Understanding Students' Mathematical Ideas and Reasoning: Problem Solving in the Online Environment of VMT Chat

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In this chapter, we examine collaborative learning mediated by a computer communication system. We trace collaborative learning as an interactive, layered building of understanding among learners. We present a study of four students who participate in collaborative, mathematical problem solving within the online environment of Virtual Math Teams-Chat (VMT Chat). Similar to other computer-meditated communication systems, VMT Chat presents communicative affordances and constraints to users that influence their discursive interactions. We are interested in how students use the affordances of the virtual environment and what mathematical ideas and lines of reasoning are evidenced in their interactions. Correspondingly, VMT Chat presents methodological affordances and constraints to researchers such as us interested to investigate how students exchange and develop emergent mathematical ideas and lines of reasoning. In this chapter, we also explore an analytic approach for inquiring into the archived interactions of students collaborating through VMT Chat on mathematical problem solving.

The Web-based, collaboration environment of VMT Chat has two interaction spaces: whiteboard and chat. Analyses of users' online problem solving typically focuses on their chat text and referenced whiteboard inscriptions. Among others, the chat and the whiteboard spaces are affordances of the environment. In the analysis that we present, for reasons that we will discuss, our analytic attention focuses on the evolution of participants' whiteboard inscriptions to gain insight into the development of their mathematical ideas and reasoning as they solve an open-ended mathematics problem.

CONCEPTUAL FRAMEWORK

In this study, key conceptual terms include discourse, student-to-student or peer mathematical discussion, collaborative interaction, problem solving and mathematical ideas. Discourse here refers to language (natural or symbolic, oral or gestic) used to carry out tasks—for example, social or intellectual—of a community. In agreement with Pirie and Schwarzenberger (1988), student-to-student or peer conversations are mathematical discussions when they possess the following four features: are purposeful, focus on a mathematical topic, involve genuine student contributions, and are interactive. Collaborative interaction is defined as individuals affecting each other and working together for a common purpose. In addition, in the context of the data of this study, the student-to-student, discursive collaborations involve minimal, substantive interaction with a teacher or researcher.

A paramount goal of mathematics education is to promote among learners effective problem solving. Mathematics teaching strives to enhance students' ability to solve individually and collaboratively problems that they have not previously encountered. Nevertheless, the meaning of mathematical "problem solving" is neither unique nor universal. Its meaning depends on ontological and epistemological stances, on philosophical views of mathematics and mathematics education. For the purposes of this chapter, we subscribe to how Mayer and Wittrock (1996) define problem solving and its psychological characteristics:

> Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver (Mayer, 1992). According to this definition, problem solving has four main characteristics. First, problem solving is *cognitive*—it occurs within the problem solver's cognitive system and can be inferred indirectly from changes in the problem solver's behavior. Second, problem solving is a *process*—it involves representing and manipulating knowledge in the problem solver's cognitive system. Third, problem solving is *directed*—the problem solver's thoughts are motivated by goals. Fourth, problem solving is *personal*—the individual knowledge and skills of the problem solver help determine the difficulty or ease with which obstacles to solutions can be overcome. (p. 47)

Coupled with these cognitive and other psychological characteristics, problem solving also has social and cultural features. Some features include what an individual or cultural group considers to be a mathematical problem (cf., D'Ambrosio, 2001; Powell & Frankenstein, 1997), the context in which an individual may prefer to engage in mathematical problem solving, and how a problem solvers understand a given problem as well as what they considers to be adequate responses (cf., Lakatos, 1976). In instructional settings, students' problem solving are strongly influenced by teachers' representational strategies, which are constrained by cultural and social factors (Cai & Lester, 2005). Moreover, in online settings, the affordances and constraints of the virtual environment provide another dimension to the social and cultural features of problem solving.

Investigating the development of mathematical ideas and reasoning, we code for instances in the data of participants' online communications of their discursive attention to any of four markers of mathematical elements—objects, relations among objects, dynamics linking different relations, and heuristics (Gattegno, 1988; Powell, 2003). In their chat text and whiteboard inscriptions, participants either communicate affirmations or interrogatives about these mathematical elements, and as such, we code for eight different types of critical events that provide insight into participants' mathematical ideas (Powell, 2003). The matrix in Table 1 contains the critical event codes.

Table 1.

Matrix of Event Types Designated as Critical

Subject and type of utterance or inscription	**O**bjects	**R**elations among objects	**D**ynamics linking different relations	**H**euristics
Affirmations	**AO**	**AR**	**AD**	**AH**
Interrogatives	**IO**	**IR**	**ID**	**IH**

It is possible that an interaction receives multiple codes. We analyze the mathematical ideas and forms of reasoning that participants produce individually (in pairs) and as a team, tracing the development of their ideas and reasoning patterns over the course of the problem-solving session.

Epistemologically, we view learning or knowledge creation as a process of conceptual change whereby individuals and groups of individuals construct new understandings of reality. Through social interactions, learners engaged with mathematics seek meaning and search for patterns, relationships, and dynamics linking relationships among objects and events of their experiential world.

METHOD

The data come from eight students from a class for undergraduate students who are elementary teacher candidates, enrolled in a course whose theme is the use of digital technologies for the teaching of mathematics in elementary schools, "Mathematics and Instructional Technology." The second author taught this class. On this particular day, he decided to have his students work on the Pizza Problem using the VMT Chat software. The problem is worded as follows:

> A local pizza shop has asked us to help them keep track of pizza sales. Their standard "plain" pizza contains cheese with tomato sauce. A customer can then select from the following toppings to add to the whole plain pizza: peppers, sausage, mushrooms, bacon, and pepperoni.
>
> How many different choices for pizza does a customer have?
>
> List all the possible different selections. Find a way to convince each other that you have accounted for all possibilities.

Because the students arrived at various times, the second author ended up grouping the students into one group of one, two groups of two, and one group of three students, with each group occupying a separate computer. The two groups of two were assigned to one chat room, while the groups of one and three were assigned to a different chat room.

For this report, we analyze the data from one chat room. Having reviewed both sets of data using the ConcertChat player, we realized that the data involving the two pairs was richer and provided an interesting analytic challenge. In what follows, we refer to each pair collectively with the screen name of the one individual of the pair who signed into the chat room. We refer to the first pair as SOSilvestre, whose participants are Sonia and Lyndsey. They used Sonia's screen name. The second pair is suzyn17, whose participants are Susan and Komal, using Susan's screen name. In our report, we consider SOSilvestre and suzyn17 as plural nouns, referring to the two participants of each pair. The pairs we asked to interact in the chat room as if they were located at distant sites.

VMT Chat maintains a persistent record of the chat room interactions, the dual-interaction spaces. This record is our data source. To analyze these data, we adapted Powell, Francisco and Maher's (2003) method for analyzing videodata. Their methodological approach is for qualitative investigations into the development of mathematical ideas with the aid of video recording. In our analysis, we treat as video the unfolding interactions in the chat and whiteboard spaces displayed through the ConcertChat player. From watching the ConcertChat player of the chat room in which these students communicated, we were able to compile an objective description and a preliminary interpretation of the actions in the chat room; that is, the actions revealed in the chat and the whiteboard areas. In the process, we familiarized ourselves with the sequence of whiteboard actions and chat texts in the VMT Chat room, as well as conducted a preliminary analysis of what has happened.

To analyze the data, we first attentively viewed the data in the ConcertChat player several times at various speeds to familiarize ourselves with the events. Afterwards, we discussed the data amongst ourselves. Also, as part of a professional development program for teacher candidates of secondary mathematics, we engaged undergraduate mathematics students in viewing and discussing the data.

After these initial viewings of the data, using the ConcertChat player, we carefully stepped through the data to create an objective description of each action that occurred in the chat and whiteboard frames. These descriptions were created for five-minute interval. Following the description, we came up with an initial interpretation of the chat room actions. Table 2 contains the first five-minute description of the chat room actions and in an adjacent column it is an interpretation of those actions.

Table 2.

Example of time-interval description and interpretation of chat room data

Time	Description	Interpretation
12:06 – 12:11	SOSilvestre creates an ellipse in the upper left side of the whiteboard, and then creates another ellipse also in the upper left side of the whiteboard. Suzyn17 creates a scribble in the upper left side of the whiteboard and then deletes this scribble. SOSilvestre creates a third ellipse in the upper left side of the whiteboard. Suzyn17 creates an ellipse in the upper left side of the whiteboard. SOSilvestre deletes her third ellipse. Suzyn17 deletes her ellipse. Suzyn17 creates a second ellipse in the upper middle part of the whiteboard. Suzyn17 creates a third ellipse in the upper left part of the whiteboard. SOSilvestre deletes one of her own ellipses. Suzyn17 creates a fourth ellipse and fifth ellipse in the upper right side of the whiteboard. Suzyn17 deletes the fifth ellipse. Suzyn17 types "Plain Pizza" into the chat window and uses the referencing tool to reference her third ellipse.	It appears that the students are testing out the software, perhaps trying to perfect their technique for representing pizzas on the whiteboard. By using the referencing tool after typing "Plain Pizza" into the chat window, suzyn17 might have been attempting to refer to one of her ellipses as a plain pizza. Suzyn17 initiates verbal interaction and further interaction by coloring and adjusting SOSilvestre's representation of the tomato and cheese pizza. (We are interpreting plain T & C to mean a plain pizza with tomato sauce and cheese.)

From the interpretations, we composed a narrative of what happened to help us better understand the actions that occurred. In both the interpretation and narrative, we attempted to better understand the motivations behind certain students' actions. For example, in the above interpretation, we inferred that Suzyn17 used the referencing tool after typing "Plain Pizza" into the chat window to refer to one of her ellipses as a plain pizza.

From the description and interpretation, we created a storyline to make sense of the actions the students take to make sense of the problem, and of the sequence of subsequent actions the students take to present and refine their solutions.

DETAILED DESCRIPTION OF DATA

What follows is an uninterrupted description of what transpires in the nearly two-hour, problem-solving session of the four chat-room participants. In the next session, we present our results, followed by a discussion.

At the beginning of the session, both suzyn17 and SOSilvestre draw ellipses on the whiteboard. Suzyn17 denote one of their ellipses as a plain pizza, using the referencing tool. Three minutes later, SOSilvestre label one of their ellipses as a plain tomato and cheese pizza. Afterward, SOSilvestre list in a textbox four two-topping pizzas all containing the letter "P."

At this point, suzyn17 and SOSilvestre occupy different sides of the whiteboard. Suzyn17 are doing all their work on the right side of the whiteboard, while SOSilvestre use the left side for theirs.

SOSilvestre create a key for the pizza toppings. P stands for pepperoni, S stands for sausage, M stands for mushrooms, B stands for bacon, and R stands for pepperoni. After creating the key, SOSilvestre list in separate columns two-, three-, and four-topping pizzas with peppers as one of the toppings. SOSilvestre list one pizza with only peppers, four pizzas with peppers and one other topping, four pizzas with peppers and two other toppings, two pizzas with peppers and three other toppings, and an additional pizza with peppers and two other toppings. Then SOSilvestre start listing pizzas with sausage in a separate textbox. They list one pizza with sausage, one pizza with sausage and three other toppings, two pizzas with sausage and two other toppings, and three pizzas with sausage and one other topping.

Suzyn17 start listing two-topping pizzas with peppers as one of the toppings by placing each combination in a separate textbox within an ellipse, and arranging these ellipses vertically, representing each topping combination as a separate pizza. They also label their column at the top with a textbox containing the word pepper. The last pizza that they create is a pizza with just peppers.

Suzyn17 use the same representational scheme to list two topping pizzas containing sausages. They places a textbox at the top to the left of their textbox labeled peppers and draws three circles for these two-topping pizzas with sausage as one of the toppings.

Returning to SOSilvestre, after they list pizzas with sausage as one of the toppings, they list in another textbox pizzas with mushrooms as one of the toppings. They list one pizza with mushroom, two pizzas with mushroom and one other topping, and one pizza with mushroom and two other toppings. Then SOSilvestre list in yet another textbox pizzas with Bacon as one of the toppings. They initially list two pizzas with bacon and one other topping, and one pizza with bacon and two other toppings, and then add to the beginning of this list one pizza with bacon. SOSilvestre then list one plain pizza with pepperoni also in a separate textbox. Altogether, they have five separate textboxes, one for each of the five toppings.

After creating pizzas with sausages, suzyn17 places a textbox labeled mushrooms to the left of their textbox labeled sausage, and list under this label two pizzas with mushroom and one other topping. They list a plain pizza with mushroom as the last pizza. Then, suzyn17 create a textbox to the left of the textbox labeled mushroom, labels it bacon, and draws two ellipses. Under this textbox, suzyn17 list one pizza with bacon and one other topping, and one pizza with just bacon. Suzyn17 then creates a textbox labeled pepperoni to the left of the textbox labeled mushroom. Below it, they draw one ellipse and, in it, place a textbox labeled pepperoni.

Suzyn17 rearrange the ellipses they have created so that they are closer together. Meanwhile, SOSilvestre rearrange their representation of the pizzas with peppers. Pizzas with peppers and one other topping are in one column, pizzas with peppers and two other toppings are in a second column, pizzas with peppers and three other toppings are in a third column, and pizzas with peppers and four other toppings are in a row below the latter three columns.

Suzyn17 now list pizzas with peppers and two other toppings, peppers and three other toppings, peppers and four other toppings, sausages and two other toppings, and sausages and three other toppings. While they do this, SOSilvestre put in parentheses the number of combinations in each of their textboxes. Thus far, they find fourteen pizzas with peppers, seven pizzas with sausages (and no peppers), four pizzas with mushrooms (and no peppers or sausages), three pizzas with bacon (and no peppers, sausages, or mushrooms), and one pizza with pepperoni (and no peppers, sausages, mushrooms, or bacon).

SOSilvestre creates an ellipse on suzyn17's side, containing a textbox listing a pizza with pepper and two other toppings: sausage and pepperoni. They fill the space within the ellipse red. Suzyn17 then list a pizza containing peppers and two other toppings as well as a pizza with mushrooms and two other toppings.

Suzyn17 note SOSilvestre's contribution to her listings and type into the chat window, "WHO COLORED MY PIZZA?" SOSilvestre respond, "i did I did." SOSilvestre then write, "pizza red right?" SOSilvestre also type, "lol [laugh out loud]." Suzyn17 then type, "WHERE'S THE CHEESE?" SOSilvestre fill the space in the textbox containing "P/S/R" yellow, and answers, "there it is."

After a nine-minute pause, SOSilvestre draw more ellipses on suzyn17's side of the whiteboard. SOSilvestre list three pizzas with peppers and two other toppings. The chat between suzyn17 and SOSilvestre continue. SOSilvestre type, "did u say u have 33 combos." Suzyn17 respond, "CAN YOU KEEP UP PLEASE-IT'S 34." SOSilvestre type, "darn I have gotten passed 29" and immediately type "haven't." SOSilvestre type, "what happen to the rest of the pizza pies… huh." Suzyn17 type, "we ate them." Suzyn17 type "what problem are rob and jenna doing?" SOSilvestre type, "mo clue prob another ICT section." Suzyn17 type "it." Suzyn17 type, "it's back to 33-we repeated…what do you want from us?" SOSilvestre type, "a clue.. lol." SOSilvestre respond, "no."

When SOSilvestre type, “what happen to the rest of the pizza pies…huh,” they are indicating that they see on whiteboard only 17 of the 34 of pizza pies that suzyn17’s claim to have. SOSilvestre ask suzyn17 for help in the chat. Suzyn17 help SOSilvestre by placing the combination of psbr into SOSilvestre’s list of pizzas with peppers as a topping. SOSilvestre incorporate this combination into their listing of pizzas with peppers, and changing the number of such pizzas from fourteen to fifteen.

Suzyn17 now list possible pizza toppings in a different manner. In a textbox, they list 10 possibilities for pizzas with two toppings, 10 possibilities for pizzas with three toppings, 5 possibilities for pizzas with four toppings, and one possibility for pizzas with five toppings.

SOSilvestre move suzyn17’s arrangement of ellipses upwards. Suzyn17 also help to move their arrangement as well. SOSilvestre then add a plain pizza into their textbox of pizzas containing peppers and change their number of pizzas combinations in that textbox from fifteen to sixteen.

In the midst of moving suzyn17’s arrangement, SOSilvestre change their method of rearrangement. They do not move the ellipse containing “Plain R” directly upwards; they move it directly below the textbox labeled “PEPPORONI.” The textboxes containing “Plain R” and “P/M/B” had been switched by suzyn17 earlier. SOSilvestre replaced them into original rearrangement.

A bit later, SOSilvestre further rearrange the ellipses, and place one-topping pizzas in the top row, two-topping pizzas in the next two rows, three-topping pizzas in the next two rows, four-topping pizzas in the next row, and five-topping pizzas in the last row. The ellipse labeled “Plain R” is beneath the textbox labeled “PEPPORONI.” While placing three-topping pizzas into rows, SOSilvestre draw some ellipses.

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

RESULTS

The student participants worked in pairs and the pairs interacted through the computer-mediated system, VMT Chat, using the two interaction spaces, the chat and the whiteboard frames. At first, the students work some on paper with their partner. After a while, the pairs used the chat room to work through the problem, with one student of each pair controlling the mouse and keyboard. While doing so, the students in each pair communicated face-to-face with each other. Since the pairs did some of their work on paper and talked with each other, it seems that the work in the chat room is a result of what each group member had thought about individually as well as what they had talked about with each other face-to-face. If, for example, each chat room member had been in different physical spaces, they would not have been able to communicate face-to-face, and perhaps the kinds of interactions that took place in the chat room would have been different.

One possible difference is the feature of the virtual environment that the students employed to interact. In this study, the two pairs interacted for the vast majority of the time through inscriptive postings on the whiteboard. In the nearly two hours of interaction, the students rarely used the chat frame to communicate. Consequently, our analysis of the mathematical ideas and reasoning that the students engaged is not based on their textual communication but rather is based on an examination of the evolution of their inscriptive whiteboard interactions.

Interaction around heuristics and mathematical ideas and reasoning is the hallmark of the pairs' chat room actions. The participants first interact around issues of heuristics. Initially on the whiteboard, SOSilvestre and suzyn17 jointly create ellipses and suzyn17 use the referencing tool to connect their chat text "Plain Pizza" to an ellipsis. They may consider this technique too cumbersome and decide to label ellipses directly using textboxes placed on the ellipses. For instance, in a textbox positioned on an ellipsis, SOSilvestre type "plain T & C," meaning a plain tomato and cheese pizza, and suzyn17 color this object red.

The chat-room participants decide how to use the workspace of the whiteboard. SOSilvestre and suzyn17 partitioned the whiteboard surface into two columns and presented their ideas on the left and right, respectively, for almost the entire session until the pairs seemingly mutually decided to use the entire space to present their work.

Another interactional move of the two pairs concerns establishing notation. In their time, each pair chooses to use a single letter to denote each available topping. When they indicate their notation in the whiteboard frame, it is common between them: P for peppers, S for sausage, M for mushroom, B for Bacon, and R for pepperoni. By publishing their notational scheme on whiteboard, besides providing an abbreviated form for their thinking about the different possible pizzas when there are five toppings from which to choose, each pair makes it possible for the other to follow and contribute to their work. These notational items represent the objects on which each pair operated.

Additionally, each pair uses slightly different objects with which to explore mathematical relationships and develop their solution. Suzyn17 create iconic objects for pizzas—ellipses with textboxes positioned within and labeled using their notation for pizza topping. They arrange these objects within columns with headings for particular toppings. For example, a column headed by "Peppers" has four pizzas each containing peppers with a different other topping and one pizza with just peppers as the topping.

In comparison, SOSilvestre use objects that are distinct from the iconic objects of suzyn17 with which to represent their thinking and display mathematical relationships they perceive among objects. In contrast, suzyn17's iconic representation of different possible pizzas, they use a symbolic representation where combination of letters are the objects with which they exhibit their thinking about different possible pizzas and relationships among these possibilities. For instance, P, S, M, B, and R stand for objects or pizza toppings and combinations of these letters such as PS or SBR indicate different possible pizzas. They present perceived relationships among these objects by listing in separate textboxes, first, all possible, different pizzas containing P or peppers; second, all possible, different pizzas containing S or sausage, expect for those that contain P since they were already accounted for; third, all possible, different pizzas containing M or mushroom, expect for those that contain P or S since they have already been accounted for; fourth all possible, different pizzas containing B, expect for those that contain P, S, or M since they have already been represented, and finally, all possible, different pizzas containing R, expect for those that contain P, S, M, or B since they have already been indicated.

In this work, SOSilvestre make evident their mathematical ideas and reasoning. They control for variables by considering in turn all possible, different pizzas that contain each of the five letters and attend to ensuring that they have no omission or repetition of sequences of letters (pizzas).

Ultimately, Suzyn17 reorganize their mathematical approach. After manipulating their iconic representations of different possible pizzas, they abandon this approach and shift to a symbolic approach, using letters as SOSilvestre did to consider different possible pizzas by cases: all possible pizzas with one topping, with two toppings, with three toppings, with four toppings, and with five toppings.

DISCUSSION

Our aims were to investigate, based on data gathered from chat-room participants' mathematical problem solving within the VMT Chat environment, how to study chat-room participants' development of mathematical ideas and lines of reasoning and, in the interactive spaces of VMT Chat, what ideas and reasoning are evident.

The data for this study provide an analytic challenge that had to be overcome to make sense of the chat room interaction of the participants. The chat-room participants hardly interacted in the chat frame of VMT and used the whiteboard almost exclusively. This meant that we had to follow the evolution of their inscriptions on the whiteboard to understand the emergence of their mathematical ideas and reasoning as the solved the Pizza Problem. Fortunately, VMT-Chat provides a persistent record a chat room's two interactional spaces—chat and whiteboard frames—which can be replayed in real time or an integral multiple of real time. This allowed us to analyze the evolution of the whiteboard inscriptions much as we would do a video recording and, therefore, adapting a videodata analytic technique for inquiring into the development of learners' mathematical ideas and reasoning (cf., Powell et al., 2003).

Interestingly, the two pairs of students immediately started to work as two separate units. In this sense, they were like two entities of a single dyad. In the psychological literature on problem solving, it is accepted that when a dyad is engaged in solving a problem that typically one entity begins to solve the problem while the other listens to the ensuing solution attempt (cf., Shirouzu, Miyake, & Masukawa, 2002). The speaker may be talking out loud while solving a problem while her partner listens. Analogously, one pair presenting their solution on the whiteboard is like a speaker talking aloud their problem-solving process. However, the data of this study shows that instead both entities of the dyad simultaneously “talked” aloud their ensuing solution and that the non-ephemeral nature of their communication medium allowed each entity to “hear” the other while “talking” aloud their problem-solving attempt. An affordance of the virtual environment may have allowed for this simultaneous solving of the problem by both entities of the dyad. In a traditional dyad, it would be difficult for both members to solve a problem out loud while paying attention to each other as well as to her own work, because two people cannot speak at once. Moreover, it is difficult to think in one way when a different way of thinking is being described aloud. In this virtual environment, perhaps because the workspace is shared, relatively large, equally visible to both pairs, and communication is non-verbal, it is easier for both pairs to go about problem solving individually while still paying attention to what the other group was doing.

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