smaller triangle is inscribed in the larger one.

As Abe drags each of the vertex points, the group notices that points A, B and D are free to move, but that C, E and F are dependent points, somehow determined by A, B and/or D. Jan asks Sam to drag E and F, but Sam finds that they cannot be dragged. This sparks Jan to express wonder about how the position of point D (as it is dragged while A, B and C remain stationary) generates the positions of E and F (line 25). This is a move to consider how the diagram must be constructed in order to display the behavior it does during dragging. Meanwhile, Abe notices in line 27 that the triangles both remain equilateral during the dragging of all their vertices.

Within about three minutes of collaborative observation, the group has systematically dragged all the available points and noted the results. They have noticed that the triangles are both inscribed and equilateral. They have also wondered about the dependencies that determine the position of E and F as D is dragged. Now they start to consider how one would construct the dynamic diagram (Log 2).

47	14:45:39	Jan	What are we thinking...
48	14:46:07	Abe	okay,we have two equilateral triangles, with the inner one constrained to the sides of the outer triangle.
49	14:46:12	Sam	I think Abe summarized what is happening nicely - that both triangles remain equilateral when any of the 3 movable points are moved.
50	14:46:26	Jan	Agreed.
51	14:47:01	Jan	The thing I'm wondering about is how to generate the specific equilateral triang.e
52	14:47:02	Sam	Yes, another good bpoint - the one is contained in the other ... further, the three points of the inner triangle are constrained by the line segments that make up the outer triangle.
53	14:47:03	Abe	let's try to construct the figures?
54	14:47:20	Jan	For example, given a point on AB and a point on AC, there exists an equilateral triangle
55	14:47:38	Jan	But that's not this sketch b/c only one point is free. The rest are constrained
56	14:49:37	Jan	I'm wondering if all the three triangles that are outside the little equilateral triangle yet inside the big one are congruent.
57	14:50:39	Abe	When you say all three triangles, do you mean the three sides of the one of the triangles?

Log 2. Group A wonders about the construction.

First they all agree on the constraint that the triangles must remain inscribed and equilateral. Abe suggests that they actually try to construct the figure (line 53); through such a trial, they are likely to gain more insight into an effective construction procedure, which will reproduce the dragging behavior they have observed. Jan first notes that an equilateral triangle can be defined by the two points of its base. However, he notes that in the given figure only one of the vertices is free and it determines the other two (line 55). This leads him to wonder, "if all the three triangles that are outside the little equilateral triangle yet inside the big one are congruent." If they are congruent, then corresponding sides will all be of equal length. Abe relates the sides of the three little congruent triangles to the three sides of the interior triangle and to the three segments on the sides of the exterior triangle. Following the excerpt in Log 2, Team A measures the three segments AE, BF and CD, discovering that they are always equal to each other, even when their numeric length changes with the dragging of any of the free points (Log 3).

72	14:53:33	Jan	That means that CD, AE, and CF also have to be the same length, bc big triangle is equilateral
73	14:53:42	Abe	did you change what is being measured? or did you resize the figure?
74	14:53:58	Jan	I just moved point D along the side of the equilateral triangle
75	14:54:35	Abe	i c
76	14:56:16	Abe	So, shall we summarize the dependencies that we notice?
77	14:57:11	Jan	Sure who wants to start?
78	14:57:45	Sam	The inner triangle is contained by the outer triangle.
79	14:58:05	Sam	segment AC is the boundary of point D
80	14:58:14	Sam	Segment CB is the boundary of point F
81	14:58:24	Sam	Segmemt AB is the boundary of point E
82	14:58:55	Jan	So I think we may want to say F is on CB a bit differenlty.
83	14:59:10	Sam	Both triangles are equalateral no matter how the three movable points -- A, B & D -- are moved.
84	14:59:14	Jan	It is not free to move on CB. It is stuck in a particular location on CB defined by where D is on CA
85	15:00:09	Abe	The line segment CB cannot move.
86	15:00:10	Jan	So I think F is CD units away from B on BC. Its not constructed as an equilateral triangle, it happens to be an equilaterl triangle because of the construction
87	15:00:26	Jan	Agreed. I meant segment of length CB
88	15:00:38	Jan	Do you all buy that...

89	15:00:39	Jan	?
90	15:00:50	Sam	@Jan - I think that's covered by saying that both triangles are always equaleteral ... it implies both points move in conjunction with the third. (D) ... Of course, I don't teach the teachers who teach math (much), so you may have a better sense of the conventions. :D
91	15:00:59	Sam	I'll buy it.
92	15:01:04	Abe	yes, i agree!
93	15:02:28	Abe	The same can be said about E, it's constructed to be CD units from A.

Log 3. Group A identifies dependencies of the inscribed equilateral triangles.

After noting the key dependency that they discovered, AE=BF=CD, they list the other dependencies involved in constructing the figure. Line 86 provides a conjecture on how to construct the inner triangle. Namely, it is not constructed using Euclid's method from Proposition 1 (the way the exterior equilateral triangle could have been). Rather, point F is located the same distance from B on side BC as D is from C on AC: a distance of CD. Jan asks the rest of his group if they agree (line 88). They do. Abe adds that the same goes for the third vertex: point E is located the same distance from A on side AB as D is from C on AC: a distance of CD. The work of the group on this problem is essentially done at this point. A few minutes later (line 110), Jan spells out how to assure that AE=BF=CD in GeoGebra: "Measure CD with compass. Then stick the compass at B and A."

We have seen that Group A went through a collaborative process in which they explored the given figure by varying it visually through the procedure of *dragging* various points and noticing how the figure responded. Some points could move freely; they often caused the other points to readjust. Some points were constrained and could not be moved freely. The group then wondered about the constraints underlying the behavior. They conjectured that certain relationships were maintained by built-in dependencies. Finally, the group figured out how to accomplish the *construction* of the inscribed equilateral triangles by defining the *dependencies* in GeoGebra.

Case study: Group B

Team B went through a similar process, with differences in the details of their observations and conjectures. Interestingly, Team B made conjectures leading to at least three different construction approaches. First, Stew wondered if the lengths of the sides of the interior triangle were related to the lengths of a segment of the exterior triangle, like DE=DA (Log 4). The group then quickly shared with each other the set of basic constraints— inscribed and equilateral—similar to Group A's list of constraints.

15	14:37:00	Stew	and it appears that the side lengths of the inner triangle are related to the length of a portion of the orignal side
16	14:37:08	Cat	so also, there must be a constraint about the segments remaining equal, no?
17	14:37:31	Cat	@Stew, why can't they just be equal to each other?
18	14:37:47	Lauren	yes, visually it sure looks like equilateral triangles
19	14:37:53	Stew	yes, I think the triangles are equilateral or something like that.?
20	14:38:00	Lauren	D is free to move on AC, but E and F cant be dragged
21	14:38:58	Lauren	constructing the outer equilateral will be easy, but how do you think we should plan the construction of the inner triangle?
22	14:39:24	Stew	you can construct an equilateral but how do you make it so that its vertices are always on the outer triangle?
23	14:39:57	Lauren	Im thinking place D on AC, and construct an equilateral from there, with intersections on the sides of the outer triangle
24	14:40:12	Lauren	should we try and see what happens?

25	14:40:13	Cat	yeah, i'm not sure about making the other points stay on their respective segments
26	14:40:27	Cat	but we can maybe see the answer when we get closer
27	14:40:35	Stew	I think we'll get intpo trouble with the third side
28	14:40:38	Lauren	yeah, that will be the tricky part, but i think if we intersect they will be constrained
29	14:40:41	Stew	but, sure, let's try it
30	14:40:53	Lauren	may I start?
31	14:40:59	Cat	go for it!

Log 4. Group B noticings while dragging points in the diagram.

The group sees that the inner triangle must remain both inscribed and equilateral. This raises difficulties because the usual method of constructing an equilateral triangle would not in general locate the dependent vertex on the side of the inscribing triangle (line 22). This group, too, decides to start construction in order to learn more about the problem (line 24). They begin by constructing triangle ABC and placing point D on AC. They anticipate problems constructing triangle DEF and ensuring that both E and F remain on the sides of the inscribing triangle while also being equidistant from D. Note that the members of the team are careful to make sure that everyone is following what is going on and agrees with the approach.

45	14:44:53	Lauren	anyone have any ideas for the inner triangle?
46	14:45:37	Stew	One thing I noticed is that the sidelength of the inner triangle appears to be the distance of the longer segment on the original triangle
47	14:46:17	Cat	i wish i could copy the board :) i know that is not ideal, though
48	14:46:40	Cat	i forget what the tools do exactly, and want to just remind myself
49	14:46:57	Stew	If you made a circle that fit inside the original triangle, then its point of tangency or intersection might be useful
50	14:47:31	Stew	the trick might be to find the center of such a circle.
51	14:48:15	Stew	There are interesting centers made by things such as Cat was suggesting, the angle bisectors, or perpendicular bisectors
52	14:48:33	Lauren	yes - the center of each triagle probably is the same - do you think?
53	14:48:54	Lauren	angle bisecotrs would work
54	14:49:05	Stew	I don't think they have the same center
55	14:49:44	Lauren	maybe not....

Log 5. Group B conjectures about the construction.

In line 46 (Log 5), Stew repeats his conjecture about side DE equaling the length of "the longer segment" of AC, i.e., either CD or AD depending on which is longer at the moment. This conjecture is visibly supported by the special cases when D is at an endpoint of AC or at its midpoint: when D is at an endpoint, DE=AC or DE=AB (and AB=AC); when D is at the midpoint of AC, DE=AD=DC because the three small triangles formed between the inscribed triangles are all equilateral and congruent.

But then the group switches to discussing a quite different conjecture that Lauren had brought up earlier and that Cat is trying to work on through GeoGebra constructions. That conjecture is that it would be helpful to locate the centers of the inscribed triangles, construct a circle around the center and observe where that circle is tangent to or intersects triangle ABC. In general, triangles have different kinds of centers, formed by constructing bisectors or the triangle's angles or by constructing perpendicular
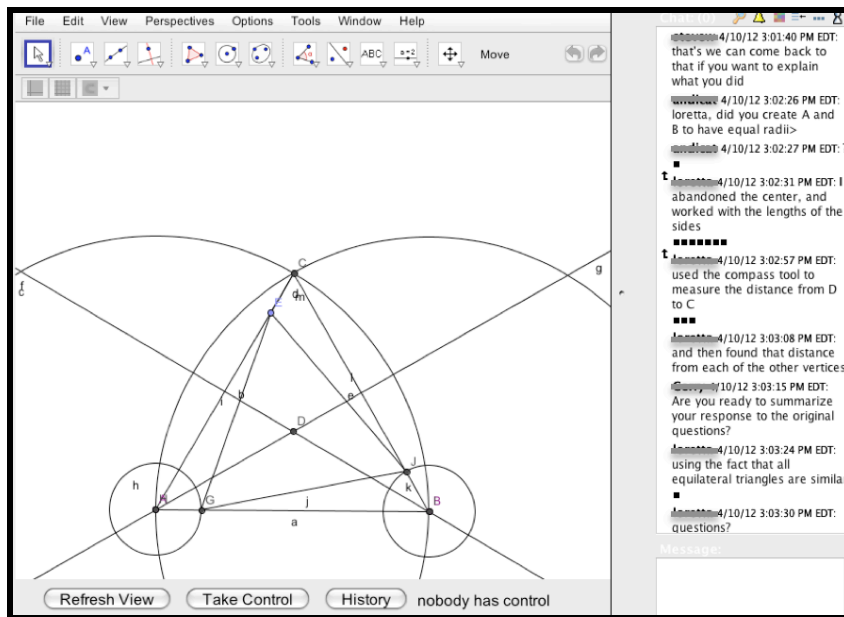
Figure 8. Finding the center and constructing equal line segments.

bisectors of the triangle's sides. The group discusses which to use and whether they might be the same center for both of the triangles.

Lauren does some construction (see Figure 8). She locates a point at which triangle ABC's angle bisectors meet. However, she then abandons this approach (Log 6).

80	15:01:40	Stew	that's we can come back to that if you want to explain what you did
81	15:02:26	Cat	Lauren, did you create A and B to have equal radii>
82	15:02:27	Cat	?
83	15:02:31	Lauren	I abandoned the center, and worked with the lengths of the sides
84	15:02:57	Lauren	used the compass tool to measure the distance from D to C
85	15:03:08	Lauren	and then found that distance from each of the other vertices
87	15:03:24	Lauren	using the fact that all equilateral triangles are similar
88	15:03:30	Lauren	questions?
89	15:04:05	Lauren	is everyone convinced the inner triangle is as it should be?

Log 6. Group B constructs the dependencies of the inscribed equilateral triangles.

Instead, she pursued a new conjecture, related to Stew's earlier observations: "we know by similar triangles, that each line of the inner is the same proportion of the outer" (Lauren, line 75). She used the GeoGebra compass tool with a radius of CD to construct circles around the other vertices of triangle ABC (Line 84, 85), just like Group A had done. This located points where the circles intersected the triangle sides for placing the other vertices of the inscribed triangle with the constraint that CD=AE=BF. She then concluded by asking if the other group members agreed that this constructed the figure properly.

Like Group A, Group B initiated a collaborative process of exploring the given diagram visually with the help of dragging points. They developed conjectures about the constraints in the figure and about what

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dependencies would have to be built into a construction that replicated the inscribed equilateral triangles. They decided to explore trial constructions as a way of better understanding the problem and the issues that would arise in different approaches. Eventually, they pursued an approach involving line segments in the three congruent smaller triangles.

It is interesting to note the role of the three small triangles formed between the two inscribed triangles. These small triangles are not immediately salient in the original diagram. Triangles ABC and DEF are shaded; the smaller triangles are simply empty spaces in between. They become focal and visible to the groups due to their relationships with the sides of the salient triangles, and particularly with the segment CD. It is the fact that these three smaller triangles are congruent that supports the insight that the necessary constraint is to make CD=AE=BF. The smaller triangles become visible through the exploratory work of dragging, conjecturing and constructing this dependency. This is precisely the kind of perception that can occur in the scaffolded inter-personal setting of collaborative dynamic geometry and then can gradually mature into increased professional vision (Goodwin, 1994) and mastery of practices of observation and discourse by the individual group members as developing students of mathematics.

Conclusion

Both Group A and Group B find a solution to the problem they address by taking advantage of the affordances of collaborative dynamic geometry. Their understanding of the problem (Zemel & Koschmann, 2013) develops gradually through dragging points, noticing how other points respond, wondering about effective constraints and conjecturing about possible dependencies to construct. Next, they begin exploratory construction. These are trial-and-error attempts in different directions. Some reach deadends or are simply put aside as more promising attempts catch the group's attention. Finally, the group agrees upon a key dependency to build into the construction. This dependency—in its connections to related geometric relationships—forms the basis for persuading the group members of a solution to the problem. This is implicitly a justification or proof of the solution. In the end, the group can construct a set of inscribed equilateral triangles, building in the dependency that CD=AE=BF. They can then prove that the triangles are inscribed and equilateral by referring to the dependency that CD=AE=BF, along with certain well-known characteristics of equilateral and congruent triangles.

Although both groups reached a similar conclusion, their paths were significantly different. First, they defined their problem differently. Group A focused on listing the constraints that they noticed from dragging points and then on proving that the given triangles were in fact equilateral. Group B, in contrast, quickly realized that it would be difficult to construct triangle DEF to be both inscribed and equilateral, since these characteristics required quite different constraints, which would be hard to impose simultaneously. Whereas Group A coordinated its work so that the members followed a single path of exploration and conjecture, Group B's members each came up with different conjectures and even engaged in some divergent explorative construction on their own before sharing their findings. Despite these differences, both groups collaborated effectively. They listened attentively and responded to each other's comments. They solicited questions and agreement. They followed a shared group approach. Together, they reached an accepted conclusion to a difficult problem, which they would not all likely have been able to solve on their own, illustrating effective group cognition (Stahl, 2006).

The case study of Groups A and B illustrates the approach of collaborative dynamic geometry. The groups took advantage of the three central dimensions of dynamic geometry—dragging, construction and dependencies—to explore the intricacies of a geometric configuration and to reach—as a group—a deep understanding of the relationships within the configuration. They figured out how to construct the diagram and they understood why the construction would work as a result of dependencies that they designed into it.

In the Virtual Math Teams Project, we are currently refining the VMT software and developing curriculum (Stahl, 2012b) to guide the use of collaborative dynamic geometry in in-service-teacher professional development and high-school geometry (Stahl, 2012a). The curriculum centers on activities like the one in Figure 6. It structures the use of dragging, constructing and dependencies, as well as effective collaborative discourse practices. The curriculum is closely aligned to the new Common Core State Standards for basic geometry and their recommended mathematical practices (CCSSI, 2011). It covers the most important propositions of Book I of Euclid's *Elements*, translating them into research-based, contemporary approaches to geometry and mathematical discourse in a computer-supported collaborative learning environment. We will continue to study the results of collaborative dynamic geometry through analysis of the discourse and the geometric explorations (Stahl, 2012c).

On the basis of a continuing series of trial studies like the one just reported, we feel that the approach of collaborative dynamic geometry can translate the geometry of Euclid into an effective tool of computer-supported collaborative learning.

References

Adorno, T. W., & Horkheimer, M. (1945). *The dialectic of enlightenment* (J. Cumming, Trans.). New York, NY: Continuum.

CCSSI. (2011). High school -- geometry. In Common Core State Standards Initiative (Ed.), *Common core state standards for mathematics.* (pp. 74-78)

Euclid. (300 BCE/2002). *Euclid's elements* (T. L. Heath, Trans.). Santa Fe, NM: Green Lion Press.

Gleick, J. (2011). *The information*. New York, NY: Vintage.

Goodwin, C. (1994). Professional vision. *American Anthropologist. 96*(3), 606-633.

Heidegger, M. (1938/1999). *Contributions to philosophy (from enowning)* (P. Emad & K. Maly, Trans.). Bloomington, IN: Indiana University.

Heidegger, M. (1979). *On time and Being* (J. Stambaugh, Trans.). New York, NY: Harper & Row.

Husserl, E. (1929/1960). *Cartesian meditations: An introduction to phenomenology* (D. Cairns, Trans.). The Hague, Netherlands: Martinus Nijhoff.

Husserl, E. (1936/1989). The origin of geometry (D. Carr, Trans.). In J. Derrida (Ed.), *Edmund Husserl's origin of geometry: An introduction.* (pp. 157-180). Lincoln, NE: University of Nebraska Press.

Hutchins, E. (1996). *Cognition in the wild*. Cambridge, MA: MIT Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, UK: Cambridge University Press.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York City, NY: Basic Books.

Latour, B. (2008). The netz-works of Greek deductions. *Social Studies of Science. 38*(3), 441-459.

Livingston, E. (1999). Cultures of proving. *Social Studies of Science. 29*(6), 867-888.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art forms*. New York, NY: Belevue Literary Press.

Long, P. O., McGee, D., & Stahl, A. M. (Eds.). (2009). *The book of Michael of Rhodes: A fifteenth-century maritime manuscript*. Cambridge, MA: MIT Press.

Merleau-Ponty, M. (1955). *Visible and invisible*. Evanston, IL: Northwestern University Press.

Merleau-Ponty, M. (1961/1964). Eye and mind (C. Dallery, Trans.). In J. M. Edie (Ed.), *The primacy of perception.* (pp. 159-190). Evanston, IL: Northwestern University Press.

Netz, R. (1999). *The shaping of deduction in Greek mathematics: A study in cognitive history*. Cambridge, UK: Cambridge University Press.

Oner, D. (2013). Analyzing group coordination when solving geometry problems with dynamic geometry software. *International Journal of Computer-Supported Collaborative Learning. 8*(1).

Ong, W. (1998). *Orality and literacy: The technologizing of the world*. New York, NY: Routledge.

Pappus of Alexandria. (340). *Synagoge [mathematical collection]*. Alexandria, Egypt.

Piaget, J. (1990). *The child's conception of the world*. New York, NY: Littlefield Adams.

Scher, D. (2000). Lifting the curtain: The evolution of the geometer's sketchpad. *The Mathematics Educator. 10*(1).

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge, UK: Cambridge University Press.

Stahl, G. (2006). *Group cognition: Computer support for building collaborative knowledge*. Cambridge, MA: MIT Press. Web: http://GerryStahl.net/mit/.

Stahl, G. (2008). Thinking as communicating: Human development, the growth of discourses and mathematizing. *International Journal of Computer-Supported Collaborative Learning. 3*(3), 361-368. Web: http://GerryStahl.net/pub/Sfardreview.pdf. Doi: 10.1007/s11412-008-9046-4.

Stahl, G. (2009). *Studying virtual math teams*. New York, NY: Springer. Web: http://GerryStahl.net/vmt/book. Doi: http://dx.doi.org/10.1007/978-1-4419-0228-3.

Stahl, G. (2012a). *Designing a learning environment for promoting math discourse*. Paper presented at the 12th International Conference on Mathematical Education. Seoul, Korea. Web: http://GerryStahl.net/pub/icme_design.pdf, Slides: http://GerryStahl.net/pub/icme2012.ppt.pdf.

Stahl, G. (2012b). *Dynamic-geometry activities with GeoGebra for virtual math teams*. Web: http://GerryStahl.net/pub/activities.pdf.

Stahl, G. (2012c). *Evaluating significant math discourse in a learning environment*. Paper presented at the 12th International Conference on Mathematical Education. Seoul, Korea. Web: http://GerryStahl.net/pub/icme_discourse.pdf.

Stahl, G., Koschmann, T., & Suthers, D. (2006). Computer-supported collaborative learning: An historical perspective. In R. K. Sawyer (Ed.), *Cambridge handbook of the learning sciences.* (pp. 409-426). Cambridge, UK: Cambridge University Press. Web: http://GerryStahl.net/elibrary/global.

Stahl, G., Ou, J. X., Weusijana, B. K., Çakir, M. P., & Weimar, S. (2010). Multi-user GeoGebra for virtual math teams. *GeoGebra: The New Language For The Third Millennium. 1*(1), 117-126. Web: http://GerryStahl.net/pub/geogebra_romania.pdf.

Sutherland, I. E. (1963). *Sketchpad: A man-machine graphical communication system*. Cambridge, MA: MIT Lincoln Labs.

Swetz, F. (1987). *Capitalism & arithmetic: The new math of the 15th century*. Chicago, IL: LaSalle University Press.

Vygotsky, L. (1930/1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Zemel, A., & Koschmann, T. (2013). Online math problem solving as a process of discovery in CSCL. *International Journal of Computer-Supported Collaborative Learning. 8*(1).