Chapter 13

Inscriptions, Mathematical Ideas and Reasoning in VMT

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Abstract: In this chapter, we trace collaborative problem solving as an interactive, layered building of meaning among learners working as a small group. Our analytic aim is to investigate how students through their inscriptive signs collaboratively build mathematical ideas, heuristics and lines of reasoning in the VMT environment.

Keywords: Discourse, heuristics, reasoning, inscription, combinatorics

Similar to other computer-mediated communication systems, the VMT environment presents communicative affordances and constraints that influence users' discursive interactions. We are interested in how students use the affordances of the virtual environment—including the shared, dynamic whiteboard space, chat feature and referencing tool—as well as what mathematical ideas, heuristics and lines of reasoning are visible in their interactions. In addition, we are interested in how constraints of the system intervene in student discursive interactions.

Online communication systems present affordances and constraints to researchers, as well. VMT presents methodological challenges and opportunities to researchers interested in investigating how students exchange and interactively develop emergent mathematical ideas, heuristics and lines of reasoning. Consequently, we explore an analytic approach for inquiring into the archived interactions of students collaborating on mathematical problem solving through the online dual-interaction space. While analyses of users' online problem solving typically focus on their chat text, in the analysis that we present, for reasons that we will discuss, our analytic attention focuses almost exclusively on the evolution of participants' whiteboard inscriptions as a means to gain insight into the interactive development of their mathematical ideas, heuristics and reasoning as they solve an open-ended mathematics problem.

Conceptual Framework

In this chapter, key conceptual terms include discourse, student-to-student or peer mathematical discussion, collaborative interaction, problem solving, heuristics, mathematical ideas and inscriptions. *Discourse* here refers to language (natural or symbolic; oral, gestic or inscriptive) used to carry out tasks—for example, social or intellectual—of a community. In agreement with Pirie and Schwarzenberger (1988), student-to-student or peer conversations are *mathematical discussions* when they possess the following four features: are purposeful, focused on mathematical notions, involve genuine student contributions and are interactive. We define *collaborative interaction* as individuals exchanging ideas and considering and challenging each other's ideas so as to affect one another's ideas and working together for a common purpose. In the context of the data of this study, the student-to-student, discursive collaborations involve only minimal substantive interaction with a teacher or researcher.

The term heuristics applied to human beings and machines has various uses and meanings in fields as diverse as philosophy, psychology, computer science, artificial intelligence, law and mathematics education. We construe *heuristics* to mean actions that human problem solvers perform that serve as means to advance their understanding and resolution of a problem task. We do not imply that when problem solvers implement a set of heuristics that they will necessarily advance toward a solution but only that their intent is to do so. Our sense of heuristics includes explicit and implicit general strategies such as categories outlined by Pólya (1945/1973, pp. xvi-xvii, 112-114) and others (Brown & Walter, 1983; Engle, 1997; Mason, 1984; Mason, 1988; Schoenfeld, 1985) and pertains to other actions such as a group of problem solvers' decision to assign subtasks to each other to later pool their outcomes to influence their progress on the larger problem at hand (Powell, 2003). Furthermore, we distinguish heuristics from *reasoning*, which we view as a broad cognitive process of building explanations for the outcome of relations, conclusions, beliefs, actions and feelings.

A paramount goal of mathematics education is to promote among learners effective problem solving. In our view, mathematics teaching strives to enhance students' ability to solve problems individually and collaboratively that they have not previously encountered. Nevertheless, the meaning of mathematical *problem solving* is neither unique nor universal. Its meaning depends on ontological and epistemological stances, and on philosophical views of mathematics and mathematics education. For the purposes of this chapter, we subscribe to how Mayer & Wittrock (1996) define problem solving and its psychological characteristics:

> Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver (Mayer, 1992). According to this definition, problem solving has four main characteristics. First, problem solving is *cognitive*—it occurs within the problem solver's cognitive system and can be inferred indirectly from changes in the problem solver's behavior. Second, problem solving is a *process*—it involves representing and manipulating knowledge in the problem solver's cognitive system. Third, problem solving is *directed*—the problem solver's thoughts are motivated by goals. Fourth, problem solving is *personal*—the individual knowledge and skills of the problem solver help determine the difficulty or ease with which obstacles to solutions can be overcome. (p. 47)

Coupled with these cognitive and other psychological characteristics, mathematical problem solving also has social and cultural dimensions. Some features include what a social or cultural group considers to be a mathematical problem (cf., D'Ambrosio, 2001; Powell & Frankenstein, 1997), the context in which individuals may prefer to engage in mathematical problem solving, and how problem solvers understand a given problem as well as what they consider to be adequate responses (cf., Lakatos, 1976). In instructional settings, students' problem solving are strongly influenced by teachers' representational strategies, which are constrained by cultural and social factors (Cai & Lester Jr., 2005; Stigler, 1999). Moreover, with online technologies, the affordances and constraints of virtual environments provide another dimension to the social and cultural features of problem solving since "such technologies are intertwined in the practices used by humans to represent and negotiate cultural experience" (Davis, Sumara & Luce-Kapler, 2000, p. 170) and how problem solvers think and act. Finally, the framing of abstract combinatorial concepts in the cultural context of a "pizza" problem (which is presented in the next section) also offers conceptual affordances and constraints.

In offline as well as online environments, users express objects, relations and other ideas graphically as text and as inscriptions. These are special instances of the more general semiotic category of signs. A *sign* is a human product—an utterance, gesture, or mark—by which a thought, command or wish is expressed. As Sfard notes, "in semiotics every linguistic expression, as well as every action, thought or feeling, counts as a sign" (Sfard, 2000, p.45). A sign expresses something and, therefore, is meaningful and as such communicative, at the very least, to its producer and, perhaps, to others. Some signs are ephemeral such as unrecorded speech and gestures, while others like drawings and monuments persist. Whether ephemeral or persistent, a sign's meaning is not static; its denotation and connotation are likely to shift over time in the course of its discursive use.

As a discursive entity, a sign is a linguistic unit that can be said to contain two, associated components. Saussure (1959) proposes that a sign is the unification of the phonic substance that we know as a "word" or *signifier* and the conceptual material that it stands for or *signified*. He conceptualizes the linguistic sign (say, the written formation) as representing both the set of noises (the pronunciation or sound image) one utters for it and the meaning (the concept or idea) one attributes to it. Examples of the written formation of a linguistic sign are "*chair*" and "*cos2(x)* "—each with associated, socially constructed meanings. Saussure observes further that a linguistic sign is arbitrary, meaning that both components are arbitrary. The signifier is arbitrary since there is no inherent link between the formation and pronunciation of a word or mathematical symbol and what it indexes. A monkey is called *o macaco* in Portuguese and *le singe* in French, and in English the animal is denoted "monkey" and not "telephone" or anything else. The arbitrariness of the signified can be understood in the sense that not every linguistic community chooses to make it salient by assigning a formation and a sound image to some aspect of the experiential world, a piece of social or perceptual reality. Consider, for example, the signifieds *cursor*, *mauve* and *zero*; they index ideas that not all linguistic communities choose to lexicalize or represent.

Signs can be considered to represent ideas. However, Sfard (2000) argues that a sign is constitutive rather than strictly representational since meaning is not only presented in the sign but also comes into existence through it. Specifically, she states,

> Mathematical discourse and its objects are *mutually constitutive*: It is the discursive activity, including its continuous production of symbols, that creates the need for mathematical objects; and these are mathematical objects (or rather the object-mediated use of symbols) that, in turn, influence the discourse and push it into new directions. (p. 47, original emphasis)

This theoretical stance on the mutually constitutive nature of meaning and sign provides a foundation for analysis of the discursive emergence of mathematical ideas, reasoning and heuristics. On the one hand, signs can represent encoded meanings that—based on previous discursive interactions—interlocutors can grasp as they decode the signs. On the other hand, through moment-to-moment discursive interactions, interlocutors can create signs and, during communicative actions, achieve shared meanings of the signs. In this sense, the sameness of meaning for interlocutors that allows for success of their communication is not something pre-existing but rather an *achievement* of the communicative act. This accomplishment may compel interlocutors to bring into existence signs to further their discourse.

Mathematical signs—objects, relations, symbols and so on—are components of mathematical discourse and are intertwined in constituting mathematical meanings. Signs exist in many different forms, and inscriptions or written signs are but one. They are produced for personal or public consumption and for an admixture of purposes: to discover, construct, investigate or communicate ideas. As mathematicians and other mathematics education researchers also emphasize (Dörfler, 2000; Lesh & Lehrer, 2000; Speiser, Walter & Maher, 2003; Speiser, Walter & Shull, 2002), building and discussing inscriptions are essential to building and communicating mathematical and scientific concepts. In a discussion of mathematics and science teaching, Lehrer, Schauble, Carpenter & Penner (2000) illustrate how learners work "in a world of inscriptions, so that, over time, the natural and inscribed worlds become mutually articulated" and illustrate the importance of a "shared history of inscription" (p. 357). In mathematics, the invention, application and modification of appropriate symbols to express and extend ideas are constitutive activities in the history of mathematics (Struik, 1948/1967). Some researchers claim that mathematical meaning only exists through symbols and that symbols constitute mathematical ideas.

For researchers in mathematics education and in computer-supported collaborative learning, the arbitrariness of signifieds is a more significant point about Saussure's observation concerning the arbitrariness of signs. The reason is that the conceptual material that a person (or a small group of people) lexicalizes—for example, with pencil and paper, with text in a chat window or with drawn objects on a shared, digital workspace—indicates to what that user attends, her insight into material reality that is external or internal to her mind. The inscriptions of individuals working online in a small group or team provide observers—who must interpret meanings constituted in the inscriptions—evidence of individual and collective thinking. The small group's inscriptions present ideas it chooses to lexicalize or symbolize. By analyzing the unfolding and use of inscriptions, researchers can understand how participants constitute their mathematical ideas, reasoning and heuristics, the meanings they attribute to their inscriptions, and how their inscriptions influence emergent meanings. As Speiser, Walter & Maher (2003) underscore, what counts as mathematical in analyzing inscriptions is not the inscription itself, which are "tools or artifacts, but rather how the students have chosen to *work*" (p. 22, original emphasis) with their inscriptions. In the specific case of this study, in an online environment that offers resources for individuals to collaborate, what work they interactively accomplish with their inscriptions reveals their ideas, heuristics and reasoning.

Although some of this conceptual framework derives from psychological theories focused on individual cognition, we have tried to show how it essentially involves group and social dimensions. Moreover, it can be interpreted in group-cognitive terms applied to small groups as the creative agents of problem-solving efforts. As will be seen in the following section, the study in this chapter looked at the interactions of a pair of dyads, rather than a small group of individuals, so the active cognizing subjects were themselves cognizing groups.

Method

The data come from a class of undergraduate teacher candidates for positions in urban schools who are enrolled in a semester course—"Mathematics and Instructional Technology"—whose theme is the use of digital technologies for the teaching of mathematics in elementary schools. This data differs from the PoW-wow and Spring Fest data in many of the other chapters of this volume in that it comes from a college classroom context where the chat was part of a larger curriculum (compare Chapters 23 and 24). The second author taught this course, which was developed by the first author. During a particular class session, students worked on an open-ended problem, the Pizza Problem, interacting in chat-room teams of four through the online, collaborative VMT environment. When students enter their assigned chat room, they are presented the problem shown in Figure 13-1.

The Pizza Problem

A local pizza shop has asked us to help them keep track of pizza sales. Their standard "plain" pizza contains cheese with tomato sauce. A customer can then select from the following toppings to add to the whole plain pizza: peppers, sausage, mushrooms, bacon, and pepperoni.

How many different choices for pizza does a customer have?

List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities.

Figure 13-1. The pizza problem.

We chose this mathematical problem for three reasons: (1) it relates to the course module, which concerned number and algebra, (2) its context is familiar to students from urban and suburban communities and (3) mathematically it affords different solution approaches, ranging from simple listing procedures to more advanced methods involving combinatorial analysis.

Epistemologically, we view learning or knowledge creation as a process of conceptual change whereby individuals and groups of individuals construct new understandings of reality. Through social interactions, learners engaged with mathematics seek meaning and search for patterns, relationships and dynamics linking relationships among objects and events of their experiential world.

Our data sources are the mathematical problem and the persistent computer log of the chat-room interactions from the dual-interaction spaces that the VMT environment provides. To investigate the online, problem-solving actions of learners so as to understand how they build mathematical ideas, heuristics and reasoning, we code for instances in the data of their discursive attention to any of four markers of mathematical elements: objects, relations among objects, dynamics linking different relations and heuristics (Gattegno, 1988; Powell, 2003). In their chat text and whiteboard inscriptions, participants either communicate affirmations or interrogatives about these mathematical elements. We attend to eight different critical events that provide insight into learners' general mathematical behavior. We use both inductive and deductive codes to make sense of the data. The matrix in Table 13-1 contains deductive codes we used to flag these critical events in the chat text and whiteboard inscriptions. We also coded the data for emergent themes as related to our research questions. These include ones about interactional behaviors (II for participant initiating an interaction) and about reasoning (RC for reasoning by cases and CV for controlling variable). We will provide an example of how we coded a version of our data in Table 13-2.

Table 13-1. Matrix of event types.

Subject and type of utterance or inscription	**O**bjects	**R**elations among objects	**D**ynamics linking different	**H**euristics

			relations	
Affirmations	**AO**	**AR**	**AD**	**AH**
Interrogatives	**IO**	**IR**	**ID**	**IH**

It is possible that an interaction receives multiple codes. We analyze the mathematical ideas and forms of reasoning that learners produce working interactively in dyads in a chat room, tracing the development of their ideas and reasoning patterns over the course of the problem-solving session.

We grouped students into teams as they arrived in the classroom. Each team consisted of four students and was assigned to a chat room. In one virtual chat room, students were grouped in dyads, each dyad at one computer. In the other chat room, three students shared a computer and one student was alone at a computer.

For this case study, we analyze data from one chat room, the one involving two dyads of students. After reviewing data of both chat rooms, using the VMT Replayer, we chose this dataset, realizing that given its paucity of chat text (compared to the wealth of whiteboard inscriptions) this chat would provide a particularly interesting analytic challenge. In what follows, we refer to the two students in each dyad collectively, using an abbreviation of the screen name of the one individual of the dyad who signed into the chat room. We refer to the first dyad as Silvestre; the participants are Sonia and Lyndsey, and they used Sonia's screen name, SOSilvestre, in the chat room. We refer to the second dyad as Suzyn; the participants are Susan and Komal, and they used Susan's screen name, suzyn17, in the chat room. In this report of our case study, although we are speaking of two dyads of students, to simplify things, we will refer to each dyad in the female singular as Silvestre and Suzyn for the sake of simplicity in our narrative. Although the dyads were co-located, they were asked to interact only through the chat room, pretending that they were located at distant sites.

In analyzing our data, we realized that the data for this particular study provided an analytic challenge that had to be overcome to make sense of the chat room interaction of the participants. Specifically, the participants hardly interacted in the chat frame of VMT and used the whiteboard almost exclusively. This meant that we had to follow the evolution of their inscriptions on the whiteboard to understand the emergence of their mathematical ideas and reasoning as they solved the Pizza Problem. To analyze the evolution of the whiteboard inscriptions, we adapted a video-data analytic technique used for qualitative investigations into the development of learners' mathematical ideas and reasoning (Powell, Francisco & Maher, 2003). This approach allows us to view our replayed data much as we would a video recording, through four recursive stages.

Our *first* analytic move was to view attentively the data in the VMT Replayer several times at various speeds to familiarize ourselves with the real-time sequence of whiteboard actions and chat text postings. Afterwards, we discussed our sense of the data amongst ourselves. Also, as part of a professional development program for teacher candidates of secondary mathematics, we engaged undergraduate mathematics students in viewing and discussing the data.[1]

After these initial viewings of the data, our *second* analytic move was to step carefully through the data with the VMT Replayer to create an objective description of actions that transpired in the chat and whiteboard spaces. We created these descriptions for each five-minute interval.

Following the descriptions, our *third* move was to code the data deductively and inductively, while also writing analytic, interpretative notes of the problem solving and other interactive accomplishments occurring in the session. For the deductive codes, we used the markers of attention to mathematical elements indicated in Table 13-1. For the inductive coding, we inquired into the heuristics and lines of reasoning evident in the data as well as to how the participants manage affordances and constraints of the virtual environment. We present the results of our coding in the next section of this report. In Table 13-2, we present an example of a description, interpretation and coding of three intervals of the chat-room actions, each less than a minute long, in three respective columns. In the three intervals (rows of the table) Silvestre contributes to Suzyn's solution, and then Suzyn subsequently critiques this addition and induces Silvestre to make further changes. In the interpretation column, for each interval, we include rationale for our coding of a particular chunk of data. The letters, EC, which stand for "explanation of code," precedes these rationales.

Table 13-2. Time interval description.

Example of time-interval description, interpretation, and coding of chat room (chat text and whiteboard inscriptions) data 12:50:02 – 12:50:16	SOSilvestre creates an ellipse filled with the color red below the ellipse containing the textbox containing "**M/B/R**." Within this red ellipse, SOSilvestre creates a textbox and types "**P/S/R.**"	SOSilvestre creates an ellipse on suzyn17's side, containing a textbox listing a pizza with pepper and two other toppings, presumably because SOSilvestre is done with her work, and wants to help out suzyn17. SOSilvestre seems to color the pizza red to have more fun with the problem. This seems to be the second attempt to collaborate since suzyn17 wrote "Plain Pizza" into the chat window. EC: (AO) SOSilvestre creates a pizza containing peppers, sausages, and pepperoni as toppings on Suzyn17's side of the whiteboard EC: (AR) By creating a pizza for Suzyn17, SOSilvestre engages in a relation among the objects on Suzyn17's side of the whiteboard. EC: (II) By creating a pizza for Suzyn17, SOSilvestre essentially initiates an interaction with Suzyn17, although the "interaction" here is not verbal.	AO, AR, II
12:56:24 – 12:56:49	Suzyn17 types into the chat window "**WHO COLORED MY PIZZA?**"	EC: (AO) SOSilvestre creates a pizza with peppers as the only topping on Suzyn17's side of the whiteboard. She then deletes this pizza.	AO, AR, II

[1] These students are teacher candidates for teaching high school mathematics in economically impoverished, urban school districts and recipients of Robert Noyce scholarships, sponsored by the US National Science Foundation and administered through a joint project of Rutgers University, New Jersey Institute of Technology, the Newark Public Schools and the Newark Museum.

	SOSilvestre types "**i did I did**". SOSilvestre types "pizza red right?" SOSilvestre types "lol".	EC: (AR) By creating an additional pizza on Suzyn17's side of the whiteboard, SOSilvestre engages in a relation among the objects on Suzyn17's side. EC: (II) By asking "**WHO COLORED MY PIZZA?**" in the chat window, Suzyn17 attempts to initiate an interaction with SOSilvestre in the chat window around the pizza that SOSilvestre has drawn for Suzyn17.	
12:57:08 – 12:57:43	SOSilvestre adjusts the size of the ellipse containing the textbox containing "**S/M/B/R**". suzyn17 types "**WHERE'S THE CHEESE?**" SOSilvestre colors the textbox containing "**P/S/R**" yellow. SOSilvestre types "**there it is**".	EC: (AO) SOSilvestre colors yellow the peppers, sausages, and pepperoni pizza on suzyn17's side.	AO

Our third analytic move proceeded from our interpretations and EC rationales. We chunk the data by reorganizing them into specific categories based on the deductive and inductive codes. This allowed us further to understand the actions the team takes to make sense of the problem and the sequence of subsequent actions the participants perform to present and refine their solutions. In this stage, we also create a story line, deciding how the data informs our research question and what other interpretive frames the data suggest. The *fourth* stage of our analytic process was to compose a narrative, the report that you are reading.

Our trajectory of analytic moves is far more recursive than the linear description we have just provided. For instance, we refined and corrected the description as we coded and composed interpretations of chunks of data. In some instances, deductive and inductive coding occurred almost simultaneously.

Results

With regards to our inquiry into the cognition of the team of participants, our investigation concerns two guiding questions: (1) How do learners interactively build (externally represented) mathematical meanings by collaborating in small groups, using a computer-mediated communication system? (2) In the process, what mathematical ideas, heuristics and reasoning do they develop? These are overarching questions of our research program. The data that we analyze here represents a small, preliminary case study. We present the results along several dimensions: interaction, heuristics, mathematical ideas, mathematical reasoning. Afterward, we discuss issues that emerge from our results and conclude with implications of our case study.

The data of this case surprised us in that the team communicates sparingly with chat text and mainly through whiteboard postings. In our experience, most teams use the chat space to a much greater extent than this team does. Consequently, our analysis of the mathematical ideas and reasoning that the students engaged is not primarily based on their textual communication, but rather mainly on an examination of the evolution of their inscriptive whiteboard interactions.

Interaction

The student participants worked in dyads and the two dyads, as a team, interacted through the VMT system using two interaction spaces, the chat and the whiteboard frames. The dyads used the chat room to work through the problem, with one student of each dyad controlling the mouse and keyboard. The two dyads interacted with each other for the vast majority of the time through inscriptive postings on the whiteboard. In the nearly two hours of interaction, the students rarely used the chat frame to communicate with the other dyad.

Our analyses of the data reveal how participants use the affordances of the VMT environment, how they managed constraints they encountered in it, and what mathematical ideas, heuristics and lines of reasoning are evident in their collaborative interactions. The initial work of the online group can be read as establishing its bearings. These include how to work within the affordances and constraints of the VMT environment, how to manage the shared workspace and how to represent the object with which they will work.

Interactive Initiation of Inscriptive Phases

The two dyads of participants, collaborating in a single chat room, develop inscriptions or, more specifically, discursive objects or artifacts that serve to simultaneously represent and beget their mathematical ideas and reasoning as they build solutions to the problem. As Sfard (2000) notes, "mathematical discourse and its objects are *mutually constitutive*" (p. 47, original emphasis). While building their solutions, the development of discursive objects occurs in what we discern as phases.

Phase 1 is initiated when the dyad of participants, Silvestre, experiments with drawing ellipses, which seem to be analogous to pizza pies. The participant dyad, Suzyn, then also experiments with drawing ellipses.

Phase 2 entails labeling ellipses. Suzyn types "**Plain Pizza**" in the chat frame and uses the reference tool to link this chat statement with an ellipse on the whiteboard. Afterward, Silvestre creates a textbox in an ellipse and types "**plain T & C**," establishing that it perhaps is more convenient to indicate a pizza and its topping such as a plain tomato and cheese pizza with a textbox superimposed onto an ellipse rather than linking a chat statement with an element—an ellipse—drawn on the whiteboard. With this action, Silvestre appears to offer an implicit proposal. Both labeling approaches seem to be cumbersome for the participants, and in the next phase, each dyad modifies their approach.

In *phase 3*, apparently influenced by Silvestre's use of a textbox superimposed onto an ellipse, Suzyn incorporates this technique into her representation. Each ellipse that Suzyn creates is labeled with a textbox and represents a specific pizza with particular toppings. Suzyn employs this iconic representation for most of the remaining time in which she works. By this point, Silvestre and Suzyn type on separate parts of the whiteboard. Silvestre uses the left side while Suzyn uses the right side.

In their modified representations, each dyad uses the symbol, P. However, what does P represent, peppers or pepperoni? Silvestre settles the question by creating a key in which she indicates what letter represents what topping: P for peppers, S for sausage, M for mushroom, B for Bacon and R for pepperoni. In a different way, Suzyn also announces what P stands for. She creates a textbox, types "**PEPPERS**" into it and lines up in a column under this heading her three pizzas that contain P: P/B, P/S and P/M. Instead of an ellipse representing a class of pizzas, each ellipse represents a different pizza, differentiated from the others by its topping. These objects or pizzas are also similar to each other in that each contains two toppings, one of which is P, indicating that Suzyn is engaged with relations among objects. This pattern is indicative of thinking about grouping different possible pizzas by cases. In this instance, the case is two-topping pizzas with each including P as a topping. Suzyn employs this iconic representation for most of the remaining time in which she works (see Figure 13-2).

While Suzyn modified her representation, also in phase 3, Silvestre changes her notational scheme and develops a symbolic inscription. To indicate the mathematical objects with which she is working, she types **P/S**, **P/M**, **P/S** and **P/R** into a single textbox superimposed on an ellipse. Now, a single circular ellipse is not a single pizza but represents a class of pizzas. Her notation's structure appears to be the following: a single pizza has two toppings and the toppings on a pizza are separated with slashes. Her inscription also indicates a relation among the objects with which she is engaged; namely, each object is a two-topping pizza with **P** as one of its toppings. Moreover, this pattern is suggestive of a strategy by which she may intend to list different possible pizzas. In this instance, it is grouping different, possible pizzas by cases. In this case, it is two-topping pizzas with **P** as one of the two toppings (see Figure 13-2).

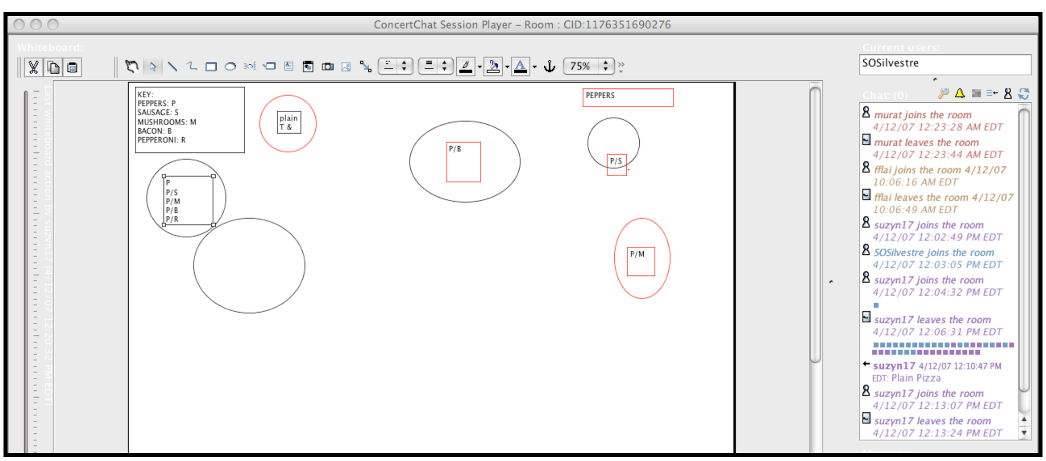

Figure 13-2. Screenshot of phase 3.

Silvestre further modifies her notational scheme by removing forward slashes. Instead of using **P/M** or **P/S** to represent pizzas with peppers and mushrooms or pizzas with peppers and sausages, respectively, Silvestre uses **PM** and **PS** to represent these pizzas. In addition, she expands upon her representation and uses it to designate pizzas with more than two toppings. For instance, a pizza with sausages, bacon and pepperoni is represented by **SBR**.

In *phase 4*, Suzyn finally seems to adopt Silvestre's notational inscription to display her way of reasoning about a solution to the problem. Suzyn develops a symbolic inscription. To display her solution, she moves Silvestre's inscriptions to the bottom of the whiteboard (see Figure 13-3, bottom center). Placing each case within a textbox, Suzyn lists and enumerates pizzas containing certain numbers of "**combinations**." She lists one pizza with "**0 Combinations**" or no toppings, five pizzas with "**1 Combination**" or one topping, ten pizzas with "**Two Combinations**" or two toppings, ten pizzas with "**Three Combinations**" or three toppings, five pizzas with "**Four Combinations**" or four toppings and one pizza with "**Five Combinations**" or five toppings. Like Silvestre, Suzyn uses combinations of letters as the objects with which she exhibits her thinking about different possible pizza pies and relationships among these possibilities, but groups her pizzas according to total number of toppings as opposed to common toppings (see Figure 13-3, textbox on left).

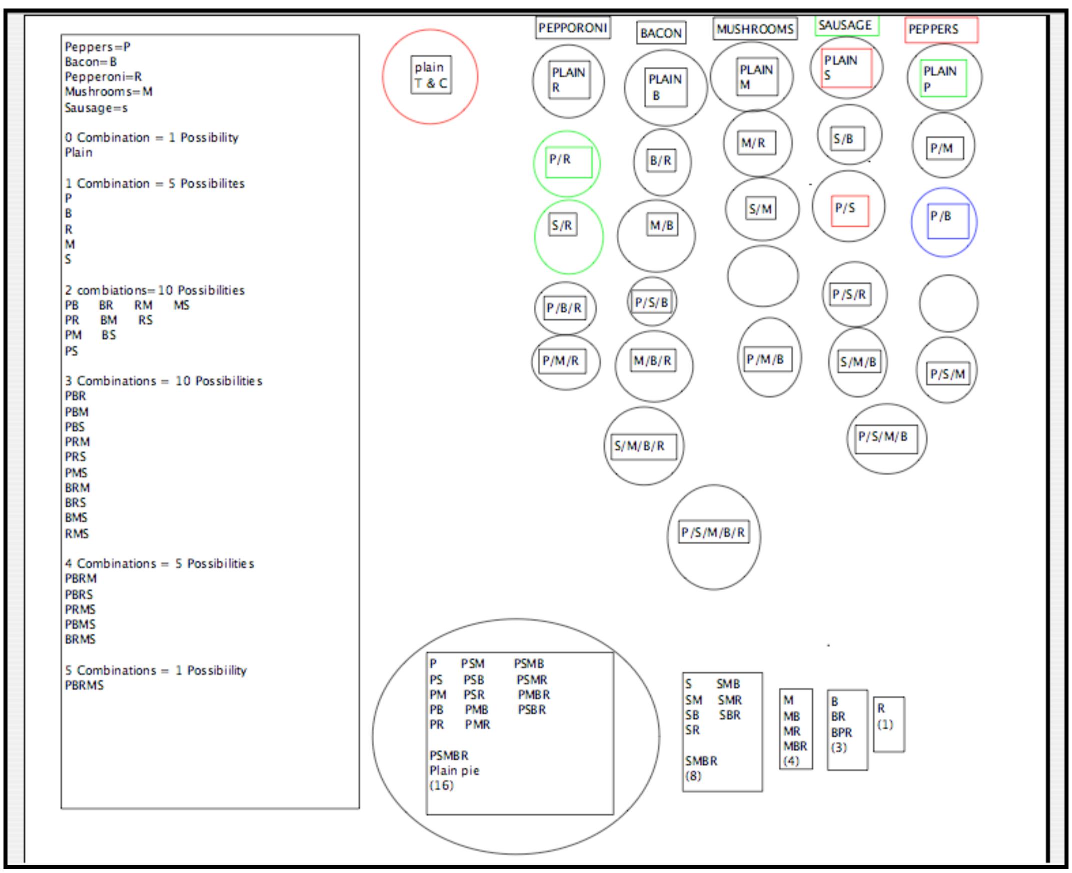

Figure 13-3. Screenshot of phase 4.

Representing Objects and Engaging with Relations among Objects

As we have seen, the groups develop two different representations for the objects with which they develop mathematical ideas and reasoning. The dyad designated by Suzyn initially uses an iconic inscription for each of their pizzas. It consists of an ellipse formed into a circle and a textbox with letters. The letters **P**, **S**, **M**, **B** and **R** are toppings and combinations of them are placed in a textbox atop an ellipse. Suzyn uses the two inscriptions—ellipse and a non-empty textbox—to represent a particular pizza pie.

Different from Suzyn's iconic representation, Silvestre develops a symbolic inscription. She uses combinations of letters as the objects with which she exhibits her thinking about different possible pizza pies and relationships among these possibilities. For instance, **P**, **S**, **M**, **B** and **R** stand for objects or pizza toppings and combinations of these letters such as **M**, **PS** or **SBR** designate different possible pizzas. In her semiotic system, Silvestre uses a letter or combination of letters to represent both particular toppings and pizza pies with particular toppings. That is, **P** can stand for one of the available toppings (peppers) or a one-topping pizza (of peppers). Unlike Suzyn's inscriptive system, where two distinct types of inscriptions represent toppings and pizzas with toppings, Silvestre's symbols play dual roles. Later Suzyn will appreciate the economy of this semiotic system and she will shift her notational usage.

Interestingly, although at the start of the group's problem-solving session Silvestre initiated constructing ellipses on the whiteboard and used textboxes to label an ellipse—such as when she created a "**plain T & C**" pizza—the chore or cumbersomeness of drawing and labeling within the whiteboard may have contributed to her development of another, more convenient representation. Drawing elliptical shapes and creating textboxes on the shared workspace are affordances of the system, which at the same time represent a constraint because of mechanical or motor difficulties involved in creating and coordinating these objects. This constraint may have impelled Silvestre to find a less representational, more symbolic and therefore computationally more powerful inscription.

With her inscriptive objects, Silvestre engages with relations among their objects. Just as Silvestre and Suzyn developed different representations, they also engage with different relations among the objects or pizzas. Suzyn indicates relations among her objects spatially by locating pizzas that contain a particular, common topping, like peppers, under a column head by the name of the common topping. The column headed by "**Peppers**" has four pizzas each containing peppers with one different other topping and one pizza with just peppers as its topping; the column headed by "**Sausage**" has three pizzas each containing sausage with one different other topping and one pizza with just sausages as the topping; the column headed by "**Mushroom**" has two pizzas each containing mushrooms with one different other topping and one pizza with just mushrooms as the topping; the column headed by "**Bacon**" has one pizza containing bacon with one different other topping and one pizza with just bacon as the topping; the column headed by "**Pepperoni**" has one pizza with just peppers as the topping. Each successive column had one less pizza than the one before it because it does not include the topping used in the previous column. Suzyn seems to realize this before labeling her pizzas since, as she went along, she drew just the right number of ellipses under each column heading.

Silvestre presents her perception of relationships among objects, the different possible pizzas. In turn, she considers each available topping and, in separate textboxes, lists all possible pizzas that contain it as a topping (see the five textboxes at the bottom of Figure 13-3). That is, first, she lists all possible different pizzas containing P or peppers; second, all possible, different pizzas containing **S** or sausage, except for those that contain **P** since they were already accounted for; third, all possible, different pizzas containing **M** or mushroom, except for those that contain **P** or **S** since they have already been accounted for; fourth all possible, different pizzas containing **B**, except for those that contain **P**, **S** or **M** since they have already been represented, and finally, all possible, different pizzas containing **R**, except for those that contain **P**, **S**, **M** or **B** since they have already been indicated.

Engaging with Dynamics Linking Different Relations

The work of Suzyn and Silvestre evidence their engagement with dynamics linking different relations or, in other words, relations among relations. Silvestre listed, for example, pizzas containing peppers, **P**, in a textbox. This listing by itself is a relation. Ultimately, she arranged the possible pizzas containing in turn each of the five available toppings into separate textboxes. The textbox containing pepper pizzas is to the left of the textbox containing sausage pizzas, which is to the left of the textbox containing mushroom pizzas, which is to the left of the textbox containing bacon pizzas, which is to the left of the textbox containing the pepperoni pizza. Her inscriptive and spatial work indicates that Silvestre views each listing as distinct from the others. In this sense, she is also engaged with dynamics linking—by distinction—different relations.

In an analogous manner, Suzyn signals her engagement with relations among relations. She lists different possible pizzas by considering cases. In the long, rectangular textbox on the left in Figure 13-3, Suzyn lists in turn all possible pizzas with 0 toppings, 1 topping, 2 toppings, 3 toppings, 4 toppings and 5 toppings. Each case indexes a relation and is distinct from the others. Her listing indicates Suzyn's engagement with dynamics linking different relations.

Each of the participants—Suzyn and Silvestre—considers different dynamics linking different relations. The structure of their thinking in this regard reveals different perceptions of the underlying mathematical structure of the problem. We elaborate on this in the discussion section below.

Inventing Heuristics

Both Suzyn and Silvestre seemed to invent heuristics based on the resources within the VMT environment. For example, both started off by drawing ellipses using the ellipse tool. It seems that Suzyn then realized, after using the referencing tool to label an ellipse as a plain pizza, that the textbox could be better used for this purpose. Thus, for the early part of her work session, Suzyn used ellipses labeled by textboxes to represent her pizzas. Her solution representation is iconic.

For Silvestre, the drawing of ellipses may have seemed too cumbersome. Silvestre used a symbolic method of representation. Specifically, she used the textbox tool to list pizza possibilities. Within separate textboxes, Silvestre listed pizzas containing peppers as a topping, pizzas containing sausage as a topping that have not already been listed, and so on.

Suzyn's evolution of heuristic use from iconic representation to symbolic representation may have been influenced by Silvestre's use of symbolic representation in her solution method. That Silvestre was able to list more pizzas with her method than Suzyn was able to list with her iconic representation may have influenced Suzyn to use a symbolic representation to complete her solution. Interestingly, although both Suzyn and Silvestre end up with symbolic representations, their solutions are quite different.

Reasoning about Possibilities

The work of the team and of each of the two dyads in the team exemplifies particular types of mathematical analysis: *reasoning by cases* and *reasoning by controlling variables.* The teams of Silvestre and Suzyn both begin their work by indicating possible pizzas with two toppings in which one is P, peppers. On the one hand, this line of reasoning continues to dominate the work of Suzyn throughout the session. *Suzyn reasons by cases by counting and listing pizzas with one topping, pizzas with two toppings, pizzas with three toppings and pizzas with four toppings*. Suzyn continues reasoning by cases by listing additional combinations in her textbox. She lists the combination of no toppings, a plain pizza and the combination of all toppings, a pizza containing peppers, sausages, mushrooms, bacon and pepperoni.

On the other hand, Silvestre shifts from reasoning by cases to reasoning by controlling for variables. When Silvestre creates a textbox and types in four pizzas containing peppers as a topping with the combinations of peppers and sausage, peppers and mushroom, peppers and sausage, and peppers and pepperoni, this is the first instance of reasoning by cases. Later on, *Silvestre controls for the variable P, as she lists one-, two-, three- and four-topping pizzas containing peppers*. Silvestre then creates a textbox and lists pizzas containing sausages, sausages and two other toppings, and sausages and three other toppings.

Within each textbox, Silvestre also engages in reasoning by cases. She adjusts her list of pizzas with peppers so that one-, two-, three- and four-topping pizzas all appear in separate columns. Before the adjustment, pizzas with three and four toppings appeared in the same column. In a similar fashion, Silvestre arranges her listing of pizzas containing sausage, not containing peppers, by grouping the possibilities according to the number of toppings.

Discussion

Our aims were to investigate—based on data gathered from chat-room participants' mathematical problem solving within the VMT environment—how to study chat-room participants' development of mathematical ideas and lines of reasoning, and what ideas and reasoning are evident in the data. In the following sections, we discuss the significance of the results in the light of our theoretical and cognitive perspectives.

Discourse Creating Objects and Objects Shaping Discourse

To explore, develop and communicate their mathematical ideas, the four students—acting as dyads Suzyn and Silvestre—interactively unfold an inscriptive system composed of objects as well as implicit relations among the objects and relations among the relations. After entering their assigned chat room in the VMT environment and after reading the statement of the Pizza Problem, the students experiment drawing circular ellipses and initially default to pictorial or iconic representations of pizzas. Suzyn types "Plain Pizza" and uses the reference tool to link this chat statement to an ellipse on the whiteboard. Immediately afterward, Silvestre creates a textbox superimposed on an ellipse and types "plain T & C," which we understand to mean a plain pizza of tomato and cheese. In subsequent actions of creating objects on the whiteboard, Suzyn incorporates Silvestre's technique of drawing an ellipse and labeling it by typing into a textbox superimposed on the ellipse. Each ellipse that Suzyn creates is labeled with a textbox and represents a specific pizza with particular toppings, with each of the toppings separated by a slash. Each ellipse also represents a different pizza, differentiated from the others by its indicated toppings. Both Suzyn and Silvestre use ellipses in their representations of pizzas perhaps because representing pizzas in a pictorial manner makes the problem more personal, less abstract and easier to work with in early stages of their thinking. Their initial discourse in the chat and whiteboard spaces concerns experiments with designs for the objects on which they will work.

After Silvestre experiments with using an ellipse labeled with a textbox as a way of indicating pizzas, she changes from an iconic to a symbolic representational scheme. The chore of drawing and labeling within the VMT system may have contributed to her development of a less pictorial, more symbolic, and therefore, computationally more powerful inscription.

To indicate the mathematical objects with which they are engaged, Silvestre's initial symbolic inscription involved a list of letters and slashes—P/S, P/M, P/S and P/R—typed into a single textbox superimposed on an ellipse, indicating pizzas and their toppings. The structure of the notation appears to be the following: each group of two letters with a slash between them is a single pizza with two toppings with each topping indicated by a letter. The ellipse is not a single pizza but indexes a class of pizzas and a relation among them. The relation that it indexes seems to be all two-topping pizzas containing peppers, P. This pattern may suggest how Silvestre intends to list different possible pizzas, distinguishing classes of pizzas by means of ellipses.

There is an interaction between Silvestre's objects and her problem-solving strategy. The objects push her discourse in new directions. Silvestre modifies and extends her inscriptive and problem-solving strategy. She uses combinations of letters without slashes as objects to represent different possible pizza pies and relationships among these possibilities. For instance, **P**, **S**, **M**, **B** and **R** stand for the five different pizza toppings and combinations of these letters such as **M**, **PS** or **SBR** designate different possible pizzas. Silvestre uses a letter or combination of letters to represent both particular toppings and pizza pies with particular toppings. That is, **P** can stand for one of the available toppings (peppers) or a one-topping pizza (of peppers).

Silvestre's development of a more cogent and computationally powerful inscription parallels shifts in her discourse. That is, this notational scheme allows her to not only illustrate pizzas with different combinations of topping but also to engage with patterns and relationships of these combinations and to use these patterns and relationships to engage with and illustrate relations among the relations. In her final solution, she presents in five different textboxes different classes of pizzas: first, all different possible pizzas containing peppers, **P**; second, all pizzas containing sausage, **S**, but not containing peppers; and so on. Examining these textboxes makes Silvestre's strategy for listing pizzas evident. In the textbox with pizzas containing peppers, a one-topping pizza containing peppers is listed first. Then, for each two-topping combination containing peppers, peppers is listed first, followed by single-topping combinations of sausages, mushrooms, bacon, or pepperoni, listed in this order. A similar systematic strategy is followed for pizzas containing other toppings. Note that the order in which pizzas containing certain toppings are presented (pizzas containing peppers, pizzas containing sausages, pizzas containing mushrooms, pizzas containing bacon, and finally pizzas containing pepperoni) is the same as the order of the toppings presented in each textbox (Figure 13-4).

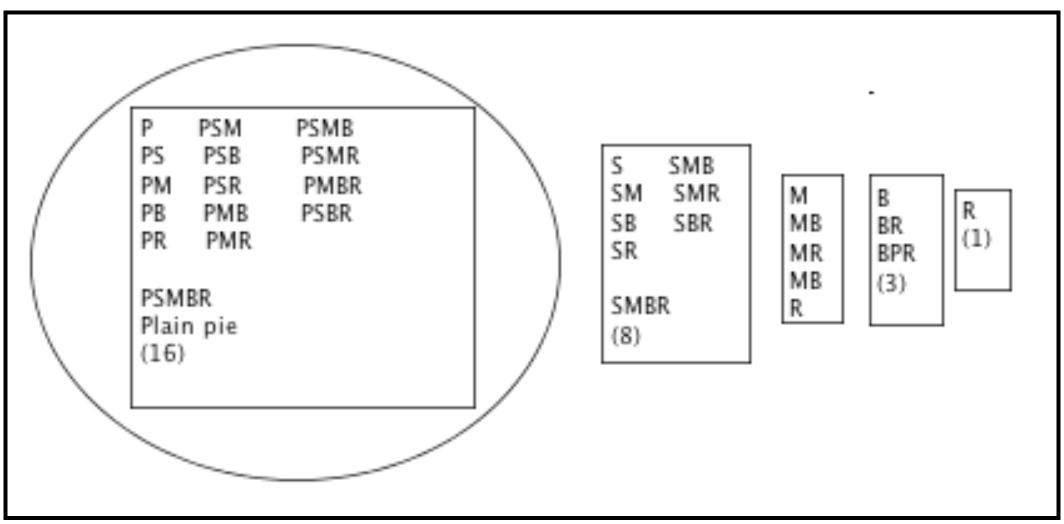

Figure 13-4. Screenshot of Silvestre's final solution.

The content of the textboxes displays particular relations among the pizzas and the different textboxes distinguish relations among these relations. This inscriptive system that Silvestre develops illustrates the theoretical point about the signs learners choose and how their signs provide an analytic window into the signified field of conceptual material or ideas with which they engage (Powell, 2003).

The team's initial and later work to create and use their mathematical objects exemplifies another theoretical point. Sfard (2000) theorizes, "mathematical discourse and its objects are *mutually constitutive*" (p. 47, original emphasis). Through their discourse, the students in our data develop approaches to represent the objects on which they work in their solution space. They consider and modify an initial proposal for how to represent their objects—pizzas with particular toppings. Each dyad elects to work with a different representation, one iconic and the other symbolic. The emergence to these inscriptive systems usher into the discourse two directions of work toward a solution of the problem. Indeed, the process of designing objects shapes their respective solution space. Silvestre's symbolic representation supports her reasoning about the different possible pizzas as collections in which they control variables, holding **P** (peppers) fixed and listing first all possible pizzas containing **P**. Though Suzyn's iconic representation supports her reasoning—cases defined by the number of toppings—it proves cumbersome and inefficient. Toward the end of the problem-solving session, she abandons it in favor of Silvestre's symbolic representation. The iconic representation communicates the physicality of a pizza—an ellipse—and in a textbox displays its toppings. The symbolic representation—concatenated letters—simultaneously lists the toppings of a pizza and stands for the pizza itself. The meaning of the objects and the meaning presented through the objects are constituted through their use. From their discursive interactions in the two interactive spaces of VMT, the teams implicitly agree that what distinguish pizzas from one another are their toppings. Therefore, it is sufficient to list their toppings without having to draw pictures of pizzas. Through their discursive interaction the team constitutes the objects and, in turn, their objects shape and advance the discourse. This point is further evidenced in the next section.

Dyads Influencing Dyads

When Suzyn and Silvestre enter the VMT space, they both begin by drawing ellipses. Suzyn uses a chat posting of "**Plain pizza**" to link to one of her ellipses as a way of labeling it as a plain pizza. Silvestre takes one of her ellipses and places a textbox inside, and labels it "**plain T&C**." Suzyn seems to be influenced by this and subsequently uses this notational scheme to develop her solution.

While both dyads use letters to represent the pizza toppings, the letter **P** can represent either peppers or pepperoni. Silvestre settles this problem by creating a key in which she indicates what letter represents what topping: **P** for peppers, **S** for sausage, **M** for mushroom, **B** for Bacon and **R** for pepperoni. In a different way, Suzyn also announces what **P** stands for. She creates a textbox, types "**PEPPERS**" into it, and lines up in a column under this heading her three pizzas that contain **P**. After Silvestre creates this key, Suzyn appears to adopt Silvestre's notation.

Unlike Suzyn's inscriptive system, where two distinct types of inscriptions represent toppings and pizzas with toppings, Silvestre's symbols play dual roles. The economy of this semiotic system is appreciated by Suzyn and she shifts her notational usage to a symbolic notational scheme.

Throughout the session, both Suzyn and Silvestre influenced each other in various ways. In the beginning of the session, Silvestre was influenced by Suzyn to use ellipses to represent pizzas, but used textboxes instead of linked chat statements to label the ellipse. Later on, Silvestre switched to a symbolic representation of pizzas, perhaps seeing that the iconic representation was not suitable for generating large numbers of possibilities. Near the end, that Silvestre was able to list more pizzas with her method than Suzyn was able to list with her iconic representation may have influenced Suzyn to use a symbolic representation to complete her solution.

Within the session, through mutual influences the gradually shared semiotic conventions were established as a system of shared meaning underlying the on-going social practices or methods jointly available to the two dyadic participants.

Dyadic Reasoning

Interestingly, Suzyn and Silvestre engage similar reasoning processes but with different inscriptive results. They both reason by cases. Moreover, within each case, they reason by controlling variables. However, their distinct inscriptive results emerge from their differentiated cases. In her final symbolic inscription, Suzyn lists her pizzas by separating them into cases, according to the number of toppings and, within each case, controls variables. Under 0 topping pizzas, she lists a plain pizza (tomato sauce and cheese). Under 1-topping pizzas, she lists a pizza containing only peppers as a topping, then a pizza containing only bacon, followed by a pizza containing only pepperoni, then a pizza containing only mushrooms, and finally a pizza containing only sausages. Under 2-topping pizzas, she lists all pizzas containing peppers and one other topping; then all pizzas containing bacon and another topping different from peppers; then lists all pizzas containing pepperoni and another topping different from peppers or bacon; followed by all pizzas containing mushrooms and another topping different from peppers, bacon, or pepperoni. At this point, she has exhausted all of the possibilities for 2-topping pizzas. She continues to engage reasoning by cases and by controlling variables to list all her 3-, 4-, and 5-topping pizzas.

As for Silvestre, she lists her pizzas by separating them into cases of pizzas containing peppers, pizzas containing sausages but excluding those previously listed, pizzas containing mushrooms but excluding those previously listed, pizzas containing bacon but excluding those previously listed, and pizzas containing pepperoni but excluding those previously listed. Within each case, she controls variables by indicating the common topping first and followed methodically varying other toppings. For example, for pizzas containing peppers, she lists the following ones: **P, PS, PM, PB, PR, PSM, PSB, PSR, PMB, PMR, PSMB, PSMR, PMBR, PSBR**. The peppers topping is always listed first, followed by variations containing sausage, mushroom, bacon, and pepperoni always listed in that order. Afterward, she methodically lists all possible pizzas containing sausage and other toppings, not including in the list those pizzas containing sausage that were already indicated in the previous list of pizzas containing peppers. Working in this fashion, each of her cases is separated in a different textbox and within each she controls variables. Since Silvestre's cases are distinct from those of Suzyn, her inscriptive result also differs.

By the end of the session, Suzyn's thinking has progressed to the point where Silvestre was earlier. Both are now thinking beyond the idea of just generating different pizzas, but rather of producing combinations of toppings to generate patterns, and to use these patterns to ensure that they have accounted for all combinations.

Viewed as individual actors, Suzyn and Silvestre (who are themselves actually each a dyad of human students) are seen to influence each other while still maintaining different perspectives on their shared problem. Viewed as an interacting small group, they develop shared meanings, a joint problem space, common methods and accepted practices. These elements of group cognition can be seen to emerge and evolve out of the situation or activity context including the driving problem, the technical environment and the unfolding interaction.

Mathematical Significance

As a small group, Suzyn and Silvestre built sophisticated cognitive structures that can provide insight into Pascal's triangle and combinatorial analyses. From an analytical viewpoint, we find these structures to be significant since they establish cognitive foundations upon which students can build and extend their understanding of binomial structures. In Figure 13-4, Silvestre's representation of her solution nearly mimics successive rows of Pascal's triangle. Her listing of pizzas with peppers almost represents the fourth row of Pascal's triangle (see Figure 13-5). First, she lists a pizza with peppers, pizzas with peppers and one other topping, pizzas with peppers and two other toppings, pizzas with peppers and three other toppings and a pizza with peppers and four other toppings. The number of pizzas in each of these sub-categories is the same as one of the numbers in the fourth row of Pascal's triangle: 1 4 6 4 1. That is, there is one pizza with peppers only, four pizzas with peppers and one other topping, six pizzas with peppers and two other toppings, four pizzas with peppers and three other toppings and one pizza with peppers and four other toppings.

Zeroth row	1
First row	1 1
Second row	1 2 1
Third row	1 3 3 1
Fourth row	1 4 6 4 1
Fifth row	1 5 10 10 5 1

Figure 13-5. The initial rows of Pascal's triangle.

Combinatorially speaking, for the case of all possible pizzas containing peppers, since each pizza must have peppers as a topping, there are four remaining toppings from which to choose. Using combinatorial notation, $\binom{n}{r}$, which means the number of ways to select r items from a collection of n of them, the following are the possible pizzas containing peppers:

- Pizzas with peppers only $=\binom{4}{0}=1$, since out of four choices of toppings none are chosen,
- Pizzas with peppers and one other topping $=\binom{4}{1}=4$, since out of four choices of topping one is chosen,
- Pizzas with peppers and two other toppings $=\binom{4}{2}=6$, since out of four choices of topping two are chosen,
- Pizzas with peppers and three other toppings $=\binom{4}{3}=4$, since out of four choices of topping three are chosen and
- Pizzas with peppers and four other toppings $=\binom{4}{4}=1$, since out of four choices of topping four are chosen.

In Silvestre's representation of all possible pizzas with peppers as a topping, she misses the pizza with peppers, bacon and pepperoni, and instead lists it as a pizza with bacon, peppers and pepperoni under her listing of pizzas with bacon as a topping (see Figure 13-2). She also places within her listing of pizzas with peppers as a topping a plain pizza. Aside from these two inconsistencies, Silvestre's listing of pizzas with sausages as a topping, mushrooms as a topping, bacon as a topping and pepperoni as a topping represent the third, second, first and zeroth rows of Pascal's triangle, respectively, and can also be described in a combinatorial fashion as above. Finally, Silvestre's solution method represents the sum of all rows of Pascal's triangle up to the fifth row.

In contrast to Silvestre's solution method, which represents the sums of each of the first five rows of Pascal's triangle, Suzyn's solution method mimics the sixth row of Pascal's triangle: 1 5 10 10 5 1. Within a textbox, she first lists at the top a key containing abbreviations for each of the toppings, and then underneath in successive rows she lists pizzas under the following headings:

- 0 Combination = 1 Possibility (one possible pizza with no toppings)
- 1 Combination = 5 Possibilities
- 2 Combinations = 10 Possibilities
- 3 Combinations = 10 Possibilities
- 4 Combinations = 5 Possibilities and
- 5 Combinations = 1 Possibility

In our analysis of the data, we did not find evidence in the students' discourse that they were aware of Pascal's triangle or of the mathematics of binomial structures. Nevertheless, if the student sessions were to be continued over a longer period of time, students could be engaged with other problems that would provide them with opportunities to construct mathematical ideas and frameworks that underlie the rich concepts and structures of Pascal's arithmetic triangle (Edwards, 1987). In a mathematics classroom or during virtual mathematics problem-solving sessions, to promote the construction of ideas and framework, a curriculum unit could be built around a sequence of open-ended, well-designed mathematics tasks that engage teams of students with binomial structures in varied contexts. To make information on Pascal's triangle available to students as they are solving these tasks, we may do one of several things: Either an online moderator can direct students while they are in VMT to websites featuring discussions of Pascal's triangle, or a time-sensitive Wiki on Pascal's triangle may be made available to students only after they have completed a certain problem within a sequence of related problems.

In our study, the sophisticated cognitive structure that the two dyads built and made available on the shared whiteboard for each other emerged from interactional work. After the first ten minutes of nearly 120 minutes of work, the two dyads of students started to work as two separate units. In this sense, the two dyads were themselves like two entities of a single dyad. In the psychological literature on problem solving, it is commonly argued that when a dyad is engaged in solving a problem one student typically begins to solve the problem while the other listens to the ensuing solution attempt (e.g., Shirouzu, Miyake & Masukawa, 2002). The speaker may be talking out loud while solving a problem while her partner listens. Analogously, one of our dyads presenting her solution on the whiteboard is like a speaker talking aloud about their problem-solving process. However, the data of this case study shows that instead both entities of the dyad simultaneously "talked" aloud their ensuing solution and that the non-ephemeral nature of their communication medium allowed each entity to "hear" the other while "talking" aloud their problem-solving attempt. An affordance of the virtual environment may have allowed for this simultaneous solving of the problem by both entities of the chat group. In a traditional dyad, it would be difficult for both members to solve a problem out loud while paying attention to each other as well as to their own work, because two people cannot speak at once face to face. Moreover, it is difficult to think in one way when a different way of thinking is being described aloud. In this virtual environment, perhaps because the workspace is shared, relatively large, equally visible to both dyads and communication is non-aural, it is easier for each dyad to go about problem solving individually while still paying attention to what the other dyad was doing.

As we have attempted to demonstrate, while solving the Pizza Problem, the interactional and collaborative work of Silvestre and Suzyn establishes important mathematical bases for their future action. These include ways of reasoning mathematically as well as particular combinatorial structures. Stahl (2006) suggests that "[t]he being-there-together in a chat is temporally structured as a world of future possible activities with shared meaningful objects" (p. 115). The interactive work of the four students in the chat room that we have analyzed leaves them with tools for future collaboration. Interactively, they have built a discursive world of mathematical entities with which to engage particular combinatorial ideas and lines of reasoning. Silvestre and Suzyn experienced interacting together in the chat room, and this leaves them prepared for further collaborative mathematical actions with sets of shared, meaningful mathematical objects, relations among the objects and dynamics linking relations.

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