# Supplemental Curriculum Unit for Online, Collaborative Problem Solving in VMT

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## *Introduction*

The problem sets of this curriculum are supplementary to any middle school to high school mathematics curriculum. The mathematics of the problem sets is intended to supplement, not replace, the curriculum of a mathematics course. For many students, the problems will be challenging in the sense that they will initially not be aware of procedural or algorithmic tools to solve the problems but will be invited to develop tools in an online, problem-solving context. We intend for these problem sets to be used in the online environment of Virtual Math Teams (VMT). There, students will collaborate in small groups or teams to discuss and solve mathematics problems, while building mathematical ideas, engaging in mathematical reasoning, and developing problem-solving heuristics.

### *Structure*

The supplemental curriculum consists of eight problem sets within the mathematical topic of combinatorics. The first seven problem sets each consists of a lead problem and extensions of it, while the eighth problem set is an assessment task. Each lead problem and its extensions are open-ended problems, in that each problem admits multiple interpretations as well as multiple solution paths. Furthermore, as the problems do not require specialized prerequisite knowledge, they are accessible to students of varying mathematical background and are amenable to a mix of representational systems. Extensive research exists on the mathematical behavior of students working on many of the problems in the problem sets and is summarized in Maher, Powell, and Uptegrove (in press). An important design feature of this supplemental curriculum is that it offers a sequence of problem sets that are interrelated to each other as well as to features of Pascal's triangle, thus enabling students to develop over time meaningful insights into elementary combinatorial concepts and procedures.

Here are the names of the problem sets:

1. The Towers Problem Set
2. The Pascal's Triangle Problem Set
3. The Pizza Problem Set
4. The Pizza with Halves Problem Set
5. The World Series Problem Set
6. The Taxicab Problem Set
7. The Cuisenaire Rods Problem Set
8. Final Compare-and-Contrast Problem Set

*Lead problems*. To address different populations of students and particular instructional styles, each lead problem is presented in three versions (1, 2, and 3). We designed each version around the amount of scaffolding you may wish to provide to your students.

- In version 1, we present a situation to be explored and invite students to develop their own questions to generate a problem on which to work.
- In version 2, we augment the situation presented in version 1 with open-ended questions that direct students' attention to a specific problem space.

- In version 3, we further expand version 2 with a set of suggestions that may help students start the problem and develop useful problem-solving heuristics (see below for explanation of heuristics).

We anticipate that you might wish to start your students with version 2 and, as they develop robust problem-solving heuristics, move ultimately to version 1. We suggest that you engage your students in discussions in which they describe, examine, and access their evolving problem-solving heuristics. We also recommend that you invite students to pose their own extensions of the lead problems.

*Problem Posing*. In each problem set, posing a problem on which to work is a major activity of version 1 of the lead problem. This version can be used with any group of secondary students without regard to their history of mathematical performance or to their experience with mathematical problem solving. Nevertheless, novice problem solvers and other students may need to be apprenticed to what Brown and Walter (2005) call "the art of problem posing." Parallel to the "'what-if-not' strategy" that Brown and Walter present, along with colleagues at The Math Forum @ Drexel <mathforum.org>, we have found it particularly helpful to assist students in developing a particular set of cognitive tools or habits of the mind. As students begin to engage the situation presented in version 1, invite them explicitly to discuss with their teammates what they *notice* and what they *wonder* about. At first, they should articulate what *quantities* or *qualities* (objects) and what *relationships* (relations among the objects and even relations among the relations) they notice. Afterward, they should list what they wonder about these objects and relations. This later work will lead each team to pose a problem on which it agrees to work. This process engages students in self-directed learning, which is ever more privileged in the age of technology (Collins & Halverson, 2009).

*Heuristics*. In each problem set, following the presentation of the lead problem, we provide possible solutions to version 2 as well as screenshots from the VMT environment of student work. These screenshots present possible directions that your students could take and heuristics they might employ to solve version 2 of the lead problem. By heuristics, we mean the strategies that students use to solve a problem. General heuristics include guessing-and-checking, pattern searching, working backward, and creating a table. Students may engage with other, more specific heuristics. For example, if a team of students is asked to create a set of four-tall towers when selecting from red and yellow blocks, and if they organize a collection of towers by a particular number of red blocks, then they are using the heuristic of controlling a variable, where the variable is the number of red blocks. For each screenshot of student work, we describe some of the heuristics displayed. By highlighting these heuristics, we indicate how while problem solving students reason mathematically.

The solution and student work that we provide with each lead problem are meant to exemplify, but not exhaust, the mathematical and instructional possibilities of each problem set. The extension problems provide further opportunities for students to explore collaboratively their mathematical ideas, forms of reasoning, and heuristics in the VMT environment.

*Making Sense and Connections*. Facilitated by VMT's online environment, students make sense of the problems through their interaction with other students. When working on a particular problem, students may create and solve a simpler version of the problem first and analyze its relationship to the given task, and then use the results of their analysis to solve the given task. They may also offer extensions or generalizations of the original problem. We have found that over time students' reasoning will broaden and deepen and became increasingly more symbolic and generalized. By collaborating on the problems, students engage in inferential, inductive, deductive, and recursive reasoning. From these activities, students collaboratively develop mathematical ideas and reasoning strategies that include the following:

1. Counting without omission or repetition
2. Symmetry
3. Powers of 2
4. Pascal's triangle
5. Binomial coefficients
6. Counting the number of distinct subsets, combinations, $\binom{n}{r} = {}_{n}C_{r}$
7. Reasoning by cases (grouping all items that share a particular attribute)
8. Reasoning by controlling variables (determining which independent variable to change and manipulating it to determine changes in the dependent variable)
9. Reasoning about isomorphism (see Table 1)

Making sense of connections among problems is also an important instructional goal of this supplemental curriculum. In all but the first problem set, to prompt noticing and wondering about connections between and among problems, we explicitly invite students to compare previous problems and solutions their team derived to them with the problem on which their team is working and may have solved. Below, Figure 1 illustrates the type a Compare-and-Contrast Activity chart that students may use to reflect in writing on connections they notice between pairs of problems.

<table>
<tr><td>Statement of Problem A<br><br><br>Team's Solution to Problem A</td><td>Statement of Problem B<br><br><br>Team's Solution to Problem B</td></tr>
<tr><td colspan="2">1. Compare and contrast above two problems and write about what you and your teammates notice.<br><br>2. Compare and contrast your team's solutions to the two problems. Write about what you and your teammates notice.</td></tr>
</table>

*Figure 1.* Compare-and-Contrast Activity chart

We want to encourage students to make sense of connections between superficially unrelated problems that are indeed isomorphically related in their underlying mathematical structure. After you review the problem sets, you will notice that across problem sets there are isomorphic problems. For instance, these three problems are isomorphic: Taxicab, Towers, and Pizza (see Table 1).

Table 1
*Taxonomy of isomorphisms among three mathematical problems*

| | Taxicab | Towers | Pizzas |
|---|---|---|---|
| Objects | East and south vectors | Red and yellow squares | Different toppings |
| Actions | Go east or south | Affix red or yellow squares | Add or omit a topping |
| Products | Distinct shortest taxicab routes | Distinct Towers | Distinct Pizzas |

*Working Collaboratively on Open-Ended Problems*

A critical goal of this supplemental curriculum is to engage students in thinking together about mathematics. To collaborate productively, students will need to communicate effectively. As students interact in the multiple communication spaces of the VMT online environment, you may remind them of the following general guidelines for collaborative work:

READ CHAT AND WORKSPACE POSTINGS TO

1. be prepared to refer to and connect to someone else's ideas.
2. get thoughts on open questions.
3. get new perspective on your thoughts.

WRITE CHAT AND WORKSPACE POSTINGS TO

1. make your thinking available for the group to use.
2. develop your thinking.
3. get feedback on your ideas.
4. give feedback to others.

Moreover, there are other specific suggestions that you may wish to provide to promote behaviors for successful collaboration. Here are a few:

1. Discuss things and ask questions.
2. Include everyone's ideas
3. Ask what your team members think and what their reasons are.
4. Cooperate to work together.
5. Listen to each other.

6. Agree before deciding.
7. Make sure all of the ideas are on the table.
8. Try out the ideas put forth, no matter how promising or relevant.
9. Voice all doubts and questions and critiques
10. Ensure everyone's contributions are valued.
11. Decide what to focus on, have ways of keeping track of and returning to other ideas and questions, and use multiple approaches

*Benefits of Collaboration*. The National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000) urges educators to use instructional tools to support mathematics learning with understanding. Contemporary trends in education call for collaborative learning and knowledge building. Now, more than ever, individuals must collaborate on technologically elaborate and multidimensional problems with information and communication technologies (Reich, 1992; Sawyer, 2006). We believe that students engaging in problem solving within the VMT environment will develop collaborative, communicative, and mathematical abilities.

By working within this environment, students will experience peer discourse with computational support while solving open-ended problems. With peer discourse, a team of students can explore a math problem from multiple perspectives and make their reasoning explicit to each other. Computational support in the form of shared, persistent media such as chat spaces, whiteboards, wikis, and browsers can help students explore mathematical ideas they build, relationships they notice, and solutions they derive. Research has shown that online collaboration with digital media such as the above can be highly effective in fostering social interest and insightful experiences in mathematics (Stahl, 2009). Furthermore, we agree with mathematicians who argue that "Mathematics is the art of explanation" and that it is important for students to have opportunities "to pose their own problems, to make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs" (Lockhard, 2009, p. 29).

Knowing the benefits of peer discourse and collaboration, we feel compelled to recommend that you resist the urge to explain the problem or to give direct instructions on a specific approach. Instead of explaining, suggest that students do the following in their teams:

- ✓ Explain to each other their understanding of the problem.
- ✓ Discuss their interpretation of the language of the problem.
- ✓ Exchange their ways of visualizing the problem.
- ✓ Describe patterns they notice.

*Benefits of Open-Ended Problems*. When solving open-ended problems, students generally have greater room to interpret and discuss. A team of students may first come to one or more interpretations of the problem statement, after which students may work out different solutions based on these interpretations. Subsequently, students may decide amongst themselves how to interpret the problem and then argue their different solutions. Ultimately, students may or may not agree on a solution or solutions. Nevertheless, this kind of interpretative process tends to be beneficial, and students who treat mathematics as a discipline that allows for interpretation and meaning construction are most likely to become flexible and creative problem solvers (Resnick, 1988).

*How Often and How Many?*

How often you use this curriculum is at your discretion. As we have, you may find that your students benefit from spending more than one session on a problem, which will allow them to revisit and improve their solutions and justifications. Going beyond the five process standards of the National Council of Teachers of Mathematics (2000), we emphasize justifications as a means for students to create their own meaning to each problem, promote student discussion and representation, enhance their powers of mathematical reasoning, connect ideas, and develop their mathematical autonomy.

It is up to you and your students to form the teams in which they will work. We recommend that students work in teams of four or five. With larger teams, ideas may become lost because of limited space on the whiteboard and the fast pace of the chat. Since the VMT environment is Internet-based, collaborative possibilities include your students working with other classrooms in and across your school district. The environment and the problem sets have been designed to encourage creativity and experimentation by both teachers and their students.

*References*

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# Problem Set I

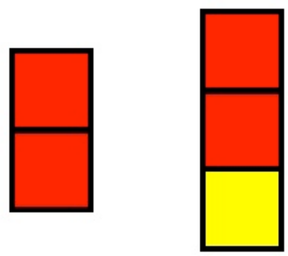
## Lead Problem: The Towers Problem

**The Problem**

In **Towers**, students will build towers using two colors of squares. Students may use many heuristics to construct and organize their towers and reason differently to justify the number of towers they have found. In time, they may notice that the towers can be grouped into the numbers found in the fourth row of Pascal's Triangle. As the students work on this combinatorial task, they will recognize patterns, further their number sense, and engage in combinatorial reasoning.

**Version 1**

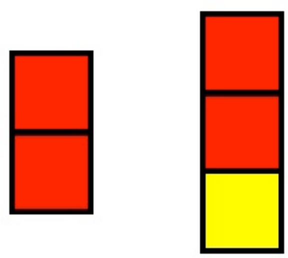
You and your team have an unlimited supply of red and yellow squares. You will use these squares to build towers of varying heights. Pictured below are two examples of towers: a tower two-squares tall and a tower three-squares tall.



Develop several questions that relate to the towers you built and post them in your summary tab.

**Version 2**

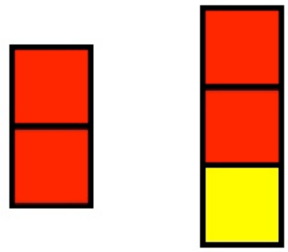
You and your team have an unlimited supply of red and yellow squares. You will use these squares to build towers. Pictured below are two examples of towers: a tower two-squares tall and a tower three-squares tall.



Work together and make as many different towers four-squares tall as possible when selecting from two colors. See whether your team can plan a good way to find all the towers four-squares tall when selecting from two colors. How can you justify to other teams that you have found all the towers?

**Version 3**

You and your team have an unlimited supply of red and yellow squares. You will use these squares to build towers. Pictured below are two examples of towers: a tower two-squares tall and a tower three-squares tall.



Work together and make as many different towers four-squares tall as possible when selecting from two colors. Your colors are red and yellow. Towers can have zero yellow squares, one yellow square, two yellow squares, and so on.

Post your answers to the questions below in the summary tab. Fill in the table according to the numbers of towers you and your teammates build.

| Number of yellow squares | 0 yellow squares | 1 yellow squares | 2 yellow squares | 3 yellow squares | 4 yellow squares |
|---|---|---|---|---|---|
| Number of towers | | | | | |

What do you notice about your completed table?

**Answer Check**

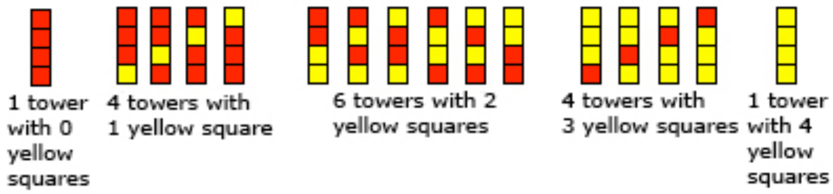
If you think you have found all possible towers, answer these questions:

- Does everyone in your team agree to the same answer? If not, has your team compared its different sets of towers?
- What patterns has your team found in its towers?
- How can your team arrange its towers to be sure that someone not on your team will agree that you have accounted for all towers?
- How did your justify its answer?
- Is your team's explanation clear and complete?

**Our Solutions**

Below are several examples of ways that students might solve version 2 of the problem. These solutions are not meant to be prescriptive or comprehensive, but meant to give you a sense of what we expect students to come up with when solving the problem.

**Strategy 1 – Looking for patterns: Grouping the towers into sets depending on the number of yellow squares**




${}_4C_0 = 1$ ${}_4C_1 = 4$ ${}_4C_2 = 6$ ${}_4C_3 = 4$ ${}_4C_4 = 1$

Students could use proof by cases by grouping their towers based on the number of yellow squares in each tower. In a proof by cases, students split their solution into different groups, where the items in each group share a common attribute. Note that when the towers are grouped in this manner, the collection of towers in each group represents the fourth row of Pascal's Triangle.

The Choose symbolism (e.g., ${}_4C_1$) used above is described at http://en.wikipedia.org/wiki/Combination.

Pascal's Triangle is described at http://mathforum.org/dr.math/faq/faq.pascal.triangle.html.

**Strategy 2 – Looking for patterns: Grouping the towers into pairs**

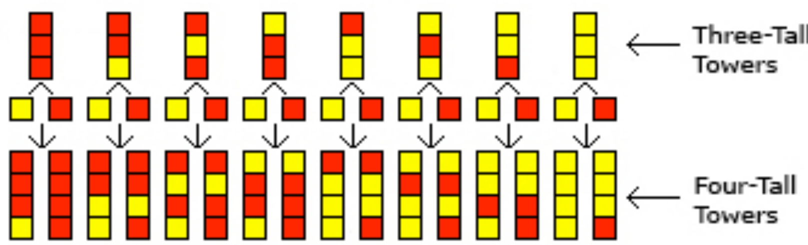
Students could use proof by cases by choosing to list their towers using letters as representations. Note that in each pair of towers, the second tower is the "opposite" of the first tower since yellow squares replace red squares and vice versa. Also notice below that this tower pattern is different from the pattern of towers in Strategy 1.

R = red, Y = yellow

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | Y | R | Y | R | Y | R | Y | R | Y | R | Y | R | Y | R | Y |
| R | Y | R | Y | Y | R | Y | R | Y | R | Y | R | R | Y | R | Y |
| R | Y | Y | R | Y | R | R | Y | Y | R | R | Y | R | Y | Y | R |
| R | Y | Y | R | Y | R | Y | R | R | Y | R | Y | Y | R | R | Y |

**Strategy 3 – Simplifying the problem: Using three-tall towers to build four-tall towers**

One may use induction to build the four-tall towers by first building the three-tall towers and adding a yellow square and a red square to each three-tall tower. This is illustrated below:




Suppose that $\text{Towers}(x) = \text{the number of towers } x \text{ squares high}$. Then,

$$\text{Towers}(4) = 2 \times \text{Towers}(3).$$

In general,

$$\text{Towers}(n) = 2 \times \text{Towers}(n-1) \text{ and } \text{Towers}(1) = 2^1, \text{ so } \text{Towers}(n) = 2^n.$$

**Strategy 4 – Binary numbers**

As a further extension, one may use the binary numbers to generate the list of towers. Set R = 0 (red) and Y = 1 (yellow).

Four-Tall Towers

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

Starting at the left column and moving to the right, we have the binary numbers (base 2) from 0 to 15 (base 10). In Column 1, we have $0000_2 = 0_{10}$, and in column 16, we have $1111_2 = 15_{10}$. This notation shows that you can either have a 0 or 1 (red or yellow) in the first position, a 0 or 1 in the second position, a 0 or 1 in the third position, and a 0 or 1 in the fourth position. By using this binary notation, we can ensure that we have all possible combinations of towers.

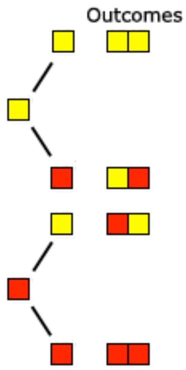
**Strategy 5 – Computational Method: *n*-tall towers**

For 1-tall towers, we have $2^1 = 2$ towers; for two-tall towers, we have $2^2 = 4$ towers; for three-tall towers, we have $2^3 = 8$ towers; for four-tall towers, we have $2^4 = 16$ towers; and in general for *n*-tall towers, we have $2^n$ towers.

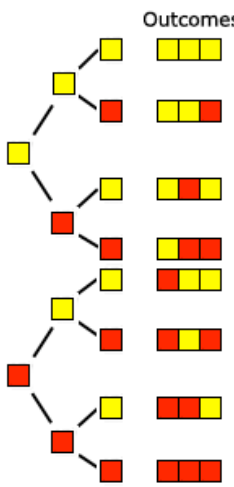
This can be seen in the following tree diagrams:

$2^1 = 2$ one-tall towers:

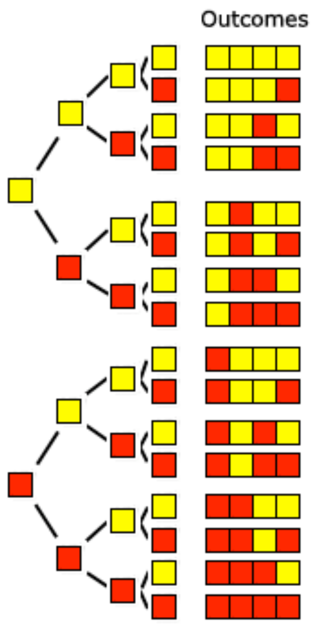

$2^2 = 4$ two-tall towers:




$2^3 = 8$ three-tall towers:



$2^4 = 16$ four-tall towers:




This pattern continues infinitely. Note that the number of towers $k$ squares tall is twice the number of towers $k-1$ squares tall. That is, if there are $2^{k-1}$ towers that are $k-1$ squares tall, there are $2\left(2^{k-1}\right) = 2^{1+(k-1)} = 2^k$ towers that are $k$ squares tall. In general, the number of towers $n$ squares tall is $2^n$.

**Sample Student Solutions**

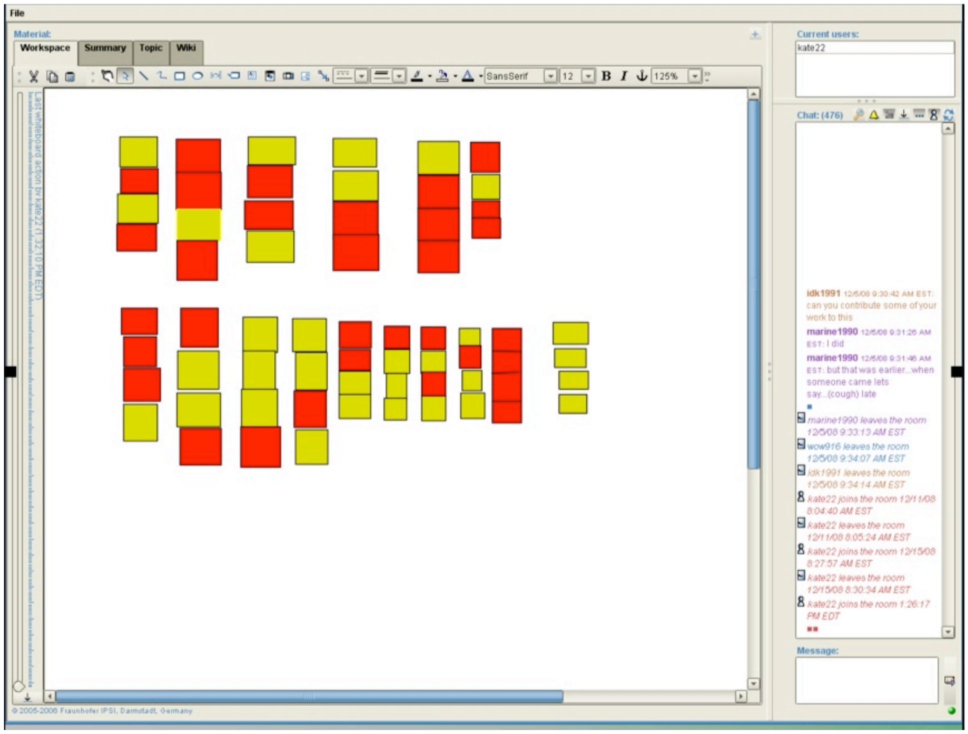
The following three solutions are screenshots, which can help you to assess the quality of a team's solution. At times, you may have to read through the chat to find evidence of your students' communication.

**Team of four students**

*This might be regarded as a novice approach. Even though they listed 16 possible towers, the whiteboard showed no*

**Sample student solution 1**

*evidence of strategy. As far as communication skills, they do little to show how they arrived at the answer.*

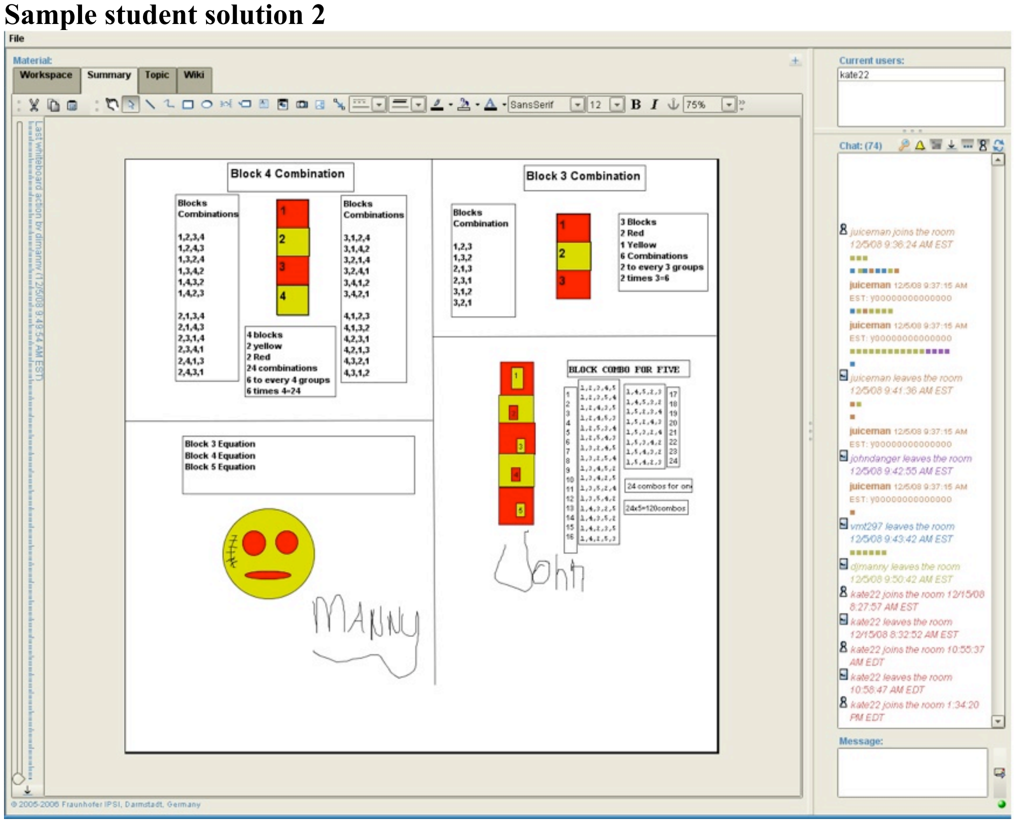



This solution shows a listing of 16 four-tall towers. The students did not seem to use any strategy to come up with this listing.

**Team of four students**

*This might be regarded as between a novice and expert approach. They choose a strategy that leads to inaccurate results, but they show evidence of finding patterns in their listing of towers.*

**Sample student solution 2**

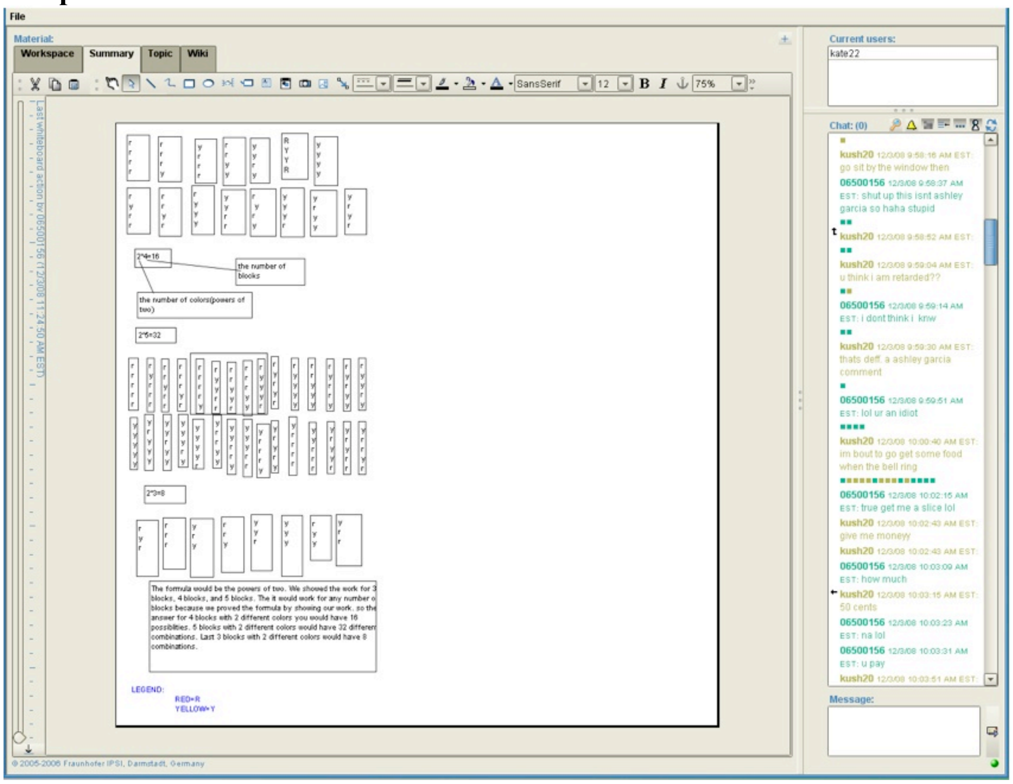



This solution shows students listing 24 four-tall, 6 three-tall, and 120 five-tall towers. Because the students interpreted the problem statement to mean that each tower had to contain at least two red squares, they ended up pursuing a different problem. In addition, the strategy that they implemented caused them to distinguish between the two red squares and between the two yellow squares, when in fact the red squares and yellow squares should be indistinguishable. Their solution could be an opportunity for further discussion within other VMT sessions.

**Team of four students**

*The work here is accurate. They explained the steps that they took to solve the problem and also provided formulas to calculate the number of three-tall, four-tall, and five-tall towers.*

**Sample student solution 3**




This solution shows towers four-tall constructed at the top of the screen, along with the formula of $2^4 = 16$ below it. Below this, the students have listed towers five-tall with the formula of $2^5 = 32$, and towers three-tall with the formula of $2^3 = 8$. They have also briefly explained their formulas.

**Teaching Suggestions**

Resist the urge to give direct instructions on a specific approach.

**Extensions**

1. Towers Five-Tall

   Work together and create as many different towers five squares tall as is possible when selecting from two colors. See if you and your teammates can plan a good way to find all the towers five squares tall.

   How many towers will have exactly two squares the same color?

2. Towers Three-Tall Written Assessment

   Your team must work together to create a set of detailed instructions to be sent to another VMT team. Your instructions must describe in detail how to build all towers possible three squares tall, when selecting from two colors. In the summary tab post your instructions.

Follow up suggestion: Post each team's summary to the Wiki. Instruct the teams to visit the Wiki and follow the given directions from another team.

3. Ankur's Challenge

   Find all possible towers that are 4 squares tall, selecting from squares available in *three* different colors, so that the resulting towers contain at least one of each color. Convince us that you have found them all.

4. Towers *n*-tall

   You have created towers four-tall, five-tall and three-tall. What about *n*-tall? Work together with your team to create a formula for towers *n*–tall using squares of two colors. Justify your solution in the summary tab. Check to make sure your solution works for towers three-tall, four-tall and five-tall.

5. Towers four-tall when selecting from three colors

   Work together and create as many different towers four squares tall as is possible when selecting from three colors. See if your team can plan a good way to find all the towers four squares tall.

6. Souleymane's Challenge

   Find all possible towers one-tall, two-tall, three-tall, and so forth when selecting from squares available in two different colors—red and yellow—so that no tower contains a yellow square touching another yellow square. You and your teammates should convince yourselves that you have found all such towers. Afterward, post to the summary tab so that other teams can read your work a description of how you found your towers and a justification for your solution so that other teams can read your work.

7. Forming Committees Problem

   A new student organization consists of four individuals: Ahmad, Beatrice, Carlos, and Daisy. From among these individuals, how many different committees are possible? List them and justify your solution.

# Problem Set II

# Lead Problem: Pascal's Triangular Array of Numbers

**The Problem**

The purpose of this lesson is to introduce students to **Pascal's Triangle**. The activities are designed to invite the students to notice patterns and relationships and to engage in mathematical discussion about what they notice.

**Version 1**

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Above is a triangular array of numbers. What do you notice about this array? What do you wonder about this array of numbers and the patterns and relationships you find? In the summary tab, describe the patterns and relationships that you found.

**Version 2**

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1. What do you predict is the next row of numbers?…and the next?…and so on?
2. See whether you can find a pattern in the sums of the numbers in the rows.
3. See if you can find the following patterns:
    - Powers of 11 (1, 11, 121,…)
    - Triangular numbers (1, 3, 6, 10,…)
    - Fibonacci Sequence (1, 1, 2, 3, 5, 8,…)
    - Square numbers (1, 4, 9, 16,…)

**Version 3**

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1. Determine the numbers for the next four rows.
2. Add the numbers in each row and record the sum. What do you notice about these sums?
3. In the above array, imagine (or actually draw) a line that is 60° to the last row of numbers. Add the numbers through which this diagonal line passes and record the sum. Move your diagonal to the left (or to the right), add the numbers through which it passes, and record this sum. Continue this process. Describe any patterns you notice.

**Answer Check**

If you think you have found all possible patterns:

- Can your team explain what happens to the powers of eleven?
- Did your team find a pattern different from the ones listed?
- What connections does your team notice between this and previous problems?

**Our Solutions**

Below are several examples of patterns that students might find. These patterns are not meant to be prescriptive or comprehensive, but meant to give you a sense of what we expect students to come up with when searching for patterns in Pascal's Triangle.

To generate the numbers in the next row of Pascal's Triangle, successively add in the row above the adjacent two numbers (see Figure 1).

1<br>
1 1<br>
1 2 1<br>
1 3 3 1<br>
1 4 6 4 1<br>
**1** **5** 10 **10** **5** 1<br>
6 15

*Figure 1.* Generating the next row from the previous one.

The sum of numbers in any row of Pascal's Triangle is twice the sum of number in the previous row and is a power of 2. If the first row of Pascal's Triangle is called the zeroth row and successive rows are consecutively numbered, then the sum of numbers in row *n* equals $2^n$ (see Figure 2).

| | |
|---|---|
| 1 | $1 = 2^0$ |
| 1 1 | $2 = 2^1$ |
| 1 2 1 | $4 = 2^2$ |
| 1 3 3 1 | $8 = 2^3$ |
| 1 4 6 4 1 | $16 = 2^4$ |
| 1 5 10 10 5 1 | $32 = 2^5$ |

*Figure 2.* The sum each row of Pascal's Triangle is a power of 2.

While we cannot show all of the fascinating patterns of number that one can find in Pascal's triangular array of numbers, there are a few that are particularly notable. One is a special category of numbers called *figurate* numbers. These are numbers that can be represented by a regular geometric arrangement of equally spaced points. Pascal's Triangle contains many groups of figurate numbers. The counting or natural numbers $(1, 2, 3, \ldots)$ appear in the second diagonal (either from the left or right). The triangular numbers (highlighted as bold in Figure 3a) appear in the third diagonal. The square numbers appear as the sum of any two adjacent triangular numbers (Figure 3b).

| a | b |
|---|---|
| 1 | 1 |
| 1 1 | 1 1 |
| **1** 2 **1** | 1 2 **1** |
| 1 **3** **3** 1 | 1 3 **3** 1 |
| 1 4 **6** 4 1 | 1 4 **6** 4 1 |
| 1 5 **10** **10** 5 1 | 1 5 10 **10** 5 1 |

*Figure 3.* Triangular and square numbers in Pascal's Triangle

Another is the powers of 11, which are found by looking at the rows as one number as opposed to viewing each individual element (see Figure 4). When an element itself has more than one digit, then carry the tens digit to the left. The number is equal to 11 to the $n^{th}$ power or $11^n$, when n is the number of the row from which the multi-digit number was taken.

| | |
|---|---|
| 1 | $11^0 = 1$ |
| 1 1 | $11^1 = 11$ |
| 1 2 1 | $11^2 = 121$ |
| 1 3 3 1 | $11^3 = 1{,}331$ |
| 1 4 6 4 1 | $11^4 = 14{,}641$ |
| 1 5 10 10 5 1 | $11^5 = 161{,}051$ |

*Figure 4.* Interpreting rows of Pascal's Triangle as powers of 11

In Figure 5, we moved all the rows to the left margin to facilitate illustrating that the Fibonacci sequence can been found by taking the sum of the numbers in successive diagonals.

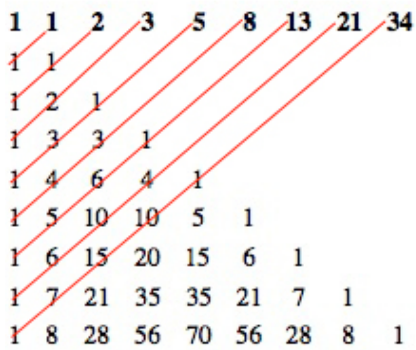



*Figure 5.* The Fibonacci sequence as the sum of successive diagonals of numbers in Pascal's Triangle

**Sample Student Solutions**

We do not have sample student solutions for Pascal's Triangle at this time.

**Teaching Suggestions**

Resist the urge to give direct instructions on a specific approach.

**Extensions**

1. Exploring Binomial Expansions: Expand a binomial expression to different successive powers. Represent and describe what you observe about the results.

   $(a+b)^0 =$ ___________

   $(a+b)^1 =$ ______________

   $(a+b)^2 =$ ___________________

   $(a+b)^3 =$ ____________________________

   $(a+b)^4 =$ ____________________________________

   . .

   . .

   . .

   $(a+b)^9 =$ ____________________________________

   How would you write the expansion of the following binomial: $(a+b)^n$?

   What relationship do you see between the expansion of a binomial and Pascal's Triangle?

# Problem Set III

## Lead Problem: Pizzas with four toppings

**The Problem**

In **Pizzas with four toppings**, students will work on a combinatorial task to develop algebraic reasoning, number sense, and pattern recognition skills. The problem may remind students of a row in Pascal's Triangle and the possibility of an isomorphic connection to Towers Four-Tall.

**Version 1**

A local pizza shop has asked you to help design a form to keep track of certain pizza choices. This form may be used by the pizza maker to check off which pizza a customer has ordered. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni.

Develop several questions that relate to this situation. Post your questions along with answers to the summary tab.

**Version 2**

A local pizza shop has asked you to help design a form to keep track of certain pizza choices. This form may be used by the pizza maker to check off which pizza a customer has ordered. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni.

Find all of the possible choices. How can you justify to other teams that you have found all possible pizzas?

**Version3**

A local pizza shop has asked you to help design a form to keep track of certain pizza choices. This form may be used by the pizza maker to check off which pizza a customer has ordered. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni.

Below is an example of the type of table you might use to keep track of the different choices. Each row represents different pizza and each column is a different available topping. To use this table, mark with a symbol in each cell whether a topping is used for a particular type of pizza. In the table, you and your teammates may need more rows. Also, as you work, discuss what you notice about your results.

Post your answers to the questions below in the summary tab. Fill in the table according to the possible pizza combinations.

| | Topping Choices | | | |
|---|---|---|---|---|
| Type of Pizza | Peppers | Sausage | Mushroom | Pepperoni |

| | | | | |
|---|---|---|---|---|
| 1st | | | | |
| 2nd | | | | |
| 3rd | | | | |
| 4th | | | | |

1. Explain how to find these numbers in Pascal's Triangle.
2. Describe how to use Pascal's triangle to help you find the number of pizzas that can be ordered with three toppings or five toppings.
3. Relate your list of pizzas to the list of towers you built in a previous problem.

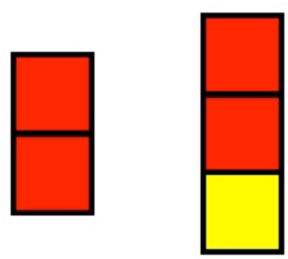
**Answer Check**

If you think you found all possible pizzas,

- Does everyone in your team agree to the same answer? If not, has your team compared its lists of pizzas?
- What representations did you use?
- What patterns did you find?
- How did you justify your solutions?
- Is your team's explanation clear and complete?
- If another team has fewer or more pizzas, would that make your team doubt its answer?
- What connections does your team notice between this and previous problems?

**Compare and Contrast**

You and your team have two colors of squares. You will use these squares to build towers. Pictured below are two examples of towers: a tower two-squares tall and a tower three-squares tall.



Work together and make as many different towers four-squares tall as possible when selecting from two colors. See whether your team can plan a good way to find all the towers four-squares tall when selecting from two colors. How can you justify to other teams that you have found all the towers?

Insert Team's Solution Here

A local pizza shop has asked us to help design a form to keep track of certain pizza choices. This form may be used by the pizza maker to check off which pizza a customer has ordered. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom and pepperoni.

How many different choices for pizza does a customer have? List all the possible choices. How can you justify to other teams that you have found all possible pizzas?

Insert Team's Solution Here

1. Compare and contrast the two problem statements listed above and

write about what you and your teammates notice.

2. Compare and contrast your team's solutions to the two problems. Write about what you and your teammates notice.

**Our Solutions**

Research has shown that from a very young age to the secondary level students approach this problem by building a proof by cases. In other words, it is likely that they will list all possible pizza combinations. Variations can be expected as the students struggle with how to represent their lists using letters, pictures, or symbols. It is deliberate that the pizza choices include peppers and pepperoni, compelling the students to create a distinction between the two variables.

**Strategy 1 – Using combinations to justify solutions.**

Plain = C; Peppers = P; Sausage = S; Mushrooms = M; Pepperoni = R

| 0-topping (or plain) pizza | 1-topping pizzas | 2-topping pizzas | 3-topping pizzas | 4-topping pizza |
|---|---|---|---|---|
| | | | | |
| C | P | PS | PSM | PSMR |
| | S | PM | PSR | |
| | M | PR | PMR | |
| | R | SM | SMR | |
| | | SR | | |
| | | MR | | |
| | | | | |
| 1 | 4 | 6 | 4 | 1 |
| ${}_4C_0$ | ${}_4C_1$ | ${}_4C_2$ | ${}_4C_3$ | ${}_4C_4$ |
| Four toppings choose zero | Four toppings choose one | Four toppings choose two | Four toppings choose three | Four toppings choose four |

There are exactly 16 pizza combinations when selecting from four toppings. As you can observe from the above list the combination of pizzas reflects the fourth row of Pascal's Triangle (1, 4, 6, 4, 1). Recall that the sum of each row is equal to $2^n$. In this combinatorial problem $n$ represents the number of toppings ($2^4 = 16$). The two (2) in this problem represents the possibility of a topping, on or off.

**Strategy 2 – Controlling variable**

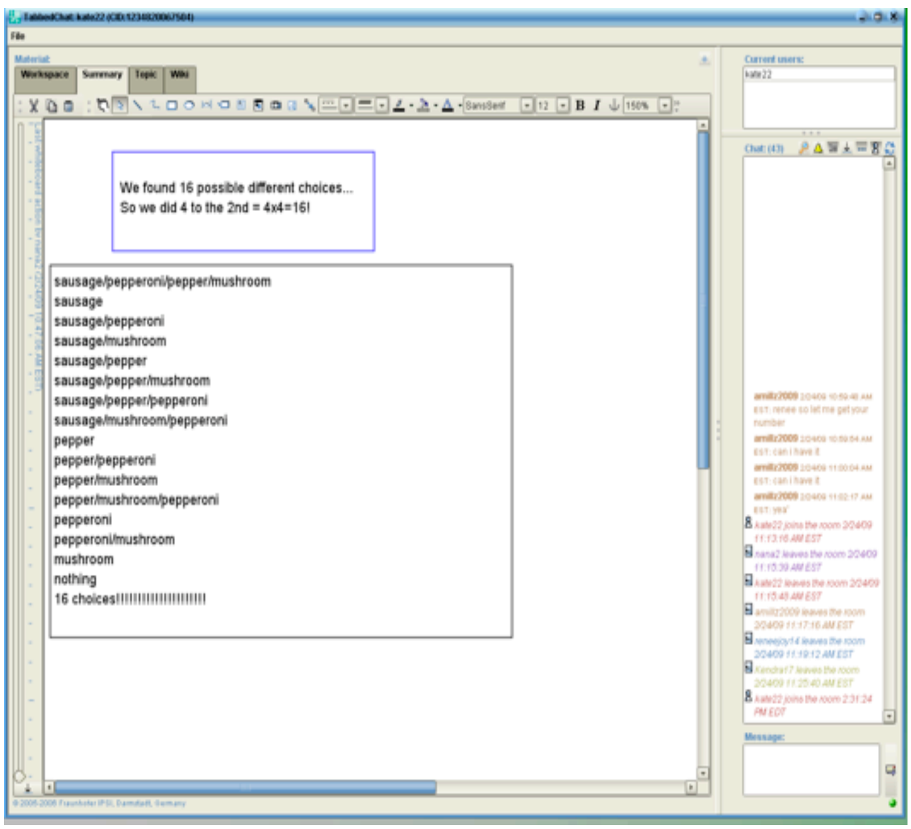
You may notice students using the heuristic of controlling variable, where one variable is held constant while the other variables change. For example, a student might list all possible pizza combinations that will have peppers on it. This would include peppers alone, peppers with one other topping, peppers with two other toppings and peppers with all three of the remaining toppings. See listing below.

Peppers = P Sausage = S Mushrooms = M Pepperoni = R

| | |
|---|---|
| 8 pizzas with peppers | P<br>PS, PM, PR<br>PSM, PSR, PMR<br>PSMR |
| 4 pizzas with sausage | S<br>SM, SR<br>SMR |
| 2 pizzas with mushrooms | M<br>MR |
| 1 pizza with pepperoni | R |
| 1 pizza | Plain |

**Group of four 12th grade students**
*Although the students listed 16 possible pizzas, the solution lacks a strong justification.*

**Sample student solution 1**

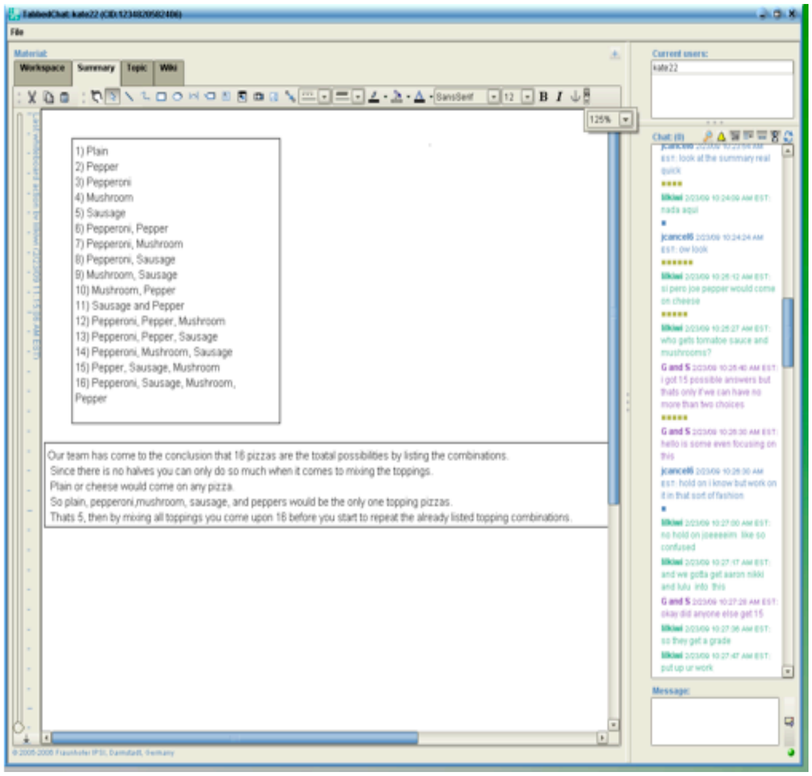



In this solution the team of students listed 16 possible pizzas. They began with a pizza using all possible toppings then held sausage as a constant. In the text box at the top, their explanation is incorrect as to how they justified 16 possible pizzas.

**Group of four 12th grade**

**Sample student solution 2**

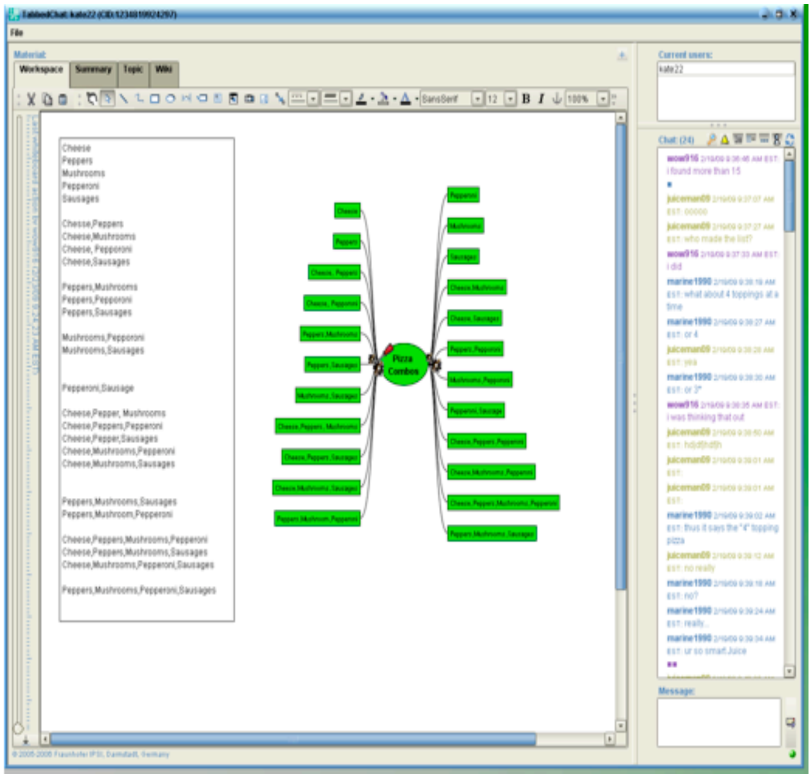
**students**
*Not only do they include an argument stating why they feel they have found all possible pizzas, notice that they begin their explanation with "Our group" which is a positive sign that they collaborated with one another.*




The solution above shows 16 possible pizza combinations. It is interesting to note that the team of students listed their pizzas in the following way: 1 plain pizza, 4 one-topping pizzas, 6 two-topping pizzas, 4 three-topping pizzas, and 1 four-topping pizza.

**Group of four 12th grade students**
*Although the presentation is clear, this group interpreted cheese as a separate topping. A five topping pizza would have 32 possible pizza combinations..*

**Sample student solution 3**




In the solution above the students illustrated their work with a mind map, one of the affordances found in the workspace. This solution differs from that of the other teams shown in the previous illustrations since this team interpreted the problem differently.

**Teaching Suggestions**

**Extensions**

1. Pizzas with 5 Toppings

   A local pizza shop has asked us to help design a form to keep track of certain pizza choices. This form may be used by the pizza maker to check off which pizza a customer has ordered. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushroom, pepperoni and onion. How many different choices for pizza does a customer have? List all the possible choices. How can you justify to other teams that you have found all possible pizzas?

2. Forming Committees

   A new student organization consists of five individuals: Ahmad, Beatrice, Carolos, Daniel, and Evelyn. From among these individuals, how many different three-person committees are possible? List them and justify your solution to other teams.

3. Forming a Committee

The student organization in the above problem gained a new member: Gwen. Now the organization is interested in forming a committee consisting of four individuals. From among its six members, in how different many ways can it compose such a committee? List them and justify your solution to other teams.

4. Pizzas with Four Toppings Chosen from 12 Toppings

   A new pizzeria has opened and has a special offer! It offers a cheese pizza with tomato sauce. A customer may order a pizza with up to four toppings from a possible selection of 12 toppings! How many different choices for pizza does a customer have? List all the possible choices. How can you justify to other teams that you have found all possible pizzas?

# Problem Set IV

## Lead Problem: Pizza with Halves

**The Problem**

In **Pizza with Halves,** students will list all possible pizza combinations when selecting from two toppings where toppings must be represented as wholes, halves, or not at all. Students may interpret this problem differently and have varying ideas about how to keep track of the choices of pizzas. These differences tend to elicit rich mathematical discussions.

**Version 1**

A local pizza shop has asked you to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or on half of the pizza. A customer can then select from two different toppings: peppers and pepperoni.

Develop several questions that relate to the pizzas you are able to create with the above toppings. Post your questions along with answers to the summary tab.

**Version 2**

A local pizza shop has asked you to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or on half of the pizza. A customer can then select from two different toppings: peppers and pepperoni.

Work together with your teammates to list all the possible different choices. With your teammates, write a report of your findings. It should include a justification that you have accounted for all possible pizzas. You want your report to convince other teams that your findings are correct. Post your report to the Summary tab.

**Version 3**

A local pizza shop has asked you to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or on half of the pizza. A customer can then select from two different toppings: peppers and pepperoni.

| Question Table | Answer |
|---|---|
| 1. How many pizzas have the same topping on both halves? | |
| 2. How many pizzas have toppings on only one half? | |
| 3. How many pizzas have different toppings on each half? | |

**Answer Check**

If you think you found all possible pizzas, answer these questions:

- Does everyone in your team agree to the same answer? If not, have you compared your lists of pizzas?
- Does order matter? In other words, is a pizza that is half pepperoni and half peppers the same as or different from a pizza that is half peppers and half pepperoni?
- Is your team's explanation clear and complete?
- What connections does your team notice between this and previous problems?

**Compare and Contrast**

Work together and create as many different towers two squares tall as is possible when selecting from three colors. See if your group can plan a good way to find all the towers two-squares tall.

Insert Team's Solution Here

A local pizza shop has asked us to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or half of a pizza. A customer can then select from two different toppings: peppers and pepperoni.

Work together with your teammates to list all the possible different choices. With your teammates, write a report of your findings. It should include a justification that you have accounted for all possible pizzas. You want your report to convince other teams that your findings are correct. Post your report to the Summary tab.

Insert Team's Solution Here

1. Compare and contrast the two problem statements listed above and write about what you and your teammates notice.

2. Compare and contrast your team's solutions to the two problems. Write about what you and your teammates notice.

**Our Solutions**

Below are several examples of ways students might solve version 2 of the problem. These solutions are neither meant to be prescriptive nor comprehensive.

**Strategy 1 – Listing all possible pizza combinations**

To find possibilities for pizzas with halves, one may list pizza possibilities by building cases, where one case is of whole pizzas and the other case concerns half topping pizzas.

| Whole Pizzas | Half Topping Pizzas |
|---|---|
| Plain<br>Pepperoni Pizza<br>Pepper Pizza<br>Pepperoni and Peppers Pizza | Half Plain/Half Pepperoni<br>Half Plain/Half Peppers<br>Half Plain/Half Pepperoni and Peppers<br>Half Pepperoni/Half Peppers<br>Half Pepperoni/Half Pepperoni and Pepper<br>Half Peppers/Half Peppers and Pepperoni |

**4 wholes + 6 half topping pizzas = 10 possible pizzas**

**Strategy 2 – Use the heuristic of making a table**

To find out the different possibilities for pizzas with halves, we can use a table:

N = Plain
P = Pepperoni
R = Peppers
PR = Pepperoni and peppers

| | N | P | R | PR |
|---|---|---|---|---|
| N | X | X | X | X |
| P | | X | X | X |
| R | | | X | X |
| PR | | | | X |

Consider the top header row to be the right side of the pizza and the left header column to be the left side of the pizza. The four X's on the diagonal represent pizzas with the same topping on each half and, therefore, are whole pizzas with a single topping. The remaining six X's represent pizzas with different toppings on each half. For instance, the X in row P and column R represents a pizza that contains pepperoni on one half and peppers on the other. The six blank cells in the table duplicate pizzas already counted. For example, a pizza with peppers on the left and pepperoni on the right is the same pizza with peppers on the right and pepperoni on the left.

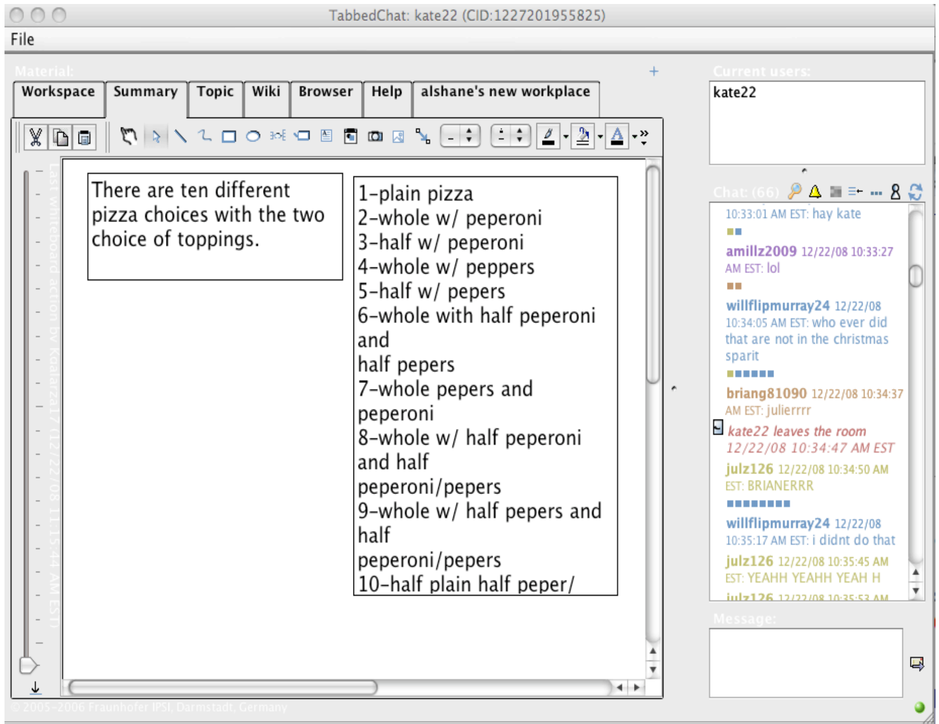
**Sample Student Solutions**

The following three solutions are screenshots, which can help you assess the quality of your students' solutions. At times, you may need to read through the chat to find evidence of student communication.

**Group of four students**

**Sample student solution**

*Although the list of pizzas is complete a justification is not given. It is important that the students practice justifying their solutions.*



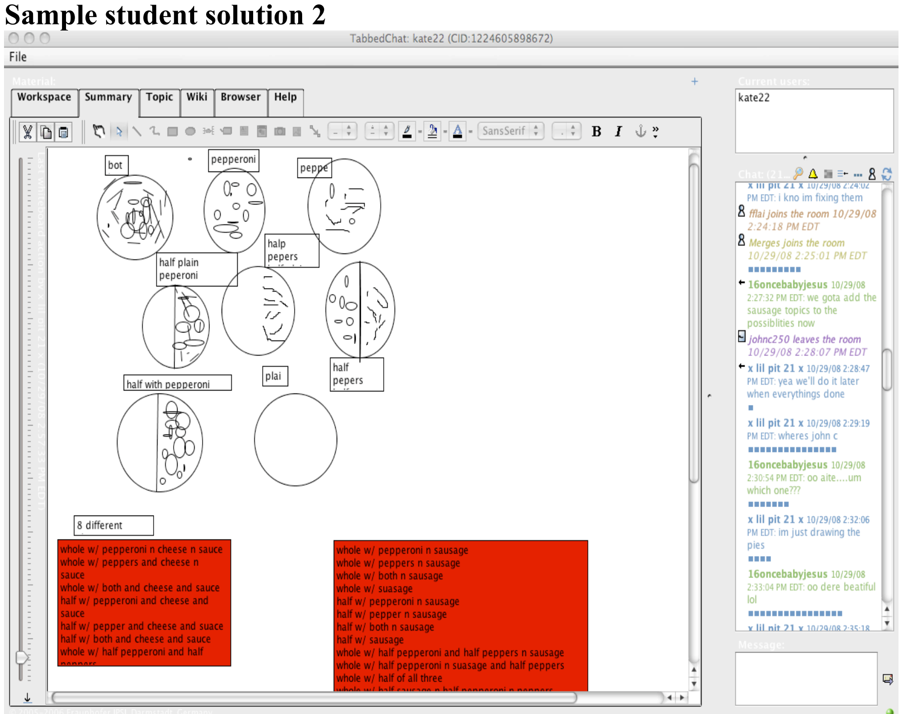

1

This solution shows a listing of 10 possible pizzas. The students did show organization by controlling variable.

**Group of four students**

*This illustrates a group working individually with three different solutions. Even though they have three different solutions, this shows little evidence of collaboration.*

**Sample student solution 2**

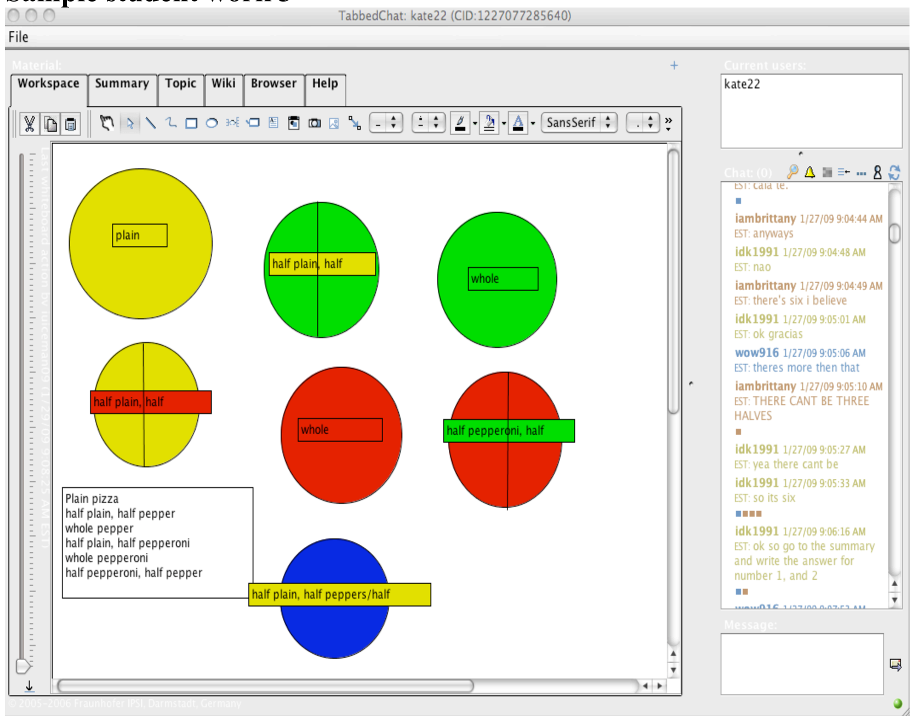



One student illustrated pizzas for his solution while two other students used textboxes to list their combinations.

**Group of students**

*The work was left in the workspace and appears incomplete. There may be evidence of mathematical discourse within the chat.*

**Sample student work 3**




This solution shows an illustration of 7 possible pizzas. Students used a textbox listing all of their pizzas to clarify their illustration.

**Teaching Suggestions**

Different interpretations of a "half" of the pizza or a plain pizza can affect the outcome of the problem. These different interpretations, if justified by the student, should be encouraged.

**Extensions**

1. Three-Topping Pizzas with Halves

   A local pizza shop has asked us to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or half of a pizza. A customer can then select from three different toppings: peppers, pepperoni, or pineapple.

   Work together with your teammates to list all the possible different selections. With your teammates, write a report of your findings. It should include a justification that you have accounted for all possible pizzas. You want your report to convince others who are not in your team that your findings are correct. Post your report to the Summary tab.

2. Pizzas with Thirds

   A local pizza shop has asked us to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or on a *third* of a pizza. A customer can then select from two different toppings: peppers and pepperoni.

Work together with your teammates to list all the possible different selections. With your teammates, write a report of your findings. It should include a justification that you have accounted for all possible pizzas. You want your report to convince others who are not in your team that your findings are correct. Post your report to the Summary tab.

3. Handshake Problem

   You are in a roomful of 35 people. Everyone is asked to shake hands with each other. How many handshakes will occur? How can you figure this out? What strategies will you use? Justify your answers and post your solutions to the Summary tab.

# Problem Set V

## Lead Problem: World Series Problem

**The Problem**

In **World Series**, students find out the probability of a baseball team winning the World Series in four, five, six, and seven games. In solving this problem, students will revisit combinatorial ideas developed in the Towers and Pizzas problems. In particular, determining an accurate sample space will be essential.

**Version 1**

In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assume that both teams are equally matched. Develop several questions that relate to this situation. Post your questions along with answers to the summary tab.

**Version 2**

In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that both teams are equally matched, what is the probability that a World Series will be won: (a) In four games? (b) In five games? (c) In six games? (d) In seven games?

**Version 3**

In a World Series, two teams, A and B, play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assume that both teams are equally matched.

You may use the table below to keep the track of the number of games played and the possible outcomes.

| Number of Games Played | All Possible Outcomes | Total Number of Possible Outcomes |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |

| | | |
|---|---|---|
| 5 | | |
| 6 | | |
| 7 | | |

Remember that a World Series will have a minimum of four games. What part of the table will you use to calculate your sample space?

For each of the possibilities you have listed, which of them represent a win for A? A win for B?

What is the probability of a win for team A after 4 games? 5 games? 6 games? 7 games?

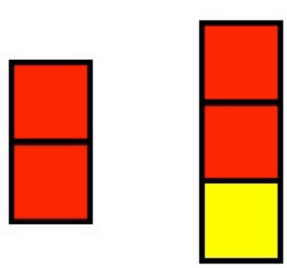
What is the probability of a win for team B after 4 games? 5 games? 6 games? 7 games?

**Answer Check**

If you are having trouble,

- Have you listed all possible outcomes for four games? For five games? For six games? For seven games? Does your team see a pattern?
- What kind of representations can your team use?
- Does your team see a pattern that connects this problem to previous problems?
- What is the sample space?
- What is the probability of winning in four games? In five games? In six games? In seven games?
- Is your team's explanation clear and complete?

**Compare and Contrast**

| | |
|---|---|
| You and your team have two colors of squares. You will use these squares to build towers. Pictured below are two examples of towers: a tower two-squares tall and a tower three-squares tall.<br> | In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that both teams are equally matched, what are all the possible outcomes for a four-series game?<br><br>Insert Team's Solution Here |

| Work together and make as many different towers four-squares tall as possible when selecting from two colors. See whether your team can plan a good way to find all the towers four-squares tall when selecting from two colors. How can you justify to other teams that you have found all the towers?<br><br>Insert Team's Solution Here | |
|---|---|
| 1. Compare and contrast the two problem statements listed above and write about what you and your teammates notice.<br><br>2. Compare and contrast your team's solutions to the two problems. Write about what you and your teammates notice. | |

**Our Solutions**

Below is an example of a way to solve the problem. This solution is not meant to be prescriptive or comprehensive but rather to give you a sense of what we expect students to come up with when solving the problem.

**Strategy – Combinations and Pascal's Rule**

a) For a team to win a 4-game series, the winning team must always win all four games. In this solution and in the next three parts, we only need to deal with the first 3, 4, 5, and 6 games since the winning team must always win the last game. In this case, we will then deal with the first 3 games.

Suppose team A is the winning team and team B is the losing team. There is ${}_3C_3 = 1$ way for team A to win the first three games.

| | |
|---|---|
| Game 1 | A |
| Game 2 | A |
| Game 3 | A |
| Game 4 | A |

To also account for the case that team B is the winning team and team A is the losing team, we double this amount, which results in $2\,{}_3C_3 = 2(1) = 2$ ways for either team to win a 4-game series.

Since there are two choices for the winner of each game, we have $2^4 = 16$ total ways to play 4 games. Therefore,

$$\text{P}(\text{World Series is won in 4 games}) = \frac{2}{2^4} = \frac{2}{16} = \frac{1}{8}.$$

b) For a team to win a 5-game series, we must account for all possible combinations of 5 games where the winning team wins four of the games and the losing team wins one of the games; from these combinations, we want to subtract the combinations where the last game is won by the losing team, since these are the combinations where the winning team will have won the series in only four games.

Suppose team A is the winning team and team B is the losing team. There are ${}_5C_4$ ways for team A to win four games and team B to win one game. There is ${}_4C_4 = 1$ way for team B to win the last game, and to have A win the first four games:

| | |
|---|---|
| Game 1 | A |
| Game 2 | A |
| Game 3 | A |
| Game 4 | A |
| Game 5 | B |

Thus, there are

$${}_5C_4 - {}_4C_4 = \frac{5!}{4!1!} - \frac{4!}{4!0!} = 5 - 1 = 4$$

ways for either team to win a 5-game series. To account for both teams, we need to double this amount, making for 8 ways.

Since from Pascal's rule we know that ${}_4C_3 + {}_4C_4 = {}_5C_4$, we must have ${}_5C_4 - {}_4C_4 = {}_4C_3$ ways for either team to win a 5-game series. This result of ${}_4C_3$ makes sense, since we can think of it in the following way. For team A to win a 5-game series, team A must win the last (fifth) game out of five games being played. If team A does not win the fifth game, team A will already have won in four games:

| | |
|---|---|
| Game 1 | A |
| Game 2 | A |
| Game 3 | A |
| Game 4 | A |
| Game 5 | B |

Thus, we need only worry about the first four games. Since we know that team A will win the last game, team A must win any three of the of the first four games. There are ${}_4C_3 = 4$ ways to do this. Again, to account for both teams, we can double this to get 8 ways.

Since there are two choices for the winner of each game, we have $2^5 = 32$ total ways to play 5 games. Therefore,

$$\text{P}\left(\text{World Series is won in 5 games}\right) = \frac{8}{2^5} = \frac{8}{32} = \frac{1}{4}.$$

c) For a team to win a 6-game series, we must account for all possible combinations of 6 games where the winning team wins four games and the losing team wins two games; from these combinations, we want to subtract the combinations where the last game is won by the losing team, since these are the combinations where the winning team will have won the series in only four or five games.

Suppose team A is the winning team and team B is the losing team. There are ${}_6C_4$ ways for team A to win four out of the six games. From these, we want to subtract ${}_5C_4 = 5$ combinations where the last game is won by team B, since these are the series where team A has won in either 4 games (if team B wins the last 2 games and team A wins the first four games) or 5 games (if team B wins the last game and the first, second, third or fourth game, and team A wins any four of the first five games not won by B).

| | | | | | |
|---|---|---|---|---|---|
| Game 1 | A | A | A | A | B |
| Game 2 | A | A | A | B | A |
| Game 3 | A | A | B | A | A |
| Game 4 | A | B | A | A | A |
| Game 5 | B | A | A | A | A |
| Game 6 | B | B | B | B | B |

Thus, we have

$$\begin{aligned} {}_6C_4 - {}_5C_4 &= \frac{6!}{2!4!} - \frac{5!}{4!1!} \\ &= \frac{6\times 5}{2\times 1} - \frac{5}{1} \\ &= 15 - 5 \\ &= 10 \end{aligned}$$

ways for a team to win a 6-game series.

From Pascal's rule, since we know that ${}_5C_3 + {}_5C_4 = {}_6C_4$, we must have ${}_6C_4 - {}_5C_4 = {}_5C_3$ ways for a team to win a 6-game series. If we ponder this result, it does indeed make sense. Out of the six games, team A must win the last game. If team A does not win the last game, the series will be won in either four or five games by team A. The first red column shows a series where team A has already won in 4 games, and the yellow columns show series where team A has won in 5 games.

| | | | | | |
|---|---|---|---|---|---|
| Game 1 | A | A | A | A | B |
| Game 2 | A | A | A | B | A |
| Game 3 | A | A | B | A | A |
| Game 4 | A | B | A | A | A |
| Game 5 | B | A | A | A | A |
| Game 6 | B | B | B | B | B |

Consequently, we need to only worry about the first five games. Since we know that team A will win the last game, team A must win three of the first five games. The number of ways to do this is ${}_5C_3 = \frac{5!}{3!2!} = 10$. Thus, there are 10 ways for team A to win a 6-game series. To account for the case where team B is the winning team and team A is the losing team, we need to double this result, giving us 20 ways for either team to win a 6-game series.

Since there are two choices for the winner of each game, we have $2^6 = 64$ total ways to play 6 games. Therefore,

$$P(\text{World Series is won in 6 games}) = \frac{20}{2^6} = \frac{20}{64} = \frac{5}{16}.$$

d) For a team to win a 7-game series, we must account for all possible combinations of 7 games where the winning team wins four games and the losing team wins three games; from these combinations, we want to subtract the combinations where the last game is won by the losing team, since these are the combinations where the winning team will have won the series in only four, five, or six games.

Suppose team A is the winning team, and team B is the losing team. There are ${}_7C_4$ ways for team A to win four out of seven games; from these, we want to subtract ${}_6C_4 = 15$ combinations where the last game is won by team B, since these are the series where team A has won in either 4 games (if team B wins the last 3 games and team A wins the first four games), 5 games (if team B wins the last game and the first, second, third or fourth game, and team A wins any four of the first five games not won by B), or 6 games (if team B wins the last game, and two of either the first, second, third, fourth, or fifth game, and team A wins any four of the first six games not won by B).

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Game 1 | A | A | A | A | B | A | A | A | B | A | A | B | A | B | B |
| Game 2 | A | A | A | B | A | A | A | B | A | A | B | A | B | A | B |
| Game 3 | A | A | B | A | A | A | B | A | A | B | A | A | B | B | A |
| Game 4 | A | B | A | A | A | B | A | A | A | B | B | B | A | A | A |
| Game 5 | B | A | A | A | A | B | B | B | B | A | A | A | A | A | A |
| Game 6 | B | B | B | B | B | A | A | A | A | A | A | A | A | A | A |
| Game 7 | B | B | B | B | B | B | B | B | B | B | B | B | B | B | B |

<u>Key</u>
Blue = combinations where winning team won in four games
Green = combinations where winning team won in five games
Orange = combinations where winning team won in six games

Thus, we have

$$\begin{aligned} {}_7C_4 - {}_6C_4 &= \frac{7!}{4!3!} - \frac{6!}{4!2!} \\ &= \frac{7\times6\times5}{3\times2\times1} - \frac{6\times5}{2\times1} \\ &= 35 - 15 \\ &= 20 \end{aligned}$$

ways for a team to win a 6-game series.

From Pascal's rule, since we know that ${}_6C_3 + {}_6C_4 = {}_7C_4$, it must be that ${}_7C_4 - {}_6C_4 = {}_6C_3$. Thus, we have ${}_6C_3$ ways for a team to win a 7-game series. If we ponder this result, it does indeed make sense. Out of the seven games, team A must win the last game. If team A does not win the last game, the series will be won in four, five, or six games by team A.

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Game 1 | A | A | A | A | B | A | A | A | B | A | A | B | A | B | B |
| Game 2 | A | A | A | B | A | A | A | B | A | A | B | A | B | A | B |
| Game 3 | A | A | B | A | A | A | B | A | A | B | A | A | B | B | A |
| Game 4 | A | B | A | A | A | B | A | A | A | B | B | B | A | A | A |
| Game 5 | B | A | A | A | A | B | B | B | B | A | A | A | A | A | A |
| Game 6 | B | B | B | B | B | A | A | A | A | A | A | A | A | A | A |
| Game 7 | B | B | B | B | B | B | B | B | B | B | B | B | B | B | B |

Key
Blue = combinations where winning team won in four games
Green = combinations where winning team won in five games
Orange = combinations where winning team won in six games

Consequently, we need to only worry about the first six games. Since we know that team A will win the last game, team A must win three of the first six games. The number of ways to do this is ${}_6C_3 = 20$. Thus, there are 20 ways for team A to win a 6-game series. To account for the case where team B is the winning team and team A is the losing team, we need to double this result, giving us 40 ways for either team to win a 7-game series.

Since there are two choices for the winner of each game, we have $2^7 = 128$ total ways to play 7 games. Therefore,

$$P(\text{World Series is won in 7 games}) = \frac{40}{2^7} = \frac{40}{128} = \frac{5}{16}.$$

**Teaching Suggestions**

- Resist the urge to give direct instructions on a specific approach.
- This problem presents an opportunity to discuss probability.

**Extensions**

8. The Problem of Points – Pascal and Fermat are sitting in a café in Paris and decide to play a game of flipping a coin. If the coin comes up heads, Fermat gets a point. If it comes up tails, Pascal gets a point. The first to get to ten points wins. They each ante up fifty francs, making the total pot worth one hundred francs. They are, of course, playing "winner take all." But then a strange thing happens. Fermat is winning, eight points to seven, when he receives an urgent message that his child is sick and he must rush to his house in Toulouse. The carriage man who delivered the message offers to take him, but only if he leaves immediately. Of course, Pascal understands, but later, in correspondence, the problem arises: how should the hundred francs be divided?

# Problem Set VI

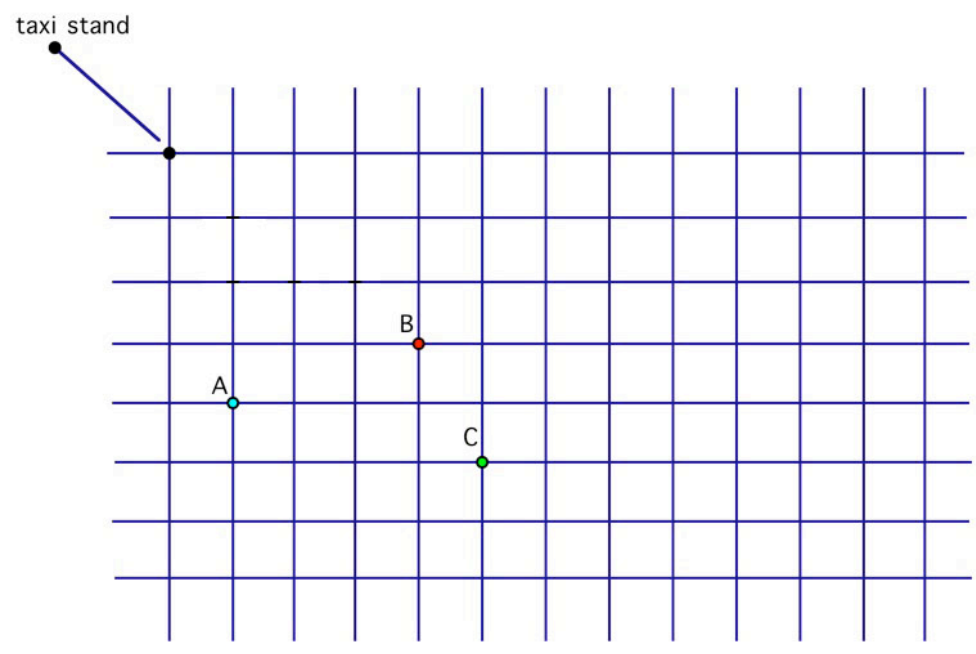
## Lead Problem: The Taxicab Problem

**The Problem**

In the **Taxicab Problem**, students will find the number of shortest routes between two points on a rectangular grid. At first glance, this problem appears to be a routine exercise; however, it has a strong mathematical structure that is a combination of geometry, problem solving, and combinatorics. The problem itself is situated on a coordinate plane, lending itself to plotting points from an origin. The origin in this case is the taxicab stand. How many paths exist between two distinct points? Students may engage with combinatorics, arithmetical progressions, binomial coefficients, and Pascal's Triangle.

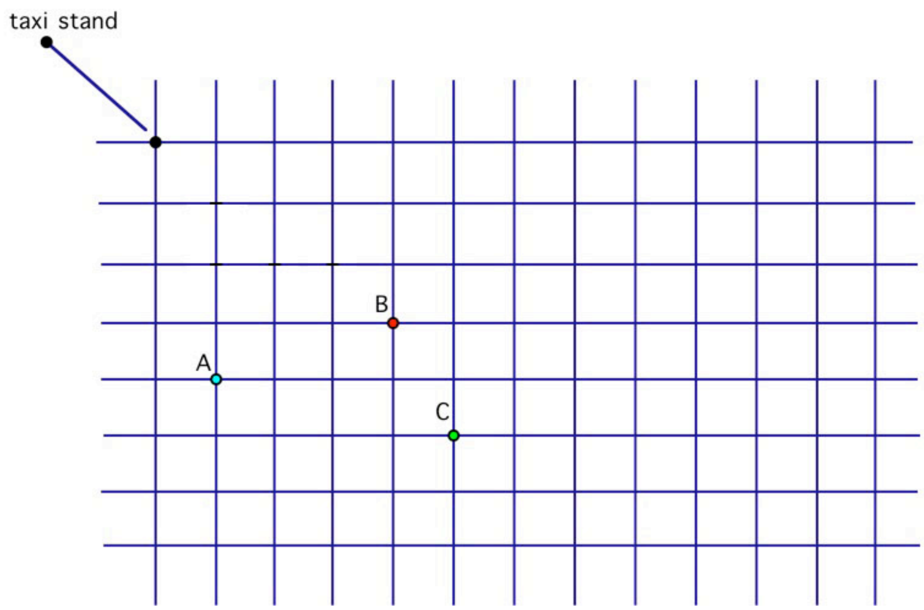
**Version 1**

A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point from the taxi stand and wonders if she could have chosen a shorter route. Develop several questions that relate to each pick-up route as well as to the entire grid and post them in your summary tab.



**Version 2**

A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point from the taxi stand and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each point? How do you and your team know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answers.




**Version 3**

A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point from the taxi stand and wonders if she could have chosen a shorter route.

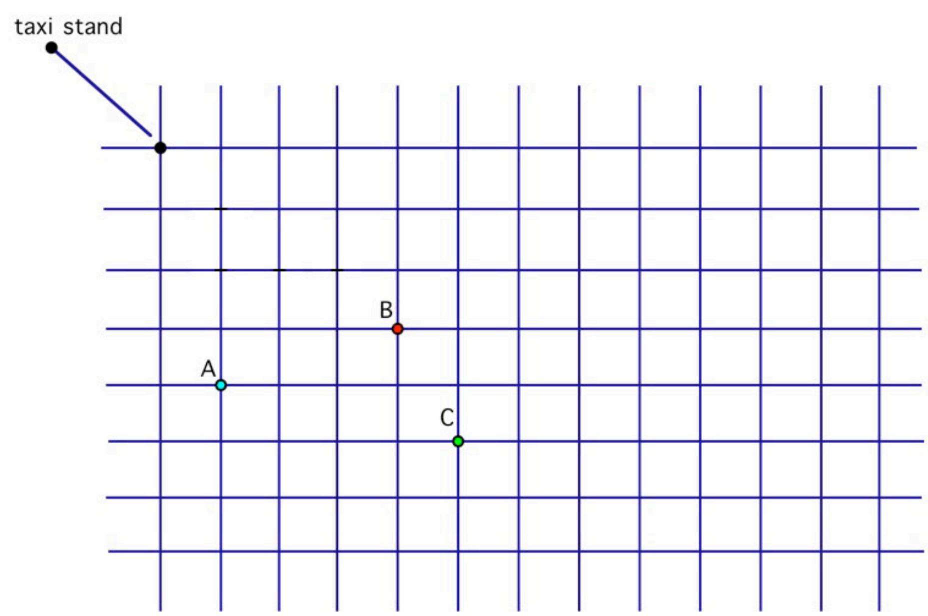
| Question | Answer |
|---|---|
| What is the shortest distance from the taxi stand to point A? | |
| How many ways can you travel from the taxi stand to point A that will be considered the shortest route? | |
| How many blocks east is the taxi stand to point A? | |
| How many blocks south is the taxi stand to point A? | |
| What is the shortest distance from the taxi stand to point B? | |

| | |
|---|---|
| How many ways can you travel from the taxi stand to point B that will be considered the shortest route? | |
| How many blocks east is the taxi stand to point B? | |
| How many blocks south is the taxi stand to point B? | |
| What is the shortest distance from the taxi stand to point C? | |
| How many ways can you travel from the taxi stand to point C that will be considered the shortest route? | |
| How many blocks east is the taxi stand to point C? | |
| How many blocks south is the taxi stand to point C? | |

Discuss as a team whether any of the answers from the above table are familiar.

Pick another point on the grid and follow the above questions. Do you notice a pattern?

Work together with your team to justify your answers.



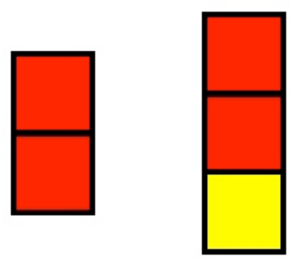
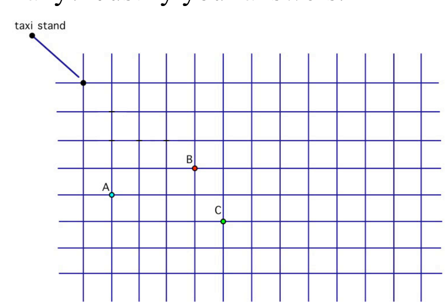

**Answer Check** If you have trouble counting the number of shortest routes for points farther away from the taxi stand, answer these questions:

- Have you counted the number of shortest routes to points closer to the taxi

stand?
- What patterns do you notice about these routes?
- How can you use these patterns to help you count the number of shortest routes to points farther away?

If you think you have counted all the shortest routes, answer these questions:
- Does everyone in your team agree to the same answer? If not, have you compared your numbers of shortest routes and the actual routes?
- What patterns has your team noticed?
- How can you be sure you have counted all the possible shortest routes?
- How can you justify this?
- Is your team's explanation clear and complete?
- What connections can you make between this problem and previous problems?

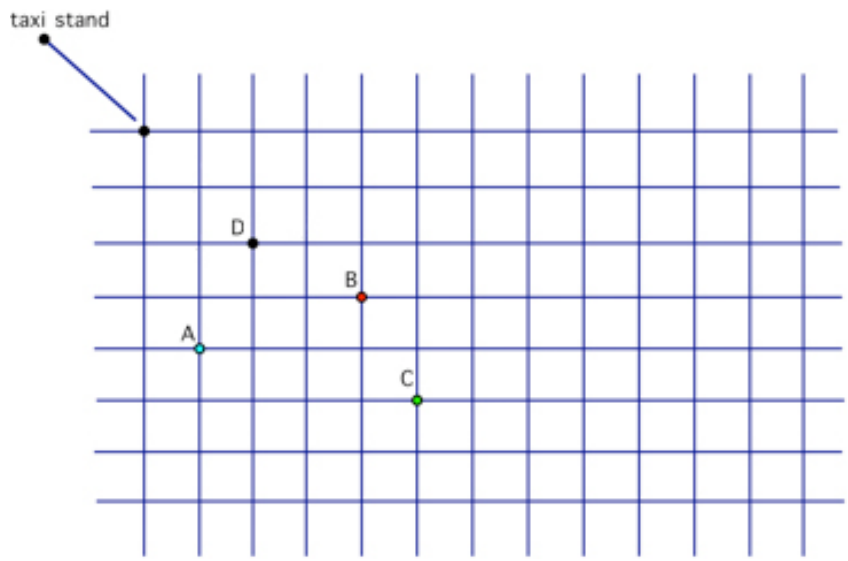
**Compare and Contrast**

| | |
|---|---|
| You and your team have an unlimited supply of red and yellow squares. You will use these squares to build towers. Pictured below are two examples of towers: a tower two-squares tall and a tower three-squares tall.<br><br>Work together and make as many different towers four-squares tall as possible when selecting from two colors. See whether your team can plan a good way to find all the towers four-squares tall when selecting from two colors. How can you justify to other teams that you have found all the towers?<br><br>Insert Team's Solution Here | A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point from the taxi stand and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answers.<br><br><br><br>Insert Team's Solution Here |

1. Compare and contrast above two problems and write about what you and your teammates notice.

2. Compare and contrast your team's solutions to the two problems. Write about what you and your teammates notice.

**Our Solutions**




**Strategy – Linking to the Towers problem and Pascal's Triangle**

Each shortest trip that the taxi driver makes to any of the destinations in the city must consist of driving east a certain number of blocks and driving south a certain number of blocks. If the taxi driver drives north or west during her trip, she would be driving away from her destination, thus making her route longer. How many blocks east and how many blocks south must the driver drive to reach a particular destination point?

Consider point D above. To drive to point D from the taxi stand, the taxi driver needs to traverse a route consisting of two blocks east and two blocks south. If we use S to represent south and E to represent east, then the following are all the possible, different shortest routes:

SSEE, SESE, ESSE, SEES, ESES, and EESS.

In total, there are six shortest routes. Note that this number is the same as all possible, different towers four tall, selecting from two different colors, with two of each color represented. Suppose the two colors are red (R) and yellow (Y). Then we have six four-tall towers with two red blocks and two yellow blocks:

RRYY, RYRY, YRRY, RYYR, YRYR, and YYRR.

This equivalence applies to all of the number of shortest routes from the taxi stand to any of the points in the grid. If we consider all of the four-tall towers when selecting from two colors, the numbers of these towers are equivalent to the numbers in the $4^{th}$ row of Pascal's triangle (1, 4, 6, 4, 1). These are equivalent to the number of different routes consisting of 4 blocks south and 0 blocks east; 3 blocks south and 1 block east; 2

blocks south and 2 blocks east; 1 block south and 3 blocks east; and 0 blocks south and 4 blocks east, respectively. The equivalence between the number of towers and routes is illustrated below:

| | |
|---|---|
| RRRR | SSSS |
| | |
| RBBB | SEEE |
| BRBB | ESEE |
| BBRB | EESE |
| BBBR | EEES |
| | |
| RRBB | SSEE |
| RBRB | SESE |
| BRRB | ESSE |
| RBBR | SEES |
| BRBR | ESES |
| BBRR | EESS |
| | |
| BRRR | ESSS |
| RBRR | SESS |
| RRBR | SSES |
| RRRB | SSSE |
| | |
| BBBB | EEEE |

When we place all of the number of the shortest routes to each point from the taxi stand onto the grid, the resulting array of numbers is isomorphic to Pascal's triangular array rotated 45 degrees, counterclockwise. That is, the each diagonal of the taxicab grid corresponds to each of the rows of Pascal's triangle.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| 1 | 4 | 10 | 20 | 35 | 56 | 84 |
| 1 | 5 | 15 | 35 | 70 | 126 | 210 |
| 1 | 6 | 21 | 56 | 126 | 252 | 462 |
| 1 | 7 | 28 | 84 | 210 | 462 | 924 |

Note that the diagonal of 1, 4, 6, 4, 1 is located fifth from the upper left-hand corner.

In general, the number of shortest routes from the taxi stand to a point on the grid that is *x* blocks east and *y* blocks west away is equivalent to the number of towers that are $x + y$ blocks tall and consists of *x* red blocks and *y* yellow blocks. These numbers can be found by calculating ${}_{x+y}C_x$ or ${}_{x+y}C_y$, as was seen in the Towers problem.

To get to point A, the taxi driver needs a shortest route that requires her to drive 1 block east and 4 blocks south. The total number of shortest routes to A is ${}_5C_1 = 5$.

To get to point B, the taxi driver needs a shortest route that requires her to drive 4 blocks east and 3 blocks south. The total number of shortest routes to B is ${}_7C_4 = 35$.

To get to point C, the taxi driver needs a shortest route that requires her to drive 5 blocks east and 5 blocks south. The total number of shortest routes to C is ${}_{10}C_5 = 252$.

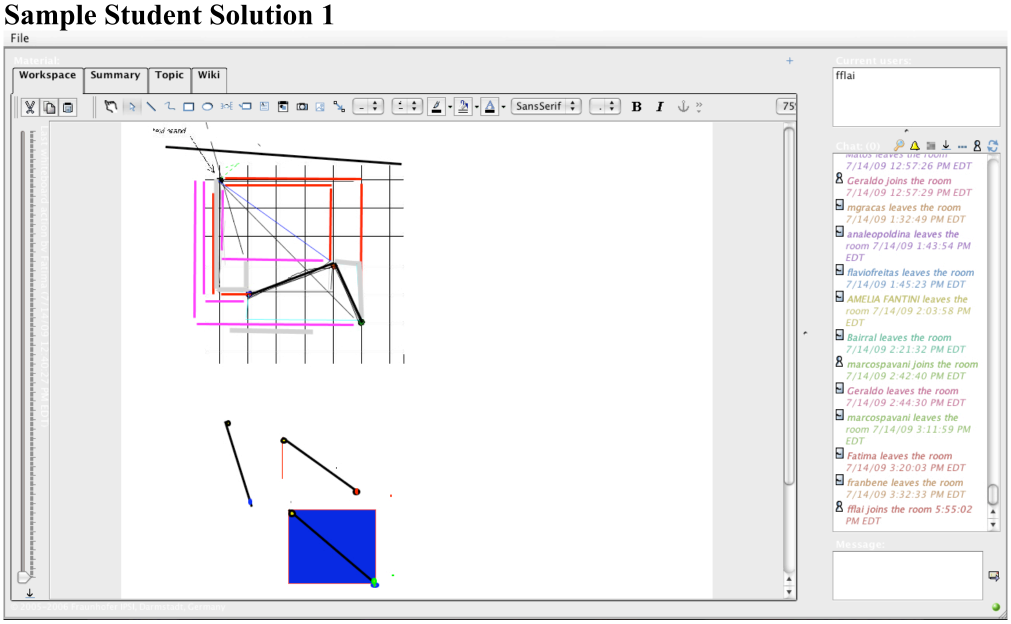
**Sample Student Solutions**

The following two solutions are screenshots of students' work on this problem, which we have provided to illustrate the kinds of ideas at which students can arrive within the environment.

As you look through your own students' work, you may have to read through the chat in addition to examining the workspace to find evidence of your students' communication.

**Group of five 14-15-year old students**

**Sample Student Solution 1**

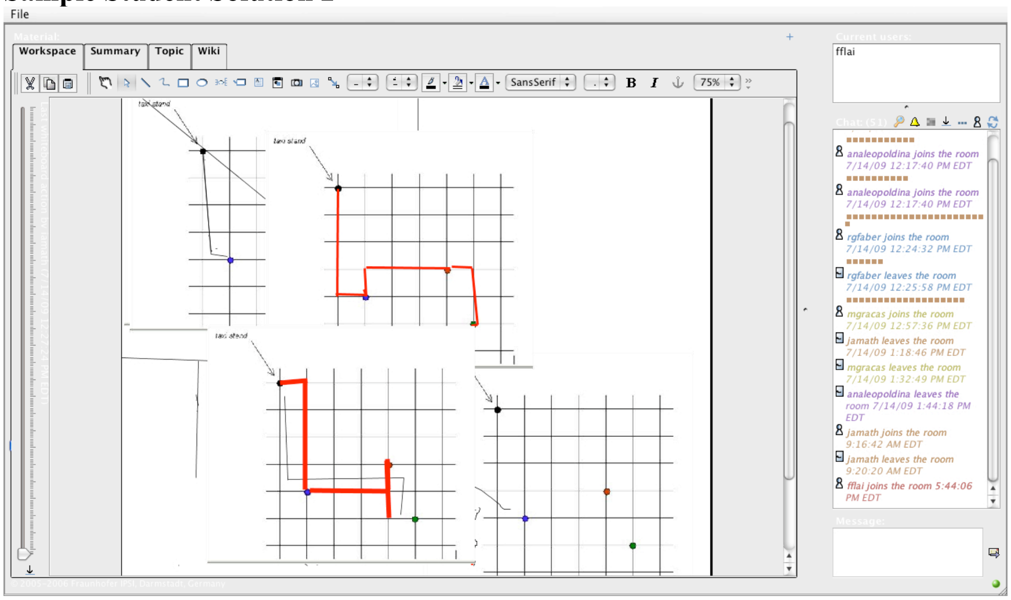



In this solution, students seem to believe that it is possible to travel off the street grid to get from one point to the next, which is actually not allowed in this problem. It is an interesting interpretation, since by travelling on the diagonal, the students are essentially creating streets that are not given in the original diagram. It would be interesting to know how students would count the "length" of a diagonal street.

It also seems that the students interpreted the problem to mean that the taxi driver started at the taxi stand and traveled to each stop successively, instead of going back to the taxi stand after every stop. It is unclear from this screenshot whether the students have come up with the number of shortest routes to each point.

**Group of three 14-15-year old students**

**Sample Student Solution 2**




In this solution, as in the first one, the students have interpreted the problem to mean that the taxi driver travels from the taxi stand to each stop in succession. They seem to have illustrated different routes to reach the three stops. It is not clear whether they have found the number of shortest routes to each point.

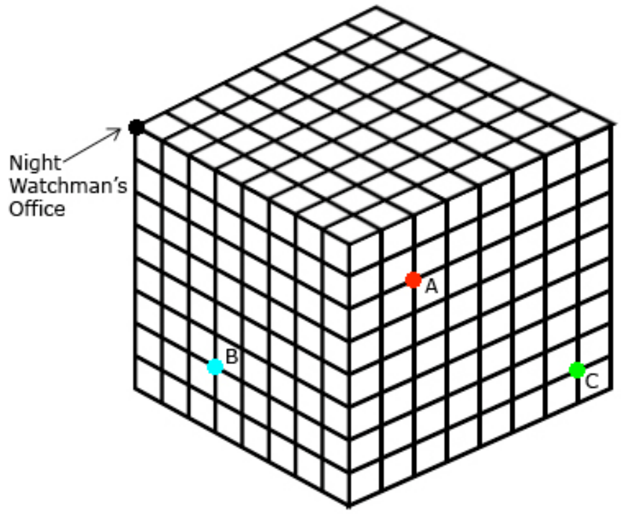
**Teaching Suggestions**

Resist the urge to give direct instructions on a specific approach. Students may notice that the number of shortest routes mapped out on the city grid is isomorphic to Pascal's triangle. When students realize that all of the shortest routes to a destination in the city contain the same number of southward moves and the same number of eastward moves, they can begin to use this information to find a solution. Each of the number of routes describes a case of the towers problem when selecting from 2 colors. If we think about moving one block south as one color, and moving one block east as another color, then when we want to have a certain number of moves south and another number of moves east, this is the same as creating a tower with a certain number of one color and another number of a second color.

**Extensions**

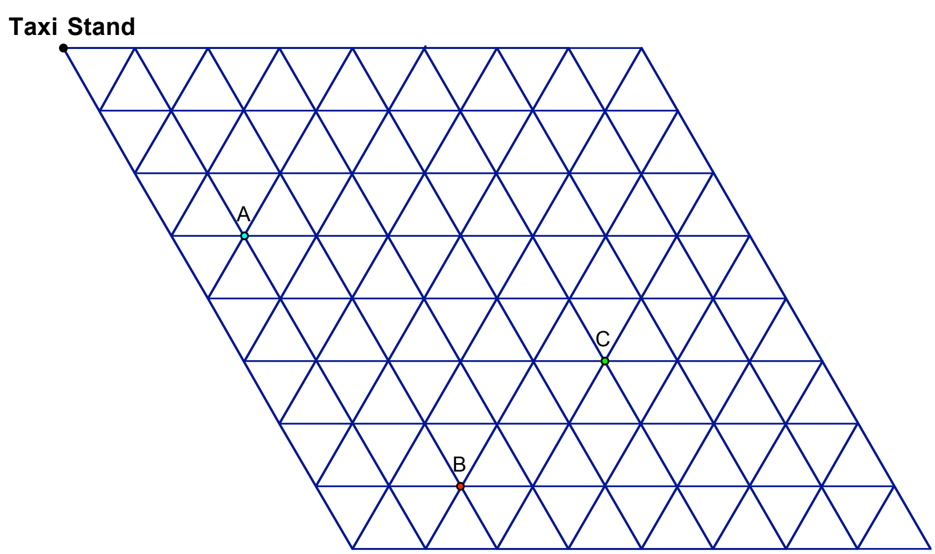
1. Night Watchman Problem (Three-Dimensional Taxicab Problem)

   A night watchman at a New York City apartment building, represented by the rectangular prism in the diagram below, must visit three different apartments for inspection (A, B, and C). Each time he visits one apartment, he goes back to his office. He has decided to consider all possible routes to each apartment. What is the shortest route from his office to each apartment? How do you know it is the shortest? Is there more than one shortest route to each point? If so, how many? Justify your answers.




2. Taxicab Problem with Triangular Street Grid.

   A taxi driver is given a specific territory of a town, represented by the grid in the diagram below. All trips originate at the taxi stand, the point in the top left corner of the grid. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated by the other points on the grid. To pass the time, she considers all the possible routes she could have taken to each pick-up point from the taxi stand and wonders if she could have chosen a shorter route. What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answers.



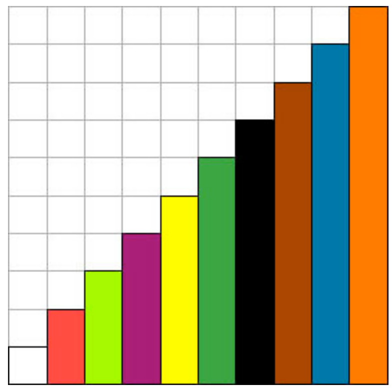
# Problem Set VII

## Lead Problem: Trains of Cuisenaire Rods Problem

**The Problem**

**Version 1**

The picture below displays one of each color of Cuisenaire rods. From left to right, the rod colors are white, red, lime, purple, yellow, green, ebony, chocolate, blue, and orange.



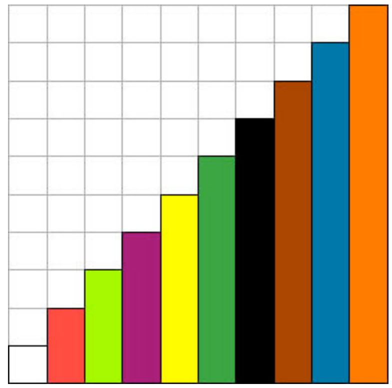
When a rod or rods are used to create a length, you can say that the rod or rods form a train. For instance, below, a red rod and a black rod form a train that is equivalent in length to a blue rod.

And below, a blue rod and light green rod form a train that is equivalent in length to a train composed of a yellow rod and black rod.

Imagine that you have as many rods of each color as you need. Create a math problem using the Cuisenaire rods and the definition of trains. Answer the problem and justify your solution in the Summary tab.

**Version 2**

The picture below displays one of each color of Cuisenaire rods. From left to right, the rod colors are white, red, lime, purple, yellow, green, ebony, chocolate, blue, and orange.



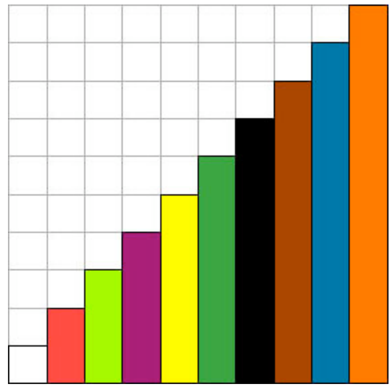
When a rod or rods are used to create a length, you can say that the rod or rods form a train. For instance, below, a red rod and a black rod form a train that is equivalent in length to a blue rod.

And below, a blue rod and light green rod form a train equivalent to the length of a train composed of a yellow rod and black rod.

Imagine that you have as many rods of each color as you need. For each color rod, create as many different trains as are possible that are equivalent in length to it. How many different trains are possible? Organize your results so that someone looking at it will not doubt that you have found all possible trains.

**Version 3**

The picture below displays one of each color of Cuisenaire rods. From left to right, the rod colors are white, red, lime, purple, yellow, green, ebony, chocolate, blue, and orange.



When a rod or rods are used to create a length, you can say that the rod or rods form a train. For instance, below, a red rod and a black rod form a train that is equivalent in length to a blue rod.

And below, a blue rod and light green rod form a train that is equivalent in length to a train composed of a yellow rod and black rod.

Imagine that you have as many rods of each color as you need. For each color rod, create as many different trains as are possible that are equivalent in length to it. How many different trains are possible? Justify your answer.

Fill in the table below to keep track of the number of trains your team creates for each color rod. For example, in the row marked L (for the lime rod), how many trains can be created that contain only one rod, then how many trains that contain two rods, then that contain three rods, and so on.

Number of Rods in each Train

| Train Length | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | W | | | | | | | | | | |
| | R | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | L | | | | | | | | | | |
| | P | | | | | | | | | | |
| | Y | | | | | | | | | | |
| | G | | | | | | | | | | |
| | E | | | | | | | | | | |
| | C | | | | | | | | | | |
| | B | | | | | | | | | | |
| | O | | | | | | | | | | |

Key:
W = white; R = red; L = lime; P = purple; Y = yellow; G = green; E = ebony; C = chocolate; B = blue; O = orange.

**Extensions**

1. How many different expressions are possible whose sum is 6, using only positive integers?

2. For each color rod, create as many different trains as are possible that are equivalent in length to it, using only white and red rods in each train. How many different trains are possible? Justify your answer.