Dialogical Perspectives on Narratives in Collaborative Mathematics Problem-Solving

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Abstract. Our approach to the study of learning of mathematical problem-solving extends the notion of narrative learning environments to include the dynamics of collaborative dialogs and related emergent narratives. This perspective favours the conception of the dialogical aspects of interaction as shared achievements of coparticipants and as central meaning-making procedures, based on our qualitative analysis of transcripts from online collaborative math problem-solving interactions. From these observations we attempt to establish a link between narrative learning environments and dialogical perspectives and explore relevant implications for the design of the Virtual Math Teams collaborative learning environment

Truth is not to be found inside the head of an individual person, it is born between people collectively searching for truth, in the process of their dialogic interaction. (Bakhtin, [1], p.110)

Introduction

Research in the field of Narrative Learning Environments (NLEs) is concerned with questions such as how to characterize the contribution of narratives and narration to learning, and how to use knowledge of narratives to design learning environments. As part of the Virtual Math Teams (VMT, see http://mathforum.org/wiki/VMT) research project, we have investigated talk-in interaction within the context of collaborative mathematical problem-solving online and have found similarities and differences between the narrative approach and a dialogical perspective on sensemaking and interaction. Therefore, we propose to extend the concept of NLEs to encompass collaborative learning environments for mathematics which, in addition to using narrative structures, offer also the possibility of joint participation and interaction with a diverse set of linguistic and extra-linguistic objects (e.g. mathematical objects and their derivative properties). In the following sections we present these perspectives and offer some ideas for future research and development. The next section briefly presents the problematic of narrative learning environments. Section 2 introduces the main ideas of the dialogical theory of Mikhail Bakhtin and its relevance for narrative learning environments. Section 3 presents in detail a qualitative analysis of a chat transcript as part of the VMT project. Section 4 concludes with some implications for design and future research.

1. Narrative Learning Environments (NLE)

Theorists of the narrative aspect of cognition (e.g. Jerome Bruner [2, 3], Walter Fisher[4, 5], Roger Schank[6], etc.) argue that the narrative form is the primary means through which human beings create and convey meanings about the world. The interest in narrative that AI and Cognitive Science have shown revolves around the ability of narratives to structure and mediate knowledge [7]. As such, major areas of AI work include story understanding and generation as well as the development of interactive environments structured as narrative spaces. Research and development on Narrative Learning Environments (NLEs), a field of work at the intersection of AI, educational technologies and narratology, is concerned with intelligent learning environments where "narrative is approached and applied" to support learning and the construction of meaning [8]. NLEs are expected to promote three main kinds of activities for learners:

(1) **co-construction**: [the ability to] participate in the construction of a narrative;

(2) *exploration*: engage in active exploration of the learning tasks, following a narrative approach and trying to understand and reason about an environment and its elements;

(3) *reflection*: engage in consequent analysis of what happened within the learning session [8].

To date, research and development in the field of narrative learning environments has concentrated on the analysis and use of narrative elements such as virtual storytelling, interactive drama, and participatory narratives, mostly within the context of literacy development and language learning (e.g. [9]). We propose to extend the idea of the usage of narratives in two dimensions: the use of co-constructed narratives in Computer-Supported Collaborative Learning (CSCL) and extending the narrativeS use also to mathematics learning.

2. The dialogical perspective on Learning

The *dialogical* perspective sees meaning-making as an interactional achievement of co-participants, rather than a property of narratives or other linguistic objects. Theorists of the dialogical aspect of language and meaning (e.g. Bakhtin [10, 11], Harré [12], Sacks [13, 14], and Schegloff [15]) point to the features of talk as action, and of shared action in itself, as the core processes of human meaning-making. These socially shared procedures might point to general sense-making strategies with applicability within particular domains (e.g. fictional storytelling, or mathematical problem-solving).

As Wegerif stresses [16], the dialogical perspective on learning attempts to access the creative space of "the interanimation of more than one perspective" that emerges in the dynamics of interactive narratives and collaborative meaning-making. Bakhtin in particular, considers any human language related activity, be it in the form of oral speech or writing, as dialogic— i.e. containing more than one voice ([10, 11]). This is of no surprise if we realize that narratives, as interaction, contain not only the voice of the narrator but also, at least, the voice of the listener. When telling a story, the narrator anticipates the listener, for instance possible aspects that might require elaboration (especially in learning contexts). This ideas are very important because they move the emphasys of learning and other sense making activities from an individual knowledge acquisition perspective (as in cognitive science) to a dialogic, collaborative, social activity of knowledge building.

From this perspective, narratives resemble, as well, processes of collaborative scientific discourse. The procedures used in structuring a narrative and, for example, writing a proof of a theorem, or presenting a solution for a problem exhibit significant similarities in their communicative structures. What is common to both narratives and theorem proving, or collaborative problem-solving is *the discourse*; the emergent sense-making of the sequencing of utterances generated within joint interactions with others and with meaningful artefacts. Furthermore, when we refer to these activities in the context of learning, it is interesting to note that "rather than speaking only about *acquisition of knowledge*", we also view learning as "*becoming a participant in a certain discourse*" [17], or of mastering a certain (e.g. mathematics) speech genre [11].

Participation in the learning processes is usually a social activity, language being an extremely important cultural artefact. As Vygotsky states in his concept of the Zone of Proximal Development [18], children's potential learning abilities are especially accessible within their interactions with others. Participatory or interactive narratives offer opportunities for co-construction of meaning precisely based on the dialogic principle (through which Bakhtin extends Vygotsky's theory) of interactivity resulting on an intermix of classical narrative structures and other frameworks of shared participation, a point we seek to illustrate within the domain of collaborative mathematical problem solving. In summary, we propose to connect narrative learning environments and collaborative learning environments by virtue of their common concern for the role of discourse and interaction in learning and its potential support via designed artefacts.

3. Collaborative Math Problem-solving: Co-construction, exploration and reflection

The Virtual Math Teams (VMT) research program investigates the innovative use of online collaborative environments to support effective K-12 mathematics learning as part of the research and development activities of the Math Forum (mathforum.org) at Drexel University. VMT extends the Math Forum's "Problem of the Week (PoW)" service by bringing together groups of 3 to 5 students in grades 6th to 11th to collaborate online in discussing and solving non-routine mathematical problems. Currently, participants interact using a computer-supported collaborative learning environment which combines quasi-synchronous text-based communication (e.g. chat) and a shared whiteboard among other interaction tools. At the core of VMT research is the premise that primarily, group knowledge arises in discourse and is preserved in linguistic artifacts whose meaning is co-constructed within group processes ([19]). Key issues addressed by the VMT include the design challenge of structuring the online collaborative experience in a meaningful and engaging way, and the methodological challenge of finding appropriate methodological approaches to study the forms of collaboration and reasoning that take place.

3.1. Data sources and Methodology

As part of the initial exploratory phase of research, the VMT offered more than 20, 1-hour online sessions in which small groups of students used AOL Instant Messenger© technologies to interact and collaboratively attempt to solve a mathematical problem provided. Through these events we have collected a corpus of chat transcripts that constitute our main source of data. The VMT implements a multidisciplinary approach to the analysis of these transcripts, which integrates quantitative modelling of students' interactions as well as ethnographic and conversation analytical studies of collaborative problem solving. A coding scheme has been developed for the quantitative analysis of the sequential organization of postings recorded in a chat log. This coding scheme includes nine content and threading dimensions (e.g. conversation, problem-solving content and threads) of each chat line (see [20] and [21] for further discussion). The analysis of these same data.

Several researchers have explored the interdependencies between narratives and mathematics (Cocking & Chipman [22]) as well as the role of narrative in mathematics learning (Burton, [23, 24]). Our qualitative analysis of collaborative mathematical problem-solving, based on conversation analysis (e.g. [14, 15]), seeks to understand the methods that co-participants use to organize their shared interactions, and further improve CSCL learning environments for facilitating their collaboration. The object of inquiry in conversation analysis (CA) is not exclusively conversation as a linguistic entity, but rather talk and social interaction. The interest of CA is "with the local production of [social] order and with 'members' methods' for doing so" ([25], p.19). Using the methods of CA, our analysis of transcripts of online collaborative problem-solving revealed, in particular instances, narrative elements—e.g. the emergence of a narrator and a narratee as well as structured sequences of events, that participants oriented to in their collaborative production of problem solutions.

3.2. Emergent Narrative Elements from Shared Participation.

The following analysis illustrates the above ideas by using data from one of the online transcripts of a VMT collaborative problem-solving session. The session presented here has three main participants, SKI, YAG and GOH. "*Press for Time*" is the problem assigned for the session, which by virtue of its presentation as a word problem, could contribute to the display of narrative elements in the dialogical interactions among participants:

The Rational Reader, a popular daily newspaper, has to be printed by 5 a.m. so that it can be distributed. Late one night, a major story broke and the front page had to be rewritten, which delayed the start of the printing process until 3 a.m. To try to get the printing done on time, the Reader used both their new printing press and their old one. The new press is three times as fast as the old one, and with both of them running, the printing was finished exactly on time. How long does it take to print a normal edition of the paper using only the new press?

From the transcript we can infer that at least two of the participants (SKI and YAG) had worked on the problem prior to their joint participation in the online collaborative session, and as a result, orient themselves to an "expository" mode of interaction in which reports of "ways" to solve the problem are offered in the form of story-like narrations. The form in which a way of solving a problem is then made accessible during this collaborative problem solving interaction is, to a certain extent, similar to that of the narration of a story. The process of narrating and the resulting narrative, however, are to be considered as an interactional achievement of all the participants despite the apparent fact of an established narrator voice or the references made by participants to the authorship of particular ways of proceeding with their joint work. On the other hand, an interactive narrative within the speech genre of mathematics problem solving (in the Bakhtinian sense [11]), has specific characteristics that govern the space of possible transformations of the different "events" of the narrative being produced. In fact, exercising narration is obvious a way of supporting learning as mastering a given speech genre. In this process, dialogues are essential (at least because mastering a speech genre implies being able to dialogize in it). The following excerpts allow us to illustrate these ideas:

1. S	SKI i started and solved	1. GOH	how come $1/x$ and
	with a system		1/y added equal
2. S	SKI of equations		1/2?
	AG let SKI explain	2. SKI	ok
	SKI lets just say x is the	3. YAG	ummm
т. 5	time for the old	4. YAG	-
			T I I I I I I
	machine and y is for	5. SKI	1/x is how much
	the new		the old one does
5. G	fOH ok		in one hour
6. S	SKI our first equation is	6. GOH	right.
	like this	7. SKI	how much of the
7. S	SKI if we atke the recip of		job it does in an
	x		hour
8. Y	AG *choughSHOWOFFchough*	8. YAG	(frac of job done)
	AG : P	9. SKI	1/y is for the new
10.Y	AG :-D		machine
11. S	SKI thats how much of the job	10. GOH	right

	the old one does in one	11. SKI	add thogo up
	hour		add those up
10 377 9		12. YAG	and since they do
12. YAG			it together at 3-5
I3. SKI	and the reciprocal of y	13. SKI	thats how much of
	is how much of the job		the job they do
	the new one does in one		together in one
	hour		hour
14.YAG	recip [of] y is the	14. YAG	it took 2 hrs
	new one	15. SKI	уа
15.SKI	ok	16. SKI	listen to [YAG]
16.SKI	recip=reciprocal		
17.SKI	anyways	1. SKI	the whole job took
18.YAG	and, recip y+ recip x		2 hours
	= 1/2	2. YAG	with both machines
19.SKI	we add $1/x$ and $1/y$	3. SKI	so in one hour
20. SKI			they did $1/2$ of
	what YAG said		the job
22. SKI		4 YAG	and in the 2nd
	in hours and fraction		hour they did the
23. 1110	of work		other half
24 VAC	needed to be done	5. GOH	Okay, I got it.
	cuz they together get	5. 0011	1/2 is how much of
23. SKI	half the job done in		the job they do
	one hour		3 1
26.YAG			together in one hour
	• -		110 41
27. SKI	are u getting our	6. SKI	rite
	first equation?	7. YAG	уеруеруер
• • •		8. SKI	u know what x and
			y represent rite?

As can be seen in these excerpts, even in this "expository" orientation, co-participants take active roles in co-constructing the explanation. Even though SKI initiates his story-like report with the form of a first person narrative ("*i started and solved with a system of equations*"), the shared narrative space of this interaction is constituted with YAG and GOH's uptake of SKI's narrator voice (lines 3 and 5) and their subsequent participation. SKI's narration seems to shift to the first person plural ("our first equation is like this") and subsequently we can observe how SKI and YAG share the narrator role by completing each other postings or interjecting new ones (e.g. lines 23 and 25). SKI and YAG have, at this point, constituted themselves as a recognizable collectivity (Lerner [27]) oriented towards the task of producing an intelligible narrative explanation for GOH (e.g. line 27).

On the other hand, by virtue of the interactional nature of the conversation being produced, GOH is by no means restricted to a passive audience role. One of the interesting peculiarities of our attempt to intersect the framework of narratology and the domain of collaborative mathematical problem-solving, results in a unique instantiation of the idea of "possible worlds." The complex world of linguistic and mathematical objects which SKI, YAG and GOH both access and co-construct (e.g. the proposition

"The new press is three times as fast as the old one" included in the problem statement, and SKI's posting "the reciprocal of y is how much of the job the new one does in one hour), their individual perspectives, and the transformations that they exert on such objects (e.g. SKI use of "*cuz*" - because - on line 25) are governed not by strict logical laws (as is sometimes assumed in narrative semantics) but by the local sense-making procedures of the co-participants and their orientation to joint-activity. For, instance, SKI in line 27 asks GOH for an assessment of her state of participation, and GOH eventually (line 57) requests that the co-constructed narrative be reoriented towards a further sense-making on the mathematical and narrative objects so far established (e.g. 1/x, "the old one," "how much of the job they do together in one hour," etc.).

In addition to the co-construction of the narrative explanation in itself, the dialogical orientation opens the space for the exploration of possibilities of the local world of mathematical objects and, what is perhaps even more interesting as far as learning is concerned, to anticipate the intelligibility of the co-constructed narrative (in Bakhtin's ideas, the narrator's voice is combining with the listener's voice, with, for example, hers possible questions, in what he utters). In line 91, SKI's question to GOH seems to represent, both an orientation towards a prerequisite for the intelligibility of the mathematical narrative being produced, as well as an anticipation of a potential problem of understanding. It is in these instances of dialogical interaction where we are able to observe the power of what Feurenstein [28], elaborating on Vygotsky, has characterized as "mediated learning experiences:" interactions through which co-participants place themselves between each other and the world, and co-construct the meaning of their joint activity (i.e. verbal or otherwise). In mediation, stimuli and responses are selected, changed, amplified and interpreted in complex ways that represent a "type of organization (which) is basic to all higher psychological processes" ([13], p. 40). Needless to say this role is also shared among co-participants.

Although we have referred to this context as collaborative problem solving, it might appear that the work being done is closer to an "explanation" than to co-construction of knowledge. Yet, the participants, perhaps influenced by the very nature of dialogic interactions, make such explanations interactive and participatory for all members of the group. The outcome of this approach is that there is a constant interchange between first person singular and third person plural narration, and a consequent change in agency and authorship embedded within certain mathematical objects: "my way" (e.g. "*YAG its kinda hard to understand ur way*"), and sometimes becoming "our way" (e.g. "*so 8 hours is 480 minute[s], divide by 3, to get 160 minutes our answer!!!!*").

Of central interest to our analysis are the methods used by co-participants to orient themselves to certain forms of participation that guide them in their collaborative sense making. The use of the "expository" mode of interaction here differs slightly from Mercer's [26] conception of the three kinds of inter-subjective talk: disputational, cumulative, and exploratory. In Mercer's framework, disputational talk is characterized by the speakers being concerned with defending their own selves, at the possible expense of any attempt at a solution. In cumulative talk, each speaker seeks to support

the other's self but fails to explore facts and solutions. Exploratory talk, according to Mercer occurs when speakers "engage critically but constructively with each other's ideas" (p.98). For a more complete analysis of the two main "participation frameworks" identified in VMT research see [20]. Although one could argue that the structure of the task itself (a word or "story" problem) might contribute to the emergence of narrative elements in the dialogical interactions among participants, similar phenomena has been observed on geometry and other non-word problems.

We have seen that two of the central elements proposed for narrative learning environments: co-construction and exploration are clearly visible in the dialogical interactions illustrated through the transcript presented. The third characteristic element of a narrative learning environment, that of *reflection* or engagement in "*consequent analysis of what happened within the learning session*" [8] seems to present itself differently in the un-moderated experiences captured in our data, a fact that would suggest a potential area where explicit support from a pedagogical environment might be specially fruitful. Having access to, at least, a partial record of the interaction in the same way that we as researchers have had through the analysis presented here might be a unique advantage of an electronic environment. In addition, we are interested in fostering reflection, particularly, at the community level, i.e. at the level where the activity of small-groups gets reified into one diverse and collective narrative, a narrative of dialogues.

4. Implications for design, future research.

The analysis presented in the previous section illustrates how certain narrative structures may emerge from the dialogical interactions and the ways participants orient themselves to their shared sense-making during mathematical problem-solving. Moreover, from a Bakhtin's, dialogistic perspective, narratives are always multi-voiced (when we build a narrative, the voice of the potential listener will be virtually present, at least, for example, by our concern for plausibility and/or usefulness of the narration).

Although we have presented a single in-depth case, we seek to identify a diverse array of patterns of participation in narratives, through discourse and conversation analysis in parallel with statistical natural language processing techniques (e.g. [21, 29]), with the goal of informing the design of the appropriate learning supports for online, collaborative math problem-solving, that integrates the ideas of NLEs and CSCL. Engagement, participation, and ultimately, learning might be emergent aspects of distributed activity systems that offer rich opportunities for the learners to construct meaning through language and interaction in true dialogical contexts [30]. Further research and development is necessary to integrate, in the design of future learning environment, theories of sense-making that account for the narrative and dialogical aspects of individual, small-group and community interactions. Additional text processing is envisioned, such as automated narrative summarization and intelligent indexing with the specific intent of facilitating the re-usability of collaborative problem-solving dialogs for specific learning purposes, including the potential support for an online community of math problem-solvers characterized as a "narrative of dialogues".

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References

- [1] Bakhtin, M.M. (1984) *Problems of Dostoevsky's Poetics*. Edited and trans. by Caryl Emerson. Minneapolis: University of Michigan Press.
- [2] Bruner, Jerome. 1986. Actual Minds, Possible Worlds. Cambridge: Harvard UP.
- [3] Bruner, J. S. (1991). The narrative construction of reality. *Critical Inquiry*, 18, 1-21.
- [4] Fisher, W. R. "The Narrative Paradigm: An Elaboration." Communication Monographs, 52, December (1985): 347-367.
- [5] Fisher, W. R. "Narrative Rationality and the Logic of Scientific Discourse." Argumentation 8 (1994): 21-32
- [6] Schank, R. C. (1995). *Tell me a story: Narrative and intelligence*. Evanston, IL: Northwestern University Press.
- [7] Herman, D., Editor. (2003) *Narrative Theory and the Cognitive Sciences*. Stanford, CA: Publications of the Center for the Study of Language and Information
- [8] Narrative Learning Environments. Workshop at The 12th International Conference on Artificial Intelligence in Education (AIED 2005). Retrieved online on April 26, 2005 from: http://gaips.inesc-id.pt/aied05-nle/
- [9] Machado, I., Brna, P., Paiva, A. (2004). 1, 2, 3 Action! Directing Real Actors and Virtual Characters. Lecture Notes in Computer Science - Proceedings of Technologies for Interactive Digital Storytelling and Entertainment: Second International Conference, TIDSE 2004, Darmstadt, Germany, June 24-26, 2004. pp. 36-41. Springer-Verlag.
- [10] Bakhtin, M.M.(1981). *The dialogic imagination: Four essays by M. M. Bakhtin* (M. Holquist, Ed.; C. Emerson & M. Holquist, Trans.). Austin: University of Texas Press.
- [11] Bakhtin, M.M.(1986). Speech genres & other late essays (Caryl Emerson and Michael Holquist Eds.; Vern W. McGee, Trans.) (pp.60–102). Austin: University Texas Press.
- [12] Harre, R., & Gillett, G. (1994). The discursive mind. Thousand Oaks, CA: Sage
- [13] Sacks, H. (1992a, b). *Lectures on conversation*, vols. I and II. Edited by Gail Jefferson with an introduction by Emanuel A. Schegloff. Oxford: Basil Blackwell.
- [14] Sacks, H., Schegloff, E. & Jefferson, G. (1974). A simplest systematics for the organization of turn-taking in conversation. *Language*, 50, 697-735.
- [15] Schegloff, E. A. (1997). "Narrative Analysis" Thirty Years Later. Journal of Narrative and Life History, 7(1-4), 97-106.

- [16] Wegerif, R. (2005). A dialogical understanding of the relationship between CSCL and teaching thinking skills. In T. Koschman, D. Suthers, & T.W. Chan (Eds.). *Computer Supported Collaborative Learning 2005: The Next 10 Years!* Mahwah, NJ: LEA ©2005 International Society of the Learning Sciences.
- [17] Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, 27, 4-13.
- [18] Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press
- [19] Stahl, G. (2004). Building collaborative knowing: Elements of a social theory of CSCL. In J.-W. Strijbos, P. Kirschner & R. Martens (Eds.), *What we know about CSCL ... and implementing it in higher education*. (pp. 53-86). Boston, MA: Kluwer.
- [20] Zemel, A., Xhafa, F., Stahl, G. (2005). Analyzing the Organization of Collaborative Math Problem-solving in Online Chats using Statistics and Conversation Analysis. Paper submitted to the 11th International Workshop on Groupware. September 25-29, 2005. Porto de Galinhas (Recife) Brazil
- [21] Cakir, M., Xhafa, F. Zhou, N., Stahl, G. (2005) Thread-based analysis of patterns of collaborative interaction in chat. Paper presented at the *12th International Conference on Artificial Intelligence in Education* (AIED 2005), Amsterdam, Netherlands.
- [22] Cocking, R. R., and Chipman, S. (1998). Conceptual Issues Related to Mathematics Achievement of Language Minority Children. In Cocking, R. R., and Mestre, J.P. *Linguistic and Cultural Influences on Learning Mathematics*. Hillsdale: Erlbaum. 1988, pp. 17-46.
- [23] Burton, L. (1996). Mathematics, and its learning, as narrative A literacy for the twenty-first century. In D. Baker, J. Clay & C. Fox (Eds.), *Changing Ways of Knowing: In English, mathematics and science*. London: Falmer Press.
- [24] Burton, L. (1999) The implications of a Narrative Approach to the Learning of Mathematics, in: L. Burton (Ed) *Learning Mathematics: From Hierarchies to Networks* (London, Falmer Press).
- [25] Ten Have, P. (1999). Doing Conversation Analysis: A Practical Guide. London: Sage Publications.
- [26] Mercer, Neil (2000) Words and Minds: How We Use Language to Think Together. Routledge.
- [27] Lerner, G. H. (1993). Collectivities in action: Establishing the relevance of conjoined participation in conversation. *Text*, 13(2), 213-245.
- [28] Feuerstein , R., Hoffman, M., & Miller, R. (1980). *Instrumental Enrichment*. Baltimore: University Park Press.
- [29] Juravsky, D., Martin, J., Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics and Speech Recognition, Prentice Hall: San Francisco 2000
- [30] Stahl, G. (2002b). The complexity of a collaborative interaction. Paper presented at the *International Conference of the Learning Sciences (ICLS '02)*, Seattle, WA. Retrieved from http://www.cis.drexel.edu/faculty/gerry/cscl/papers/ch02.pdf.