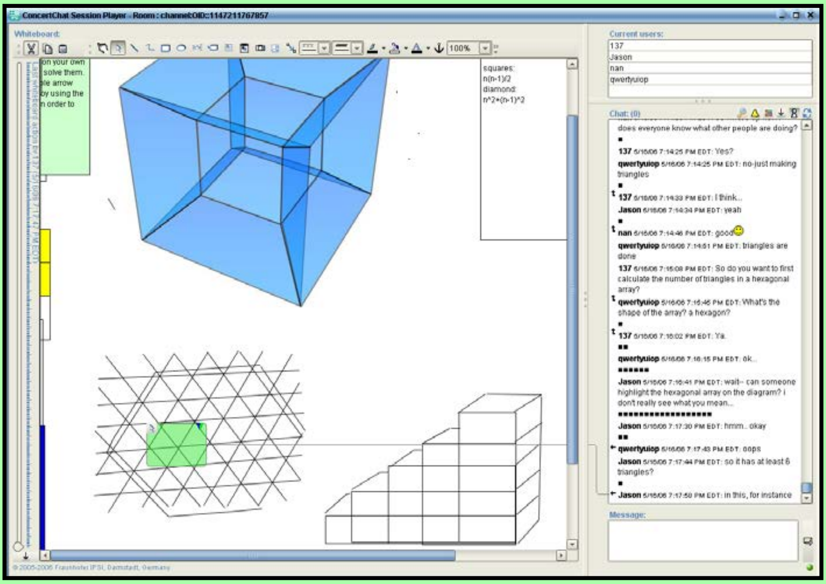
Gerry Stahl's assembled texts volume #12

Essays in Online Mathematics Interaction



Gerry Stahl

Gerry Stahl's Assembled Texts

1. *Marx and Heidegger*
2. *Tacit and Explicit Understanding in Computer Support*
3. *Group Cognition: Computer Support for Building Collaborative Knowledge*
4. *Studying Virtual Math Teams*
5. *Translating Euclid: Designing a Human-Centered Mathematics.*
6. *Constructing Dynamic Triangles Together: The Development of Mathematical Group Cognition*
7. *Essays in Social Philosophy*
8. *Essays in Personalizable Software*
9. *Essays in Computer-Supported Collaborative Learning*
10. *Essays in Group-Cognitive Science*
11. *Essays in Philosophy of Group Cognition*
12. *Essays in Online Mathematics Interaction*
13. *Essays in Collaborative Dynamic Geometry*
14. *Adventures in Dynamic Geometry*
15. *Global Introduction to CSCL*
16. *Editorial Introductions to ijCSCL*
17. *Proposals for Research*
18. *Overview and Autobiographical Essays*
19. *Theoretical Investigations*
20. *Works of 3-D Form*
21. *Dynamic Geometry Game for Pods*

Gerry Stahl's assembled texts volume #12

Essays in Online Mathematics Interaction

Gerry Stahl

Gerry Stahl
Gerry@GerryStahl.net
www.GerryStahl.net

Published by Gerry Stahl at Lulu.com
Printed in the USA
ISBN 978-1-387-85960-3 (paperback)
ISBN 978-1-329-86393-4 (ebook)

Introduction

The essays in this volume are based on research during the early years of the Virtual Math Teams project, when the topics were taken from the mathematical domain of combinatorics. Drawings were done in a generic whiteboard. In particular, two teams that interacted on the same problems during 2006 provide a variety of insights into the nature of CSCL. These papers were written with close colleagues.

References

The essays in this volume were originally published as: (Çakir, Stahl & Zemel 2010; Çakir & Stahl 2013; Çakir, Zemel & Stahl 2009; Koschmann, Stahl & Zemel 2009; Sarmiento & Stahl 2007; 2008; Trausan-Matu, Stahl & Sarmiento 2006; Stahl 2008; Stahl, Zemel, Koschmann, 2009)

Çakir, M. P., Stahl, G., & Zemel, A. (2010). *Interactional achievement of shared mathematical understanding in virtual math teams.* In the proceedings of the International Conference of the Learning Sciences (ICLS 2010). Chicago, IL. ISLS. Web: http://GerryStahl.net/pub/icls2010cakir.pdf .

Çakir, M. P., & Stahl, G. (2013). The integration of mathematics discourse, graphical reasoning and symbolic expression by a virtual math team. In D. Martinovic, V. Freiman & Z. Karadag (Eds.), *Visual mathematics and cyberlearning.* (pp. 49-96). New York, NY: Springer. Web: http://GerryStahl.net/pub/visualmath.pdf .

Çakir, M. P., Zemel, A., & Stahl, G. (2009). The joint organization of interaction within a multimodal CSCL medium. *International Journal of Computer-Supported Collaborative Learning. 4*(2), 115-149.

Koschmann, T., Stahl, G., & Zemel, A. (2009). *"You can divide the thing into two parts": Analyzing referential, mathematical and technological practice in the VMT environment.* In the proceedings of the international conference on Computer Support for Collaborative Learning (CSCL 2009). Rhodes, Greece. Web: http://GerryStahl.net/pub/cscl2009tim.pdf .

Stahl, G., Zemel, A., & Koschmann, T. (2009). *Repairing indexicality in virtual math teams.* In the proceedings of the International Conference on Computers and Education (ICCE 2009). Hong Kong, China. Web: http://GerryStahl.net/pub/icce2009.pdf.

Sarmiento, J., & Stahl, G. (2007). *Group creativity in virtual math teams: Interactional mechanisms for referencing, remembering and bridging.* In the proceedings of the

Creativity and Cognition Conference. Baltimore, MD. Web: http://GerryStahl.net/vmtwiki/johann2.pdf .

Sarmiento, J., & Stahl, G. (2008). Group creativity in inter-action: Referencing, remembering and bridging. *International Journal of Human-Computer Interaction (IJHCI).* 492–504. Web: http://GerryStahl.net/pub/ijhci2007.pdf .

Trausan-Matu, S., Stahl, G., & Sarmiento, J. (2006). Polyphonic support for collaborative learning. In Y. A. Dimitriadis (Ed.), *Groupware: Design, implementation, and use: Proceedings of the 12th international workshop on groupware, CRIWG 2006, Medina del Campo, Spain, September 17-21, 2006. LNCS 4154.* (pp. 132-139). Berlin: Springer Verlag. Web: http://GerryStahl.net/pub/interanimation.pdf .

Stahl, G. (2008). Thinking as communicating: Human development, the growth of discourses and mathematizing. *International Journal of Computer-Supported Collaborative Learning. 3*(3), 361-368.

Contents

1. Interactional Achievement of Shared Mathematical Understanding in a Virtual Math Team

Murat Perit Cakir, Gerry Stahl, Alan Zemel

Abstract: Learning mathematics involves specific forms of social practice. In this paper, we describe socially situated, interactional processes involved with collaborative learning of mathematics in a special online collaborative learning environment. Our analysis highlights the methodic ways group members enact the affordances of their situation (a) to visually explore a mathematical pattern, (b) to co-construct shared mathematical artifacts, (c) to make visible the meaning of the construction, (d) to translate between graphical, narrative and symbolic representations and (e) to coordinate their actions across multiple interaction spaces, while they are working on open-ended math problems. In particular, we identify key roles of referential and representational practices in the co-construction of deep mathematical group understanding. The case study illustrates how mathematical understanding is built and shared through the online interaction.

Introduction

Developing pedagogies and instructional tools to support learning math with understanding is a major goal in mathematics education (NCTM, 2000). A common theme among various characterizations of mathematical understanding in the math education literature involves *constructing relationships* among mathematical facts and procedures (Hiebert & Wearne, 1996). In particular, math education practitioners treat recognition of connections among *multiple realizations* of a math concept encapsulated in various inscriptional forms as evidence of deep understanding of that subject matter (Kaput, 1998; Sfard, 2008; Healy & Hoyles, 1999). For instance, the concept of function in the modern math curriculum is introduced through its graphical, narrative, tabular, and symbolic realizations. Hence, a deep understanding of the function concept is ascribed to a learner to the extent he/she can demonstrate how seemingly different graphical, narrative, and symbolic forms are interrelated as realizations of each other in specific problem-solving circumstances that require the use of functions. On the other hand, students who demonstrate difficulties in realizing such connections are considered to perceive actions associated with distinct forms as isolated sets of skills, and hence are said to have a shallow understanding of the subject matter (Carpenter & Lehrer, 1999).

Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text-chat and a shared graphical workspace—have been widely employed in CSCL research and in commercial collaboration suites such as Elluminate and Wimba to support collaborative learning activities of small groups online (Dillenbourg & Traum, 2006; Soller, 2004; Suthers et al., 2001). The way such systems are designed as a juxtaposition of several technologically independent online communication tools not only brings various *affordances* (i.e. possibilities-for and/or constraints-on actions), but also carries important interactional consequences for the users (Cakir, Zemel & Stahl, 2009; Suthers, 2006; Dohn 2009). Providing access to a rich set of modalities for action allows users to demonstrate their reasoning in multiple semiotic forms. Nevertheless, the achievement of connections that foster the kind of mathematical understanding desired by math educators is conditioned upon team members' success in devising shared methods for coordinated use of these rich resources.

Although CSCL environments with multimodal interaction spaces offer rich possibilities for the creation, manipulation, and sharing of mathematical artifacts online, the interactional organization of mathematical meaning-making activities in such online environments is a relatively unexplored area in CSCL and in math education. In an effort to address this gap, we have designed an online environment with multiple interaction spaces called Virtual Math Teams (VMT), which allows users to exchange textual as well as graphical contributions online (Stahl, 2009). The VMT environment also provides additional resources, such as explicit referencing and special awareness markers, to help users coordinate their actions across multiple spaces. Of special interest to researchers, this environment includes a Replayer tool to replay a chat session as it unfolded in real time and inspect how students organize their joint activity to achieve the kinds of connections indicative of deep understanding of math.

In this paper we focus on the practical methods through which VMT participants achieve the kinds of connections across multiple semiotic modalities that are often taken as indicative of deep mathematical understanding. We take the math education practitioners' account of what constitutes deep learning of math as a starting point, but instead of treating understanding as a mental state of the individual learner that is typically inferred by outcome measures, we argue that deep mathematical understanding can be located in the practices of collective multimodal reasoning displayed by teams of students through the sequential and spatial organization of their actions. In an effort to study the practices of multimodal reasoning online, we employ an ethnomethodological case-study approach and investigate the methods through which small groups of students coordinate their actions across multiple interaction spaces of the VMT environment as they collectively construct, relate and reason with multiple forms of mathematical artifacts to solve an open-ended math problem. Our analysis has identified key roles of referential and representational practices in the co-construction of deep mathematical understanding.

Data & Methodology

The excerpts we analyze in this paper are obtained from a problem-solving session of a team of three upper-middle-school students who participated in the VMT Spring Fest 2006. This event brought together several teams from the US, Singapore, and Scotland to collaborate on an open-ended math task on combinatorial patterns. Students were recruited anonymously through their teachers. Members of the teams generally did not know each other before the first session. Neither they nor we knew anything about each other (e.g., age or gender) except chat handle and information that may have been communicated during the sessions. Each group participated in four sessions during a two-week period, and each session lasted over an hour. Each session was moderated by a Math Forum staff member; the facilitators' task was to help the teams when they experienced technical difficulties, not to instruct or participate in the problem-solving work. Figure 6 below shows a screenshot of the VMT Chat environment that hosted these online sessions.

During their first session, all the teams were asked to work on a particular pattern of squares made up of sticks (see Figure 1 below). For the remaining three sessions the teams were asked to come up with their own shapes, describe the patterns they observed as mathematical formulae, and share their observations with other teams through a wiki page. This task was chosen because of the possibilities it afforded for many different solution approaches ranging from simple counting procedures to more advanced methods, such as the use of recursive functions and exploring the arithmetic properties of various number sequences. Moreover, the task had both algebraic and geometric aspects, which would potentially allow us to observe how participants put many features of the VMT software system into use. The open-ended nature of the activity stemmed from the need to agree upon a new shape made by sticks. This required groups to engage in a different kind of problem-solving activity as compared to traditional situations where questions are given in advance and there is a single "correct" answer—presumably already known by a teacher. We used a traditional problem to seed the activity and then left it up to each group to decide the kinds of shapes they found interesting and worth exploring further (Moss & Beatty, 2006; Watson & Mason, 2005).

(1) 4 sticks, 1 square

(2) 10 sticks, 3 squares

(3) 18 sticks, 6 squares

N	Sticks	Squares
1	4	1
2	10	3
3	18	6
4	?	?
5	?	?
6	?	?
...	...	...
N	?	?

Session I

Draw the pattern for N=4, N=5, and N=6 in the whiteboard. Discuss as a group: How does the graphic pattern grow?

Fill in the cells of the table for sticks and squares in rows N=4, N=5, and N=6. Once you agree on these results, post them on the VMT Wiki

Can your group see a pattern of growth for the number of sticks and squares? When you are ready, post your ideas about the pattern of growth on the VMT Wiki.

Sessions II and III

Discuss the feedback that you received about your previous session.

WHAT IF? Mathematicians do not just solve other people's problems - they also explore little worlds of patterns that they define and find interesting. Think about other mathematical problems related to the problem with the sticks. For instance, consider other arrangements of squares in addition to the triangle arrangement (diamond, cross, etc.). **What if** instead of squares you use other polygons like triangles, hexagons, etc.? Which polygons work well for building patterns like this? How about 3-D figures, like cubes with edges, sides and cubes? What are the different methods (induction, series, recursion, graphing, tables, etc.) you can use to analyze these different patterns?

Go to the VMT Wiki and share the most interesting math problems that your group chose to work on.

Figure 1: Task description for Spring Fest 2006

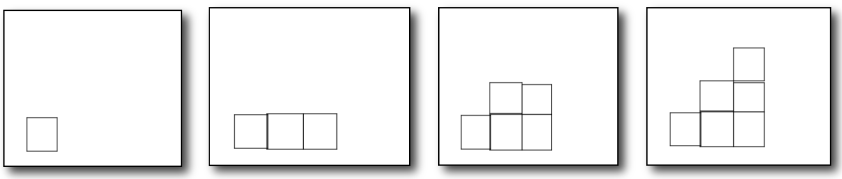
Studying the collective meaning-making practices enacted by the users of CSCL systems requires a close analysis of the process of collaboration itself (Stahl, Koschmann & Suthers, 2006; Koschmann, Stahl & Zemel, 2007). In an effort to investigate the organization of interactions across the dual-interaction spaces of the VMT environment, we consider the small group as the unit of analysis (Stahl, 2006), and we apply the methods of Ethnomethodology (EM) (Garfinkel, 1967; Livingston, 1986) and Conversation Analysis (CA) (Sacks, 1962/1995; ten Have, 1999) to conduct case studies of online group interaction. Our work is informed by studies of interaction mediated by online text-chat with similar methods (Garcia & Jacobs, 1998; O'Neill & Martin, 2003), although the availability of a shared drawing area and explicit support for deictic references in our online environment as well as our focus on mathematical practice significantly differentiate our study from theirs.

The goal of Conversation Analysis is to make explicit and describe the normally tacit commonsense understandings and procedures group members use to organize their conduct in particular interactional settings. Commonsense understandings and procedures are subjected to analytical scrutiny because they "enable actors to recognize and act on their real world circumstances, grasp the intentions and motivations of others, and achieve mutual understandings" (Goodwin & Heritage, 1990, p. 285). Group members' shared competencies in organizing their conduct not only allow them to produce their own actions, but also to interpret the actions of others (Garfinkel & Sacks, 1970). Since members enact these understandings and/or procedures in their situated actions, researchers can discover them through detailed analysis of members' sequentially organized conduct (Schegloff & Sacks, 1973).

We subjected our analysis of VMT data to intersubjective agreement by conducting numerous CA data sessions (ten Have, 1999). During the data sessions we used the VMT Replayer tool, which allows us to replay a VMT chat session as it unfolded in real time based on the timestamps of actions recorded in the log file. The order of actions—chat postings, whiteboard actions, awareness messages—we observe with the Replayer as researchers exactly matches the order of actions originally observed by the users. This property of the Replayer allows us to study the sequential unfolding of events during the entire chat session. In short, the VMT environment provides us a perspicuous setting in which the mathematical meaning-making process is *made visible* as a joint practical achievement of participants that is "observably and accountably embedded in collaborative activity" (Koschmann, 2001, p. 19).

Analysis

The following sequence of drawing actions (Figures 2 to 6 below) is observed at the beginning of the very first session of a team in the VMT environment. Shortly after a greeting episode, one student, Davidcyl, begins to draw a set of squares on the shared whiteboard. He begins by drawing three squares that are aligned horizontally with respect to each other, which is made evident through his careful placement of the squares side by side (see Figure 2 below). Then he adds two more squares on top of the initial block of three, which introduces a second layer to the drawing. Finally, he adds a single square on top of the second level, which produces the stair-step shape displayed in the last frame of Figure 2. Note that he builds the pattern row-by-row here.

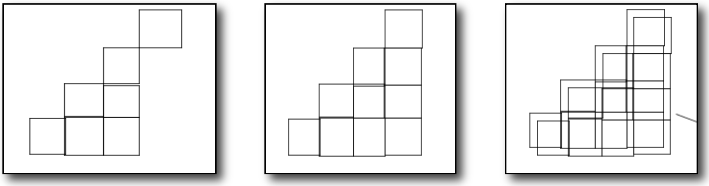


6:24:30 6:24:37 6:24:48 6:24:50

Figure 2: First stages of Davidcyl's drawing activity.

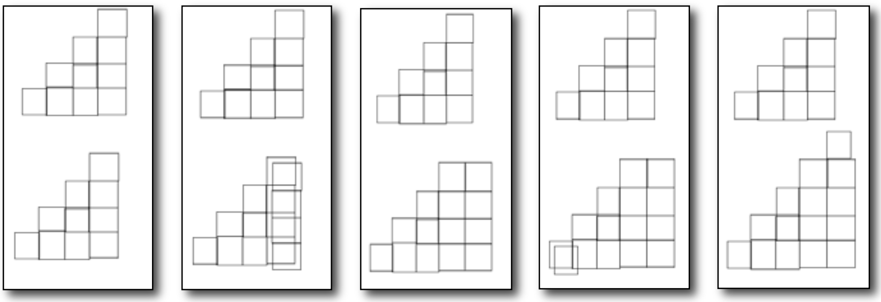
Next, Davidcyl starts adding a new column to the right of the drawing (see Figure 3). He introduces a new top level by adding a new square first, and then he adds 3 more squares that are aligned vertically with respect to each other and horizontally with respect to existing squares (see second frame in Figure 3). Then he produces a duplicate of this diagram by using the copy/paste feature of the whiteboard (see the last frame in Figure 3). Here, he builds the next iteration by adding a new column to the previous stage, starting the new column by making visible that it will be one square higher than the highest previous column.

Afterwards, Davidcyl moves the pasted drawing to an empty space below the copied diagram. As he did earlier, he adds a new column to the right of the prior stage to produce the next stage. This time he copies the entire 4th column, pastes a copy next to it, and then adds a single square on its top to complete the new stage (Figure 4). Next, Davidcyl produces another shape in a similar way by performing a copy/paste of his last drawing, moving the copy to the empty space below, and adding a new column to its right (see Figure 5). Yet, this time the squares of the new column are added one by one, which may be considered as an act of counting. In Figure 4, the new column is explicitly shown to be a copy of the highest column plus one square. In Figure 5, the number of squares in the new column are counted individually, possibly noting that there are N of them. The likelihood that the counting of the squares in the new column is related to the stage, N, of the pattern is grounded by Davidcyl's immediately subsequent reference to the diagrams as related to "*n=4,5,6*".



6:24:51 6:25:00 6:25:07

Figure 3: Davidcyl introduces the 4th column and pastes a copy of the whole shape.



6:25:13 6:25:42 6:25:45 6:25:47 6:25:52

Figure 4: Davidcyl uses copy/paste to produce the next stage of the pattern

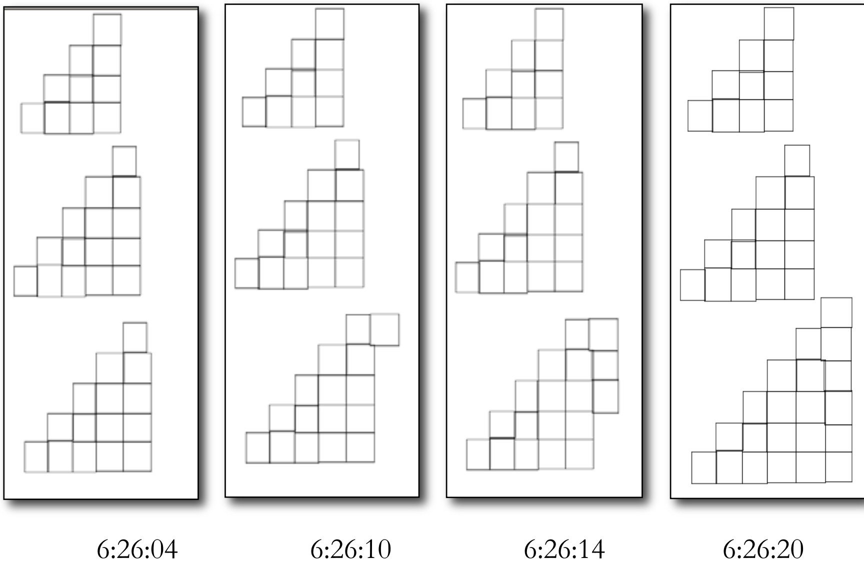


Figure 5: Davidcyl's drawing of the 6th stage

Shortly after his last drawing action at 6:26:20, Davidcyl posts a chat message stating, "*ok I've drawn n=4,5,6*" at 6:26:25. Figure 6 shows the state of the interface at this moment. The "*ok*" at the beginning of the message could be read as some kind of a transition move (Beach, 1995). The next part "I've drawn" makes an explicit verbal reference to his recent (indicated by the use of past perfect tense) drawing actions. Finally, the expression "*n=4,5,6*" provides an *algebraic gloss* for the drawings, which specifies how those drawings should be seen or treated. Once read in relation to the task description, Davidcyl's recent actions across both spaces can be treated as a response to the first bullet under session 1, which states "Draw the pattern for N=4, N=5, and N=6 in the whiteboard" (see Figure 1 above for the task description). The discussion that immediately followed Davidcyl's drawings and his last chat statement is displayed in Table 1 below.

Davidcyl's posting at line 26 is stated as a declarative, so it can be read as a claim or assertion. The references to "*n*" (i.e., not to a particular stage like 2^{nd} or 5^{th}) invoke a variable as a gloss for referring to the features of the general pattern. Moreover, the use of the clause "more…than" suggests a comparison between two things, in particular the two cases indexed by the phrases "*nth pattern*" and "*(n-1)th pattern*" respectively. Hence, Davidcyl's posting can be read as a claim about how the number of squares changes between the $(n\text{-}1)^{th}$ and n^{th} stages of the pattern at hand. The two cases compared in the posting correspond to two consecutive stages of the staircase pattern. Davidcyl's prior drawing work included similar transitions among pairs of particular stages. For instance, while he was drawing the 4th stage, he added a column of 4 new squares to the right of the 3rd stage. Hence, Davidcyl's narrative uses the drawings for particular cases as a resource to index the properties of the general pattern, which is implicated in the regularity/organization projected by his prior drawing actions.

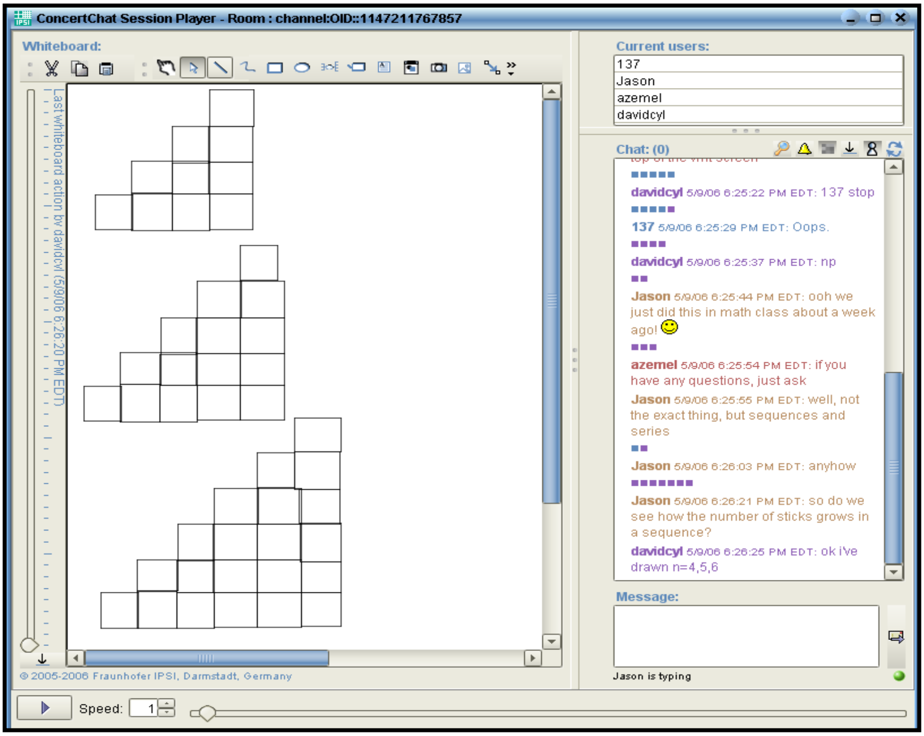


Figure 6: The state of the VMT environment when Davidcyl posted "ok I've drawn n=4,5,6" at 6:26:25.

Table 1: Chat discussion following the drawing activity

Chat Index	Time Start Typing	Time of Posting	Author	Content
26	18:27:13	18:27:32	davidcyl	the nth pattern has n more squares than the (n-1)th pattern
	18:27:30	18:27:47	137	*[137 has fully erased the chat message]*
	18:27:47	18:27:52	137	*[137 has fully erased the chat message]*
27	18:27:37	18:27:55	davidcyl	basically it's 1+2+..+(n-1)+n for the number of squares in the nth pattern
	18:27:57	18:27:57	137	*[137 has fully erased the chat message]*

28	18:28:02	18:28:16	137	so n(n+1)/2
29	18:27:56	18:28:24	davidcyl	and we can use the gaussian sum to determine the sum: n(1+n)/2
30	18:28:27	18:28:36	davidcyl	137 got it

In the next line, Davidcyl elaborates on his description by providing a summation of integers that accounts for the number of squares required to form the n^{th} stage. In particular, the expression "*1+2+..+(n-1)+n*" describes a method to count the squares that form the n^{th} stage. Since Davidcyl made his orientation to columns explicit through his prior drawing work while he methodically added a new column to produce a next stage, this expression can be read as a *formulation* of his column-by-column counting work in algebraic form. In other words, Davidcyl achieves a (*narrative*) transition from the *visual* to the *algebraic*, which is informed by his methodic construction of specific stages of the staircase pattern that allowed him to isolate relevant components of the general pattern and derive a systematic counting method.

As Davidcyl composes a next posting, 137 posts a so-prefaced math expression at line 28, "*So n(n+1)/2*" that (a) shows 137 has been attending to the organization of Davidcyl's ongoing exposition, (b) displays 137's recognition of the next problem-solving step projected by prior remarks, (c) offers an algebraic realization of the procedure described by Davidcyl, and (d) call on others to assess the relevance and validity of his claim. Davidcyl's message at line 29 (which is produced in parallel with line 28 as indicated by the typing times) is a more elaborate statement that identifies how his prior statements, if treated as a Gaussian sum, yield the same expression that 137 put forward at line 28 (viz. "n(n+1)/2"). Given that 137 anticipated Davidcyl's Gaussian sum, Davidcyl announces in the very next posting that "*137 got it,*" which recognizes the relevance of 137's posting at that particular moment in interaction, and treats 137's coordinated contribution as an act of understanding.

Discussion

Given the characterization of deep mathematical understanding in the math education literature, methodic ways through which participants coordinate their actions across the whiteboard and chat spaces are of particular interest to our investigation of mathematical understanding or meaning making at the small-group level. The episode we analyzed above includes a situation where a user, who has been active in the whiteboard, moves on to the other interaction space and posts a message referring to his prior drawing work. The chat message sequentially followed the drawings, and hence presumed their availability as a shared referential resource, so that the interlocutors can make sense of what is possibly referred to by the indexical expression "*n=4,5,6*" included in the posting. Davidcyl's explicit orientation to timing or sequencing is further evidenced by his use of the past perfect tense and his temporal positioning of the message immediately after the final step of the drawing. Moreover, the chat posting reflexively gave further specificity to the prior drawing work by informing everyone that the diagrams should be seen as specific cases of the staircase pattern described in the task description. This suggests that temporal proximity among actions can serve as an interactional resource/cue for the participants to treat those actions in reference to each other, especially when the actions are performed across different interaction spaces. In short, Davidcyl has demonstrated a *method* that one can call *verbal referencing*, which is employed by VMT users when they need to communicate to each other that a narrative/symbolic account needs to be read in relation to a whiteboard object.

Davidcyl's use of the algebraic reference "*n=4,5,6*" at this moment in interaction is also informative in terms of respective limitations of each medium and their mutually constitutive function for communication. Davidcyl's chat message not only provided further specificity to the recently produced diagrams, but also marked or announced the completion of his drawing work. This is revealing in terms of the kinds of *illocutionary acts* (Austin, 1962) that can be achieved by users in this dual-media online environment. In particular, although a drawing and its production process may be available for all members to observe in the shared whiteboard, diagrams by themselves cannot fulfill the same kind of interactional functions achieved by text postings such as "asking a question" or "expressing agreement." In other words, whiteboard objects are made interactionally relevant through chat messages that either (a) project their production as a next action or (b) refer to already produced objects. This can also be seen as members' orientation to a limitation of this online environment as a communication platform; one can act only in one space at a given time, so it is not possible to perform a simultaneous narration of a drawing as one can do in a face-to-face setting. Therefore, each interaction space as a communicative medium seems to enable and/or hinder certain kinds of actions, i.e., they carry specific *communicative affordances* (Hutchby, 2001) for collaborative problem solving online.

The way Davidcyl has put some of the features of the whiteboard —like dragging and copy/paste—into use in the episode described above demonstrates some of its key affordances as a medium for producing shared drawings. In particular, we have observed how copying and pasting is used to avoid additional drawing effort, and how collections of objects are selected, dragged, and positioned to produce specific stages of a geometric pattern. Such possibilities for action are supported by the object-oriented design of the whiteboard. Davidcyl's drawing actions show that, as compared to other physical drawing media such as paper or blackboard, the electronic whiteboard affords unique possibilities for constructing and modifying shared mathematical diagrams in ways that have mathematical, collaborative, semantic and communicative power.

It is analytically significant that Davidcyl changed from building the pattern row-wise in Figure 2 to building it column-wise subsequently and that he "computed" the height of the new column in several different ways in Figures 3, 4 and 5. This indicates that he did not have an explicit solution "in his head"—a mental model that he just had to illustrate in the world with the whiteboard. Rather, he worked out the solution gradually through emergent whiteboard activities and his recognition of what appeared in the whiteboard. Significantly, the other group members could observe the same thing.

An important concern for our group-cognitive approach is to investigate how students make use of the technological features available to them to explore mathematical ideas in an online environment like VMT. Drawing features such as copy/paste, dragging, coloring, etc. are important affordances of the shared whiteboard not simply because of their respective advantages as compared to other drawing media. The mathematical significance of these features relies on the way single actions like copy/paste or dragging are sequentially organized as part of a broader drawing activity that aims towards constructing a shared mathematical artifact. For instance, through such a sequence of drawing actions Davidcyl demonstrated to us (as analysts) and to his peers (a) how to construct a stair-step pattern as a spatially organized assemblage of squares, and (b) how to derive a new stage of the stair-step pattern from a copy of the prior stage by adding a new column of squares to its right. Moreover, Davidcyl's engagement with the squares (rather than with the sticks that make up the squares) displays his explicit orientation to this particular aspect of the shared task (i.e., finding the number of squares at a given stage). Hence, the availability of these drawing actions as a sequence of changes unfolding in the shared visual space allows group members to witness the *reasoning* process embodied in the sequential and spatial organization of those actions. In other words, the sequentially unfolding details of the construction process provide specificity (and hence meaning) to the mathematical artifact that is being constructed.

Besides figuring out ways to connect their own actions across dual-interaction spaces, VMT users also coordinate their actions with the actions of their peers to be able to meaningfully participate in the ongoing discussion. The ways participants produce and deploy mathematical artifacts in the shared space implicate or inform what procedures and methods may be invoked next to produce other mathematical artifacts, or to modify existing ones as the discussion progresses towards a solution to the task at hand. For instance, 137's competent contribution to Davidcyl's sequentially unfolding line of reasoning in Table 1 shows that shared mathematical understanding at the group level is an interactional achievement that requires coordinated *co-construction* of mathematical artifacts. The *co* prefix for the term "co-construction" highlights the *intersubjective* nature of the mathematical artifacts produced during collaborative work; they are not mere mental constructs easily ascribed to certain individuals. As we have just observed in the excerpt above, *intersubjectivity* is evidenced in the ways participants organize their actions to display their relevance to prior actions. 137's anticipation and production of the next relevant step in the joint problem-solving effort serves as strong evidence of mutual understanding between him and Davidcyl. Moreover, the term *construction* signals that mathematical artifacts are not simply passed down by the mathematical culture as ready-made Platonic entities external to the group. Once enacted in group discourse, culturally transmitted artifacts such as "Gaussian Sum" need to be made sense of and appropriated in relation to the task at hand. Hence, our use of the combined term *co-construction* implies an interactional process of sense making by a group of students—even in an excerpt like the present one in which one individual takes an extended turn in the group discourse to develop a complex presentation. The fact that it is a visible construction worked out in collaborative media and designed for reception by others makes it a co-construction from which the speaker is as likely to learn as are the other group members.

When co-construction takes place in an online environment like a chat tool, the construction process must take place through observable interactions within technical media. This requires student groups to invent, adapt or appropriate methods to co-construct mathematical artifacts. It also makes it possible for them to explicitly reflect on the *persistent traces* of their co-constructions by investigating the persistent content provided by the technology. Therefore, the persistent nature of actions provides the necessary infrastructure for joint action, and hence is a key affordance of CSCL environments like VMT, where actors work at a distance in a disembodied environment. In addition, the persistent records of interactions also allow researchers to analyze the co-construction process as it unfolded in real-time, as this paper demonstrates.

Through similar case studies of other VMT sessions, we observed that students make use of additional resources (such as the explicit referencing tool, locational pronouns, color names in chat) to methodically achieve referential relationships between shared diagrams and chat messages (Cakir, Zemel & Stahl, 2009). Chat postings use a broad and sophisticated array of such methods to refer to matters constructed graphically. Due to their recurrent appearance as a practical concern for the participants in this dual-media online environment, we refer to the collection of these methods as *referential practices*. Referential practices are of particular importance to the study of mathematical understanding as a group-cognitive phenomenon, because they are enacted in circumstances where participants explicitly orient to the task of achieving relationships between the textual and graphical contributions that they have been exchanging online—a phenomenon that is given significance in the math education literature as characterizing deep mathematical understanding. Likewise, one can use the term *representational practices* to refer to the spatial and temporal organization of whiteboard actions that produce shared diagrams, which simultaneously give further specificity to the mathematical artifacts that the team has been working on—e.g., Davidcyl's methodical sequencing of copy/paste operations to indicate growth patterns. Through referential and representational practices, participants co-construct mathematical artifacts that reify mathematical understandings. The understanding or meaning is not simply located inside students' individual brains or in the chat/drawing artifacts themselves. The meaning is embodied in the sequentially organized and coordinated actions through which those artifacts were co-constructed. To sum up, group referential and representational practices play a key role in the ways mathematical artifacts are (a) appropriated by active teams from historically developed cultural tools, and (b) emergent from ways of communicating and symbolizing within local collectivities as shared, meaningful resources for mathematical discourse, collaborative learning and group understanding.

References

Austin, J. L. (1962). *How to Do Things with Words*. Oxford: Clarendon Press.

Beach, W. A. (1995). Conversation analysis: "Okay" as a clue for understanding consequentiality. In S. J. Sigman (Ed.), *The consequentiality of communication* (pp. 121- 162). Hillsdale, NJ: LEA.

Cakir, M. P., Zemel, A., & Stahl, G. (2009). *The joint organization of interaction within a multimodal CSCL medium.* International Journal of Computer-Supported Collaborative Learning, 4(2), 155-190.

Dillenbourg, P., & Traum, D. (2006). Sharing Solutions: Persistence and Grounding in Multimodal Collaborative Problem Solving. *The Journal of the Learning Sciences, 15*(1), 121-151.

Dohn, N. B. (2009). Affordances revisited: Articulating a Merleau-Pontian view. *International Journal of Computer-Supported Collaborative Learning, 4*(2), 151-170.

Garcia, A., & Jacobs, J. B. (1998). The interactional organization of computer mediated communication in the college classroom. *Qualitative Sociology, 21*(3), 299-317.

Garfinkel, H. (1967). *Studies in Ethnomethodology*. Englewood Cliffs, NJ: Prentice Hall.

Garfinkel, H., & Sacks, H. (1970). On formal structures of practical actions. In J. Mckinney & E. Tirvakian (Eds.), *Theoretical sociology: Perspectives and developments* (pp. 337-366). New York: Appleton.

Goodwin, C., & Heritage, J. (1990). Conversation Analysis. *Annual Review of Anthropology, 19*, 283-307.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers. *Mathematical Thinking and Learning, 1*(1), 59-84.

Hiebert, J., & Wearne, D. (1996). Instruction, Understanding, and Skill in Multidigit Addition and Subtraction. *Cognition and Instruction, 14*(3), 251-283.

Hutchby, I. (2001). *Conversation and technology: From the telephone to the internet.* Cambridge, UK: Polity.

Kaput, J. (1998). Representations, Inscriptions, Descriptions and Learning: A Kaleidescope of Windows. *Journal of Mathematical Behavior, 17*(2), 265-281.

Koschmann, T. (2001). Revisiting the paradigms of instructional technology. In G. Kennedy, M. Keppell, C. McNaught & T. Petrovic (Eds.), *Meeting at the crossroads. Proceedings of the 18th Annual Conference of the Australian Society for Computers in Learning in Tertiary Education* (pp. 15-22).

Koschmann, T., Stahl, G., & Zemel, A. (2007). The video analyst's manifesto. In R. Goldman, R. Pea, B. Barron & S. Derry (Eds.), *Video research in the learning sciences*. Mahwah, NJ: LEA.

Livingston, E. (1987). *Making sense of ethnomethodology*. New York, NY: Routledge.

Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *International Journal of Computer-Supported Collaborative Learning, 1*(4), 441-465.

Mühlpfordt, M., & Wessner, M. (2005). Explicit Referencing in Chat Supports Collaborative Learning. In T. Koschmann, D. Suthers & T. Chan (Eds.), *Proceedings of the 2005 conference on Computer support for collaborative learning (CSCL): the next 10 years!* (pp. 460-469). Taipei, Taiwan: ISLS.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

O'Neil, J., & Martin, D. (2003). Text chat in action. In K. Schmidt & M. Pendergast (Eds.), *GROUP '03: Proceedings of the ACM SIGGROUP on Supporting group work* (pp. 40-49). New York, NY: ACM.

Sacks, H. (1962/1995). *Lectures on Conversation*. Oxford, UK: Blackwell.

Schegloff, E. A., & Sacks, H. (1973). Opening up closings. *Semiotica, 7*, 289-327.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York, NY: Cambridge University Press.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Soller, A. (2004). Understanding knowledge sharing breakdowns: A meeting of quantitative and qualitative minds. *Journal of Computer Assisted Learning, 20*, 212-223.

Stahl, G. (2006). *Group Cognition: Computer Support for Building Collaborative Knowledge*. Cambridge, MA: MIT Press.

Stahl, G. (Ed.). (2009). *Studying virtual math teams*. New York, NY: Springer.

Stahl, G., Koschmann, T., & Suthers, D. D. (2006). Computer-Supported Collaborative Learning. In R. K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 409-426). New York: CUP.

Suthers, D. D. (2006). Technology affordances for intersubjective meaning making: A research agenda for CSCL. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 1*(3), 315-337.

Suthers, D., Connelly, J., Lesgold, A., Paolucci, M., et al. (2001). Representational and Advisory Guidance for Students Learning Scientific Inquiry. In K. D. Forbus & P. J. Feltovich (Eds.), *Smart Machines in Education: The Coming Revolution in Educational Technology* (pp. 7-35). Menlo Park: AAAI Press.

ten Have, P. (1999). *Doing conversation analysis: A practical guide*. Thousand Oaks, CA: Sage Publications.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Associates.

2. The Joint Organization of Interaction within a Multimodal CSCL Medium

Murat Perit Cakir, Alan Zemel and Gerry Stahl

Abstract In order to collaborate effectively in group discourse on a topic like mathematical patterns, group participants must organize their activities in ways that share the significance of their utterances, inscriptions, and behaviors. Here, we report the results of a micro-ethnographic case study of collaborative math problem-solving activities mediated by a synchronous multimodal online environment. We investigate the moment-by-moment details of the interaction practices through which participants organize their chat utterances and whiteboard actions as a coherent whole. This approach to analysis foregrounds the sequentiality of action and the implicit referencing of meaning making—fundamental features of interaction. In particular, we observe that the sequential construction of shared drawings and the deictic references that link chat messages to features of those drawings and to prior chat content are instrumental in the achievement of intersubjectivity among group members' understandings. We characterize this precondition of collaboration as the co-construction of an indexical field that functions as a common ground for group cognition. Our analysis reveals methods by which the group co-constructs meaningful inscriptions in the dual-interaction spaces of its CSCL environment. The integration of graphical, narrative, and symbolic semiotic modalities in this manner also facilitates joint problem solving. It allows group members to invoke and operate with multiple realizations of their mathematical artifacts, a characteristic of deep learning of mathematics.

Computer-supported collaborative learning is centrally concerned with the joint organization of interaction by small groups of students in online environments. The term "collaborative learning" is a gloss for *interaction that is organized for the joint achievement of knowledge-building tasks* such as problem solving in domains like school mathematics. Rather than using the term "collaborative learning," which carries vague and contradictory connotations, we coined the term "group cognition" to refer to activities where several students organize their joint interaction to achieve such collective cognitive accomplishments as planning, deducing, designing, describing, problem solving, explaining, defining, generalizing, representing, remembering, and reflecting as a group.

We have argued in *Group Cognition* (Stahl, 2006) that CSCL interactions should be analyzed at the group level of description, not just at the individual or the community levels, as is done in other theoretical approaches influential in CSCL research. During the past six years, we have conducted the Virtual Math Teams (VMT) Project to explore group cognition in a prototypical CSCL setting and to analyze it at the group level. We have used our analyses of interaction to drive the design of the technology.

In this paper, we present a case study of an 18-minute-long excerpt from the VMT Project. We look at some ways in which the students organized their joint efforts. Our observations here are consistent with our impressions from more than a hundred student-hours of interaction in the VMT data corpus. Many of the broader theoretical and practical issues surrounding the analysis here are addressed by CSCL researchers in a new edited volume on *Studying Virtual Math Teams* (Stahl, 2009b) in the Springer CSCL book series.

The issue that we address in the following pages is: *How do the students in our case study organize their activity so they can define and accomplish their tasks as a group within their online environment?* This is necessarily a pivotal question for a science of CSCL (Stahl, 2009a). It involves issues of meaning making, shared understanding and common ground that have long been controversial in CSCL.

The problem of coordination is particularly salient in the VMT software environment, which is an instance of a dual-interaction space (Dillenbourg, 2005; Mühlpfordt & Stahl, 2007), requiring organization across multiple media, each with their own affordances. We have found that the key to joint coordination of knowledge building is sequential organization of a network of indexical and semantic references within the group discourse (Stahl, 2007). We therefore analyze sequential interaction at the group level of description, using ethnomethodologically inspired chat interaction analysis rather than quantitative coding, in order to maintain and study this sequential organization. Thereby, we arrive at a view of mathematical knowledge building as the coordination of visual, narrative, and symbolic inscriptions as multiple realizations of co-constructed mathematical objects.

While we have elsewhere presented theoretical motivations for focusing on *group discourse organization* as fundamental for CSCL, in this paper we foreground our *analysis of empirical data* from a VMT session. We derive a number of characteristics of the joint organization of interaction from the details of the case study. The characteristics we describe are to some extent specific to the technological affordances of the VMT environment, to the pedagogical framing of the chat session, and even to the unique trajectory of this particular group interaction. Nevertheless, the characteristics are indicative of what takes place—with variations—in similar settings. After the analytic centerpiece of the paper, we discuss *methodological implications* for CSCL analysis, including what it means to take the *group* as the unit of analysis. We then contrast our approach to leading *alternative approaches* in CSCL. This discussion focuses particularly on multimodal interaction in a *dual-interaction space* and on related conceptions of *common ground*, concluding with summary remarks on *sequential analysis*. The paper proceeds through the following topics:

- The problem of group organization in CSCL
- A case study of a virtual math team
- Implications for CSCL chat interaction analysis
- The group as the unit of analysis
- Other approaches in CSCL to analyzing multimodal interaction
- Grounding through interactional organization
- Sequential analysis of the joint organization of interaction

The Problem of Group Organization in CSCL

A central issue in the theory of collaborative learning is how students can solve problems, build knowledge, accomplish educational tasks, and achieve other cognitive accomplishments *together*. How do they share ideas and talk about the same things? How do they know that they are talking about, thinking about, understanding, and working on things in the same way? Within CSCL, this has been referred to as the problem of the "attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995), "building common ground" (Baker et al., 1999; Clark & Brennan, 1991) or "the practices of meaning making" (Koschmann, 2002). We have been interested in this issue for some time. *Group Cognition* (Stahl, 2006) documents a decade of background to the VMT research reported here: Its Chapter 10 (written in 2001) argued the need for a new approach and its Chapter 17 (written in 2002) proposed the current VMT Project, which includes this case study. Since 2002, we have been collecting and analyzing data on how groups of students in a synchronous collaborative online environment organize their interaction to achieve intersubjectivity and shared cognitive accomplishments in the domain of school mathematics.

Knowledge building in CSCL has traditionally been supported primarily with asynchronous technologies (Scardamalia & Bereiter, 1996). Within appropriate educational cultures, this can be effective for long-term refinement of ideas by learning communities. However, in small groups and in many classrooms, asynchronous media encourage mere exchange of individual opinions more than co-construction of progressive trains of joint thought. We have found informally that synchronous interaction can more effectively promote group cognition—the accomplishment of "higher order" cognitive tasks through the coordination of contributions by individuals within the discourse of a small group. We believe that the case study in this paper demonstrates the power of group interaction in a largely synchronous environment; the coordination of interaction in an asynchronous interaction would be quite different in nature as a result of very different interactional constraints.

In CSCL settings, interaction is mediated by a computer environment. Students working in such a setting must enact, adapt, or invent ways of coordinating their understandings by means of the technological affordances that they find at hand (see Dohn, this issue). The development and deployment of these methods is not usually an explicit, rational process that is easily articulated by either the participants or analysts. It occurs tacitly, unnoticed, taken-for-granted. In order to make it more visible to us as analysts, we have developed an environment that makes the coordination of interaction more salient and captures a complete record of the group interaction for detailed analysis. In trying to support online math problem solving by small groups, we have found it important to provide media for both linguistic and graphical expression. This resulted in what is known within CSCL as a *dual-interaction space*. In our environment, students must coordinate their text chat postings with their whiteboard drawings. A careful analysis of how they do this reveals as well their more general methods of group organization.

The analysis of our case study focuses on episodes of interaction through which an online group of students co-constructs mathematical artifacts across dual-interaction spaces. It looks closely at how group members put the multiple modalities into use, how they make their chat postings and drawing actions intelligible to each other, and how they achieve a sense of coherence among actions taking place across the modalities to which they have access. We base our discussion, analysis, and design of the affordances of the online environment on the methodical ways the features of the software are put into use by the students.

In another VMT case study (Sarmiento & Stahl, 2008), we have seen how the problem-solving work of a virtual math team is accomplished through the co-construction and maintenance of a *joint problem space* (Teasley & Roschelle, 1993). This figurative space—that supports group interaction and the shared understanding of that interaction by the participants—not only grounds the *content* of the team's discourse and work, but also ties together the *social* fabric of the relations among the team members as actors. In addition, we saw that the joint problem space has a third essential dimension: *time* or sequence. The construction of the joint problem space constitutes a shared temporality through bridging moves that span and thereby order discontinuous events as past, present, and future (Sarmiento-Klapper, 2009). This can be seen, for instance, in the use of tenses in group-remembering discourses. More generally, the joint problem space provides a framework of sequential orderings, within which temporal deictic references, for example, can be resolved.

In this paper, we further investigate how a virtual math team achieves a group organization of its activities such that the group can proceed with a sense of everyone understanding each other and of working collaboratively as a group. We do this through a fine-grained analysis of the group's interaction in a VMT session in which they formulate, explore, and solve a geometry problem. Their work takes place in graphical, narrative, and symbolic media—supported technologically by the shared whiteboard, text chat, and wiki pages of the VMT environment. We pay particular attention to how graphical inscriptions, textual postings, and symbolic expressions in the different media are closely coordinated by the group members, despite the differences of the media.

We pursue a micro-ethnographic approach to analyzing the activities of the group members in their own terms. They set themselves a task, propose how to proceed step by step, and explain to each other how to understand their actions. We try to follow the explanations, which are available in the inscriptions, postings, and expressions—particularly when the sequentiality of these allows the complex references among them to be followed.

The establishment of group order in small-group interaction is always strongly dependent upon the media, which mediate interaction. In the case of VMT chats, there is an intricate set of technological media, including text chat, a shared whiteboard, a community wiki, and graphical references from chat to whiteboard. The central part of this paper explores the different characteristics of the VMT media by observing how the students use them. Of particular interest are the ways in which a group coordinates activities in the different graphical and textual media. From a math-education perspective, it is also insightful to see how the visual and narrative understandings feed into the development and understanding of symbolic expressions.

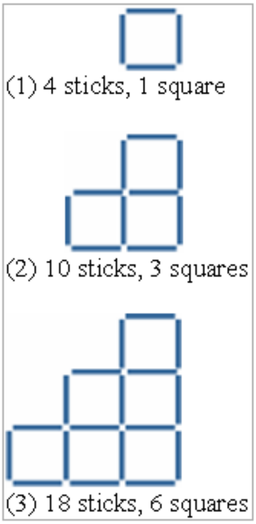
By the end of the paper, we will see how the group organization of graphical, narrative, and symbolic interactions continuously produce the joint problem space of the group's effort. This coordination is revealed through sequential analysis, in which the consequence of one action in one medium following another in another medium is seen as mutually constitutive of the meaning of those actions. The sequential web of activity across the VMT media—woven by semantic and indexical references among them—forms the joint problem space within which problem content, participant relationships, and temporal progress are all defined in a way that is shared by the group. We can see the "indexical field" formed by the group activities as the source of grounding that supports the intersubjectivity of the group effort. In contrast to psychological or psycholinguistic models of common ground, the fact that team members believe they have understandings in common about what each other is saying and doing is not a result of exchanging individual mental opinions, but is a function of the indexical organization of the group interaction.

The joint problem space—as the foundation of group cognition—is not a mental construct of a set of individuals who achieve cognitive convergence or common (identical) ground through comparing mental models anymore than it is a figment of some form of group mind. Rather, it is a system of interconnected meanings formed by a weaving of references in the group discourse itself (Stahl, 2007). In this paper, we analyze the methods the students used to co-construct this indexical field.

In our case study, the organization of group meaning making takes place across media—in accordance with the specific affordances of the different media. Furthermore, the grounding of the students' symbolic mathematical understanding can be seen as related to their visual and narrative understandings—or, rather, the various understandings are intricately interwoven and support each other. We trace this interweaving through our approach to the interactional analysis of sequential coordination at the group unit of analysis.

A Case Study of a Virtual Math Team

The excerpts we present in this paper are obtained from a problem-solving session of a team of three students who participated in the VMT Spring Fest 2006. This event brought together several teams from the US, Scotland, and Singapore to collaborate on an open-ended math task on geometric patterns. Students were recruited anonymously through their teachers. Members of the teams generally did not know each other before the first session. Neither they nor we knew anything about each other (e.g., age or gender) except chat handle and information that may have been communicated during the sessions. Each group participated in four sessions during a two-week period, and each session lasted over an hour. An adult from the research project moderated each session; the facilitators' task was to help the teams when they experienced technical difficulties, not to participate in the problem-solving work.



N	Sticks	Squares
1	4	1
2	10	3
3	18	6
4	?	?
5	?	?
6	?	?
...	...	...
N	?	?

Session I

1. Draw the pattern for N=4, N=5, and N=6 in the whiteboard. Discuss as a group: How does the graphic pattern grow?

2. Fill in the cells of the table for sticks and squares in rows N=4, N=5, and N=6. Once you agree on these results, post them on the VMT Wiki

3. Can your group see a pattern of growth for the number of sticks and squares? When you are ready, post your ideas about the pattern of growth on the VMT Wiki.

Sessions II and III

1. Discuss the feedback that you received about your previous session.

2. WHAT IF? Mathematicians do not just solve other people's problems — they also explore little worlds of patterns that they define and find interesting. Think about other mathematical problems related to the problem with the sticks. For instance, consider other arrangements of squares in addition to the triangle arrangement (diamond, cross, etc.). What if instead of squares you use other polygons like triangles, hexagons, etc.? Which polygons work well for building patterns like this? How about 3-D figures, like cubes with edges, sides and cubes? What are the different methods (induction, series, recursion, graphing, tables, etc.) you can use to analyze these different patterns?

3. Go to the VMT Wiki and share the most interesting math problems that your group chose to work on.

Figure 1. Task description.

During their first session, all the teams were asked to work online on a particular pattern of squares made up of sticks (see Figure 1). For the remaining three sessions the teams were asked to come up with their own shapes, describe the patterns they observed as mathematical formulas, and share their observations with other teams through a wiki page. This task was chosen because of the possibilities it afforded for many different solution approaches ranging from simple counting procedures to more advanced methods involving the use of recursive functions and exploring the properties of various number sequences.

Moreover, the task had both algebraic and geometric aspects, to allow us to observe how participants put many features of the VMT software system into use. The open-ended nature of the activity stemmed from the need to agree upon a new shape made by sticks. This required groups to engage in an open-ended problem-solving activity, as compared to traditional situations where questions are given in advance and there is a single "correct" answer—presumably already known by a teacher. We used a traditional pattern problem (Moss & Beatty, 2006; Watson & Mason, 2005) to seed the activity and then left it up to each group to decide the kinds of shapes they found interesting and worth exploring further.

All the problem-solving sessions were conducted in the VMT environment. The VMT online system has two main interactive components that conform to the typical layout of systems with dual-interaction spaces: a shared drawing board that provides basic drawing features on the left, and a chat window on the right (Figure 2).

The online environment has features specifically designed to help users relate the actions happening across dual-interaction spaces (Stahl, 2009b, chap.15). One of the unique features of this chat system is the referencing support mechanism (Mühlpfordt & Wessner, 2005) that allows users to visually connect their chat postings to previous postings or objects on the whiteboard via arrows (see the last posting in Figure 2 for an example of a message-to-whiteboard reference). The referential links attached to a message are displayed until a new message is posted. Messages with referential links are indicated by an arrow icon in the chat window, and a user can see where such a message is pointing by clicking on it at any time.

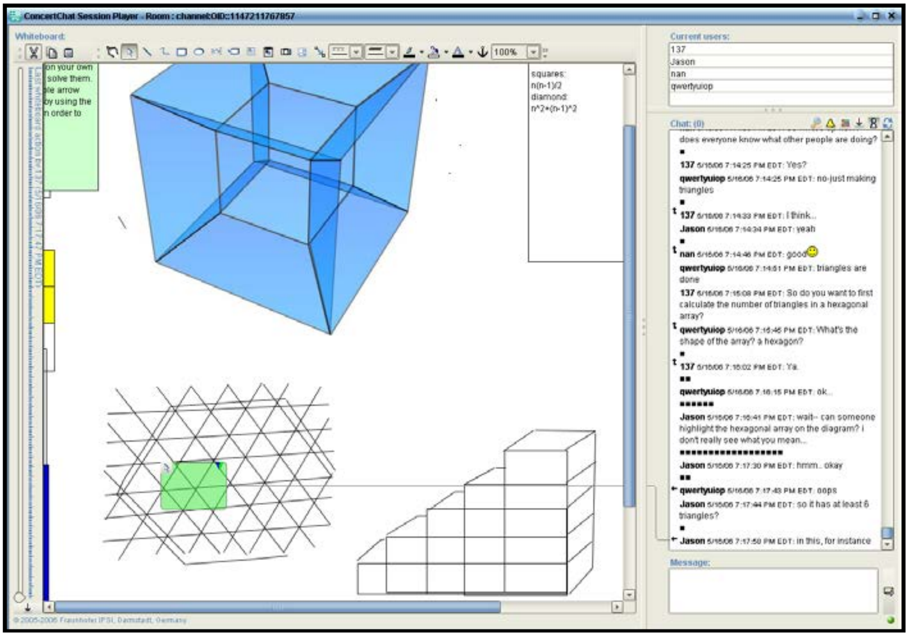


Figure 2. A screen-shot of the VMT environment.

In addition to the explicit referencing feature, the system displays small boxes in the chat window to indicate actions performed on the whiteboard. This awareness mechanism allows users to observe how actions performed in both interaction spaces are sequenced with respect to each other. Moreover, users can click on these boxes to move the whiteboard back and forth from its current state to the specific point in its history when that action was performed. Chat messages and activity markers are color coded to help users to keep track of who is doing what in the online environment. In addition to standard awareness markers that display who is present in the room and who is currently typing, the system also displays textual descriptions of whiteboard actions in tool-tip messages that can be observed by holding the mouse either on the object in the whiteboard or on the corresponding square in the chat window.

Studying the meaning-making practices enacted by the users of CSCL systems inevitably requires a close analysis of the process of collaboration itself (Dillenbourg et al., 1996; Stahl, Koschmann & Suthers, 2006). In an effort to investigate the organization of interactions across the dual-interaction spaces of the VMT environment, we consider the small group as the unit of analysis (Stahl, 2006), and we appropriate methods of ethnomethodology and conversation analysis to conduct sequential analysis of group interactions at a microlevel (Psathas, 1995; Sacks, 1962/1995; ten Have, 1999). Our work is informed by studies of interaction mediated by online text-chat with similar methods (Garcia & Jacobs, 1998, 1999; O'Neill & Martin, 2003), although the availability of a shared drawing area and explicit support for deictic references in our online environment substantially differentiate our study from theirs.

The goal of this line of analytic work is to discover the commonsense understandings and procedures group members use to organize their conduct in particular interactional settings (Coulon, 1995). Commonsense understandings and procedures are subjected to analytical scrutiny because they are what "enable actors to recognize and act on their real world circumstances, grasp the intentions and motivations of others, and achieve mutual understandings" (Goodwin & Heritage, 1990, p. 285). Group members' shared competencies in organizing their conduct not only allow them to produce their own actions, but also to interpret the actions of others (Garfinkel & Sacks, 1970). Because group members enact these understandings visibly in their situated actions, researchers can discover them through detailed analysis of the members' sequentially organized conduct (Schegloff & Sacks, 1973).

We conducted numerous VMT Project data sessions, where we subjected our analysis of the excerpts below to intersubjective agreement (Psathas, 1995). This paper presents the outcome of this group effort together with the actual transcripts so that the analysis can be subjected to external scrutiny. During the data sessions we used the VMT Replayer tool, which allows us to replay a VMT chat session as it unfolded in real time based on the time stamps of actions recorded in the log file. The order of actions—chat postings, whiteboard actions, awareness messages—we observe with the Replayer as researchers exactly matches the order of actions originally observed by the users. This property of the Replayer allowed us to study the sequential unfolding of events during the entire chat session, which is crucial in making sense of the complex interactions mediated by a CSCL environment (Koschmann, Stahl, & Zemel, 2007).

In this case study, we focus on a sequence of excerpts obtained from a single problem-solving session of a virtual math team. We are concerned with how the actors contribute to the group meaning making as they proceed. This example involves the use and coordination of actions involving both the whiteboard and chat environment. It therefore served as a useful site for seeing how actors, in this local setting, were able to engage in meaningful coordinated interaction.

The team has three members: Jason, 137 and Qwertyuiop, who are upper-middle-school students (roughly 14 years old) in the US. In the following subsections, we will present how this team co-constructed a mathematical artifact they referred to as the "hexagonal array" through a coordinated sequence of actions distributed between the chat and whiteboard spaces, and how they subsequently explored its properties by referring to and annotating shared drawings on the whiteboard. In particular, we will highlight how whiteboard objects and previous chat postings were used as semiotic resources during the collaborative problem-solving activity. This will show how chat and whiteboard differ in terms of their affordances for supporting group interaction. We will see how these differences are enacted and used in complementary ways by team members to achieve mutual intelligibility of their actions across multiple interaction spaces.

Availability of Production Processes

Log 1 is taken from the beginning of the team's third session. The team has already explored similar patterns of sticks and become familiar with the features of the VMT online environment during their prior sessions. The drawing actions at the beginning of this excerpt were the first moves of the session related to math problem solving.

Log 1

Line	**Time**	**Chat handle**	**Chat message or <whiteboard action>**
	7:07:52 - 7:11:00	**137**	**<137 draws a hexagon shape and then splits it up into regions by adding lines. Figure 3 shows some of the key steps in 137's drawing performance>**
1	**7:11:16**	**137**	**Great. Can anyone m ake a diagram of a bunch of triangles?**
	7:11:16 - 7:11:49	**137**	**<137 deletes the set of lines he has just drawn>**
2	**7:11:51**	**Qwertyuiop**	**just a grid?....**
	7:11:54 - 7:12:01	**137**	**<137 moves some of the older drawings away>**
3	**7:12:07**	**137**	**Yeah...**
4	**7:12:17**	**Qwertyuiop**	**ok…**
	7:12:23 - 7:14:07	**Qwertyuiop**	**<Qwertyuiop draws a grid of triangles in the space opened up by 137. Figure 4 shows some of the steps in Qwertyuiop's drawing actions>**

At the beginning of this excerpt, 137 performs a series of drawing actions. 137's actions on the whiteboard include the drawing of a hexagon first, then three diagonal lines and finally lines parallel to the diagonals and to the sides of the hexagon whose intersections eventually introduce some triangular and diamond-shaped regions. Moreover, 137 also performs some adjustment moves—for instance between the 4th and 5th snapshots in Figure 3—to ensure that three non-parallel lines intersect at a single point, and the edges of the hexagon are parallel to the lines introduced later as much as possible. Hence, this sequence of drawing actions suggests a particular organization of lines for constructing a hexagonal shape. (Figure 3 shows six snapshots corresponding to intermediary stages of 137's drawing actions: 137 initiates his drawing actions with six lines that form the hexagon in stage 1. Then he adds three diagonal lines in step 2. The 3rd snapshot shows the additional two lines drawn parallel to one of the diagonals. The 4th snapshot shows a similar set of two parallel lines added with respect to another diagonal. The 5th snapshot shows slight modifications performed on the new set of parallel lines to ensure intersections at certain places. The 6th snapshot shows the final stage of 137's drawing.)

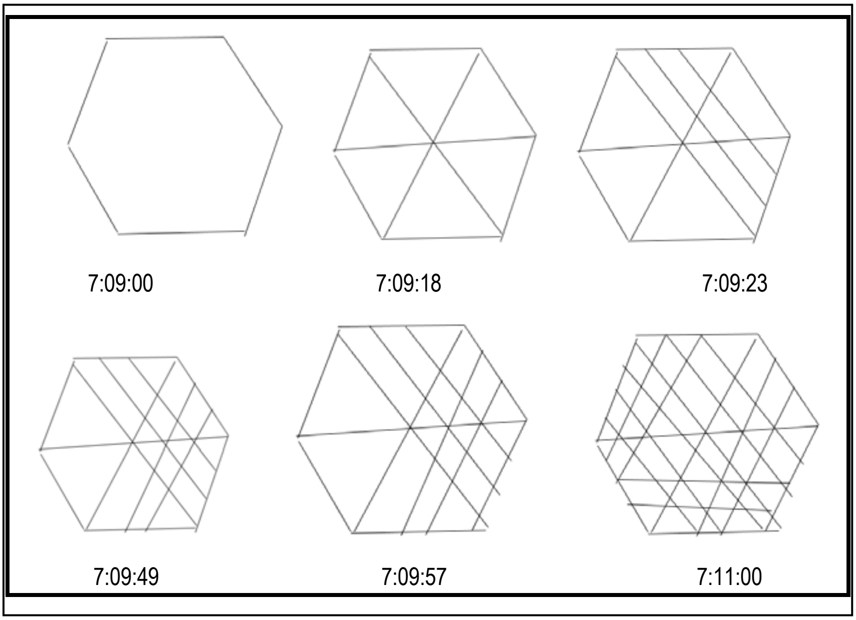


Figure 3. Six stages of 137's drawing actions obtained from the Replayer tool. The time stamp of each stage is displayed under the corresponding image. Snapshots focus on a particular region on the whiteboard where the relevant drawing activity is taking place.

137's chat posting in line 1 that follows his drawing effort (which can be read as a self-critical, sarcastic "great") suggests that he considers his illustration inadequate in some way. He makes this explicit by soliciting help from other members to produce "a diagram of a bunch of triangles" on the whiteboard, and then removing the diagram he has just produced (the boxes following this posting in Figure 5 correspond to deletion actions on the whiteboard). By removing his diagram, 137 makes that space available to other members for the projected drawing activity.

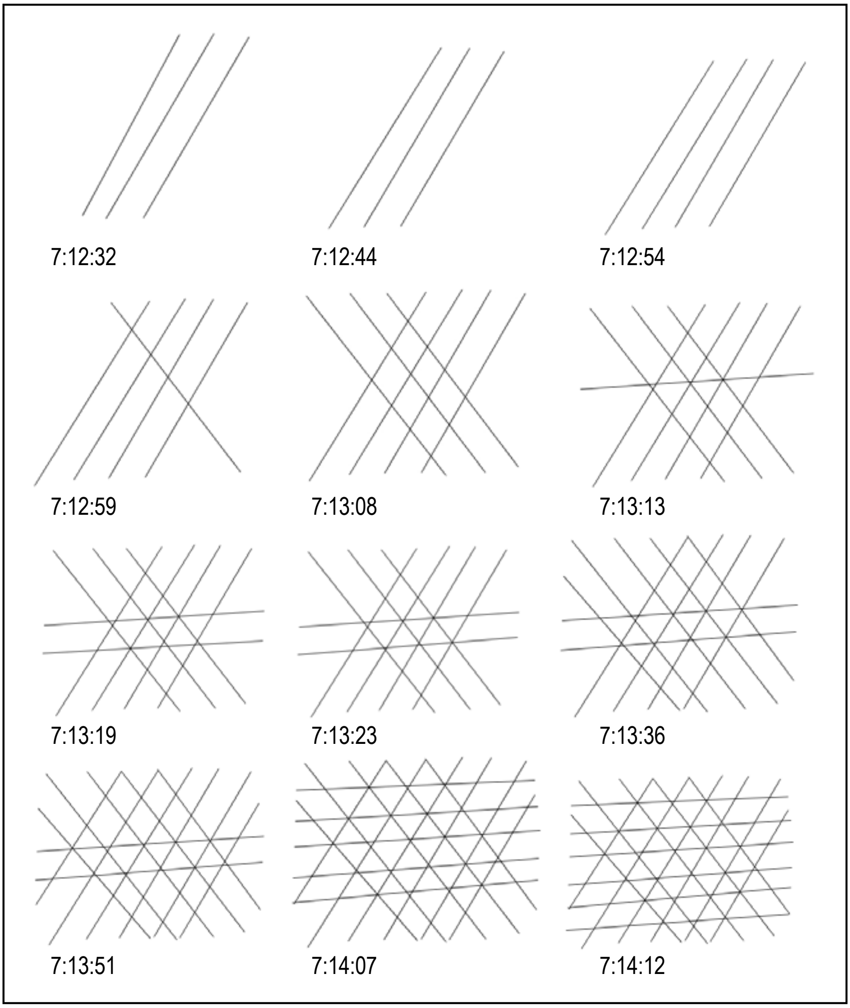


Figure 4. The evolution of Qwertyuiop's drawing in response to 137's request.

Qwertyuiop responds to 137's query with a request for clarification regarding the projected organization of the drawing ("just a grid?"). After 137's acknowledgement, Qwertyuiop performs a series of drawing actions that resemble the latter stages of 137's drawing actions, namely starting with the parallel lines tipped to the right first, then drawing a few parallel lines tipped to the left, and finally adding horizontal lines at the intersection points of earlier lines that are parallel to each other (see Figures 4 and 5). Having witnessed 137's earlier actions, the similarity in the organizations of both drawing actions suggest that Qwertyuiop has appropriated some key aspects of 137's drawing strategy, but modified/reordered the steps (e.g., he did not start with the hexagon at the beginning) in a way that allowed him to produce a grid of triangles as a response to 137's request.

The key point we would like to highlight in this episode is that *the availability of the sequencing of the drawing actions that produces a diagram on the shared whiteboard can serve as a vital resource for collaborative sense-making*. As seen in Log 1, 137 did not provide any explanation in chat about his drawing actions or about the shape he was trying to draw. Yet, as we have observed in the similarity of Figures 3 and 4, the orderliness of 137's actions has informed Qwertyuiop's subsequent performance. The methodical use of intersecting parallel lines to produce triangular objects is common to both drawing performances. Moreover, Qwertyuiop does not repeat the same set of drawing actions, but selectively uses 137's steps to produce the relevant object (i.e., a grid of triangles) on the whiteboard. Qwertyuiop does not initially constrain his representational development by constructing a hexagon first, but allows a hexagon (or other shapes made with triangles) to emerge from the collection of shapes implied by the intersecting lines. Thus, Qwertyuiop's performance shows us that he is able to *notice a particular organization* in 137's drawing actions, and he has *selectively appropriated and built upon* some key aspects of 137's drawing practice. As we will see in the following logs,[1] the group's subsequent use of this drawing will provide us additional evidence that Qwertyuiop's diagram serves as an adequate response to 137's request.

This excerpt highlights a fundamental difference between the two interaction spaces: whiteboard and chat contributions differ in terms of the availability of their production process. As far as chat messages are concerned, participants can only see who is currently typing,[2] but not what is being typed until the author decides to send the message. A similar situation applies to *atomic* whiteboard actions such as drawing an individual line or a rectangle. Such actions make a single object appear in the shared drawing area when the user releases the left mouse button; in the case of editable objects such as textboxes, the object appears on the screens of the computers of all chat participants when the editor clicks outside the textbox.

[1] For instance, after Qwertyuiop declares the completion of the grid in line 11, 137 anchors Qwertyuiop's drawing to the background at 7:15:47 (see Log 3). Because such a move preserves the positions of the selected objects and the objects affected by the move include only the lines recently added by Qwertyuiop, 137's anchoring move seems to give a particular significance to Qwertyuiop's recent drawing. Hence, 137's anchoring move can be treated as an (implicit) endorsement of Qwertyuiop's drawing effort in response to his previous request.

[2] While a participant is typing, a social awareness message appears under the chat entry box on everyone else's screen stating that the person "is typing" (see Figure 5). When the typist posts the message, the entire message appears suddenly as an atomic action in everyone's chat window.

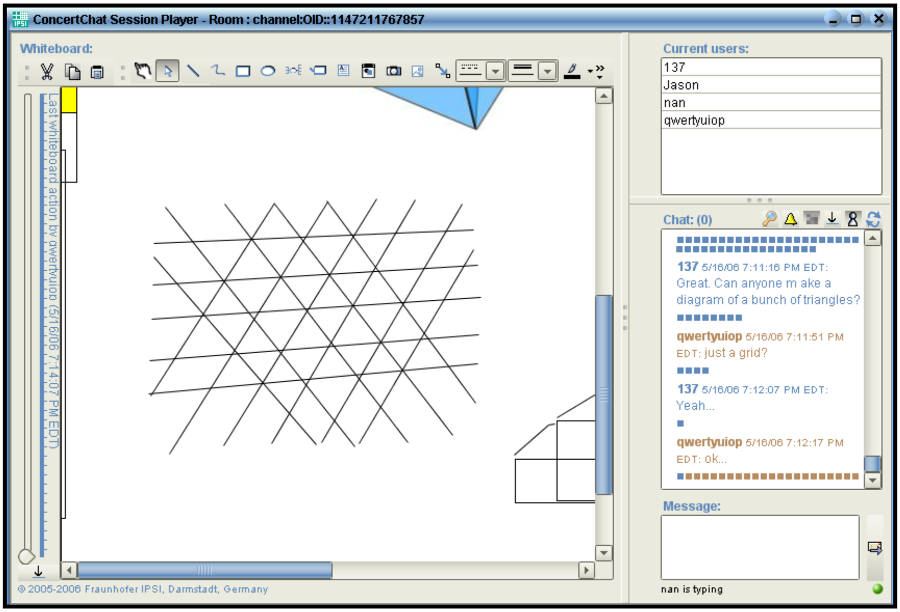


Figure 5. The interface at the 12th stage of Figure 4.

However, the construction of most shared diagrams includes the production of multiple atomic shapes (e.g., many lines), and hence the sequencing of actions that produce these diagrams is available to other members. As we have observed in this excerpt, the availability of the drawing process can have interactionally significant consequences for math-problem-solving chats due to its instructionally informative nature. In short, the whiteboard affords an *animated evolution* of the shared space, which makes the *visual reasoning process* manifest in drawing actions *publicly available* for other members' inspection. For instance, in Figure 4, transitions from stages 1 to 2 and 7 to 8 show modifications performed to achieve a peculiar geometric organization on the shared workspace.

Mutability of Chat and Whiteboard Contents

Another interactionally significant difference between the chat and the whiteboard interaction spaces, which is evidenced in the excerpt above, is the difference in terms of the mutability of their contents. Once a chat posting is contributed, it cannot be changed or edited. Moreover, the sequential position of a chat posting cannot be altered later on. If the content or the sequential placement of a chat posting turns out to be interactionally problematic, then a new posting needs to be composed to repair that. On the other hand, the object-oriented design of the whiteboard allows users to reorganize its content by adding new objects and by moving, annotating, deleting, and reproducing existing ones. For instance, the way 137 and Qwertyuiop repaired their drawings in the excerpt above by repositioning some of the lines they drew earlier to make sure that they intersect at certain points and/or that they are parallel to the edges of the hexagon illustrates this difference. Such demonstrable tweaks make the mathematical details of the construction work visible and relevant to observers, and hence, serve as a vital resource for joint mathematical sense making. By seeing that Qwertyuiop successively and intentionally adjusts lines in his whiteboard drawing to appear more parallel or to intersect more precisely, the other group members take note of the significance of the arrangement of lines as parallel and intersecting in specific patterns.

While both chat and whiteboard in VMT support persistence, visibility, and mutability, they do so in different ways. A chat posting scrolls away only slowly and one can always scroll back to it, whereas a drawing may be erased by anyone at any time. Chat conventions allow one to replace (i.e., follow) a mistyped posting with a new one, and conversational conventions allow utterances to be retracted, repaired, or refined. The mechanisms of the two mediational technologies are different and the characteristics of their persistence, visibility, and mutability differ accordingly. Collaborative interaction in the dual-space environment is sensitively attuned to these intricate and subtle differences.

Monitoring Joint Attention

The excerpt in Log 2 immediately follows the one in Log 1, where the team is oriented to the construction of a triangular grid after a failed attempt to embed a grid of triangles inside a hexagon. As Qwertyuiop is adding more lines to the grid, the facilitator (Nan) posts two questions addressed to the whole team in line 5. The question not only queries about what is happening now and whether everybody knows what others are currently doing, but the placement of the question at this point in interaction also problematizes the relevance of what has been happening so far. 137's response in lines 6 and 8 treat the facilitator's question as a problematic intervention. Qwertyuiop's response indicates he is busy with making triangles, and hence may not know what others are doing. Jason acknowledges that he is following what has been going on in line 9. These responses indicate that the team members have been following (perhaps better than the facilitator) what has been happening on the whiteboard so far as something relevant to their task at hand.

Log 2

5	7:14:09	nan	so what's up now? does everyone know what other people are doing?
	7:14:12	Qwertyuiop	< *Qwertyuiop adds a line to the grid of triangles*>
6	7:14:25	137	Yes?
7	7:14:25	Qwertyuiop	no-just making triangles
	7:14:32	Qwertyuiop	< *Qwertyuiop adds a line to the grid of triangles*>

8	7:14:33	137	I think... [*REF to line 6*]
9	7:14:34	Jason	Yeah
	7:14:36	Qwertyuiop	< *Qwertyuiop adds a line to the grid of triangles*>
10	7:14:46	nan	good :-)
11	7:14:51	Qwertyuiop	Triangles are done
12	7:15:08	137	So do you want to first calculate the number of triangles in a hexagonal array?

In this excerpt, the facilitator calls on each participant to report on his/her understanding of the activities of other participants. There was an extended duration in which no chat postings were published while whiteboard actions were being performed by Qwertyuiop. Because it is not possible for any participant to observe other participants directly, it is not possible to monitor a class of actions others may perform that (1) are important for how we understand ongoing action but (2) do not involve explicit manipulation of the VMT environment, actions like watching the screen, reading text, inspecting whiteboard constructs, and so forth. The only way to determine if those kinds of actions are occurring is to explicitly inquire about them using a chat posting.

Past and Future Relevancies Implied by Shared Drawings

Following Qwertyuiop's announcement in line 11 of Log 2 that the drawing work is complete, 137 proposes that the team calculate "the number of triangles" in a "hexagonal array" as a possible question to be pursued next. Although a hexagon was previously produced as part of the failed drawing, this is the first time someone explicitly mentions the term "hexagonal array" in this session. What makes 137's proposal potentially intelligible to others is the availability of referable resources such as whiteboard objects, and the immediate history of the production of those objects such that the proposal can be seen to be embedded in a sequence of displayed actions. 137's use of "So" to introduce his proposal presents it as a consequence of, or a making explicit of, what preceded. His suggestion of it as a "first" (next) move implies that the drawings opened up multiple mathematical tasks that the group could pursue, and that the proposed suggestion would be a candidate for a next move. In other words, the objects on the whiteboard and their visually shared production index a horizon of past and future activities. The indexical terms in 137's proposal (like "hexagonal array") not only rely on the availability of the whiteboard objects to propose a relevant activity to pursue next, but also modify their sense by using linguistic and semantic resources in the production to label or gloss the whiteboard object and its production. This allows actors to orient in particular ways to the whiteboard object and the procedures of its co-construction—providing a basis for coordinated joint activity. The joint activity acquires a temporal structure that is defined by the details of chat wording, the animation of graphical construction, and the sequentiality of proposing.

Methods for Referencing Relevant Objects in the Shared Visual Field

Bringing relevant mathematical objects to other members' attention often requires a coordinated sequence of actions performed in both the chat and whiteboard interaction spaces. The episode following 137's proposal (Log 3) provides us with an appropriate setting to illustrate how participants achieve this in interaction. Following 137's proposal in line 12, both Qwertyuiop and Jason post queries for clarification in lines 13 and 16, respectively, which indicate that the available referential resources were insufficient for them to locate what 137 is referring to with the term "hexagonal array." Jason's query in the chat is particularly important here because it explicitly calls for a response to be performed on the shared diagram, that is, in a particular field of relevance in the other interaction space. Following Jason's query, 137 begins to perform a sequence of drawing actions on the shared diagram. He adds a few lines that gradually begin to enclose a region on the triangular grid[3] (see Figure 6).

Log 3

11	7:14:51	Qwertyuiop	Triangles are done
12	7:15:08	137	So do you want to first calculate the number of triangles in a hexagonal array?
13	7:15:45	Qwertyuiop	What's the shape of the array? a hexagon? *<REF to 12>*
	7:15:47	137	*<137 locks the triangular grid that Qwertyuiop has just drawn>*
14	7:16:02	137	Ya *<REF to line 13>*
15	7:16:15	Qwertyuiop	ok….
	7:16:18 - 7:16:35	137	*<137 performs a few drawing actions and then erases them>*
16	7:16:41	Jason	wait-- can someone highlight the hexagonal array on the diagram? i don't really see what you mean...
	7:16:45 - 7:17:28	137	*<137 adds new lines to the grid on the whiteboard which gradually forms a contour on top of the grid. Figure 6 shows some of the steps*

---

3 In the meantime, Qwertyuiop also performs a few drawing actions near the shared drawing, but his actions do not introduce anything noticeably different because he quickly erases what he draws each time.

			performed by 137>
17	7:17:30	Jason	Hmm.. okay
18	7:17:43	Qwertyuiop	Oops <*REF to Whiteboard*>
19	7:17:44	Jason	so it has at least 6 triangles?
20	7:17:58	Jason	in this, for instance <*REF to Whiteboard*>
	7:18:03 - 7:18:17	137	<*137 completes the contour by adding more lines, which forms a hexagon*>
21	7:18:53	137	How do you color lines?
22	7:19:06	Jason	There's a little paintbrush icon up at the top
23	7:19:12	Jason	it's the fifth one from the right
	7:19:13 - 7:19:20	137	<*137 begins to change the color of the lines that form the contour to blue*>
24	7:19:20	137	Thanks.
25	7:19:21	Jason	There ya go :-)
	7:19:25 - 7:19:48	137	<*137 finishes the coloring. Now the contour is highlighted in blue*>
26	7:19:48	137	Er... That hexagon.
27	7:20:02	Jason	so... should we try to find a formula i guess

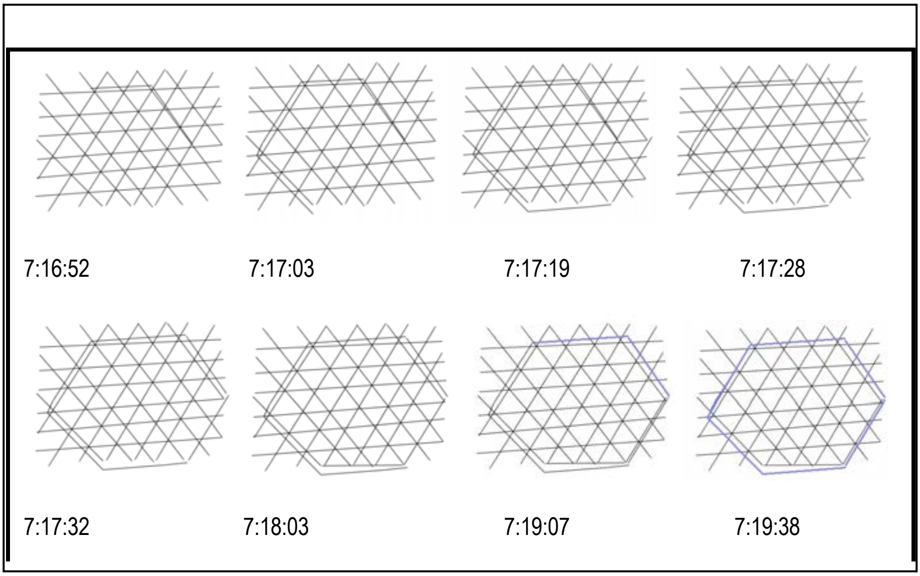


Figure 6. Snapshots from the sequence of drawing actions performed by 137.

When the shared diagram reaches the stage illustrated by the 4th frame in Figure 6, Jason posts the message "hmmm... okay" in line 17, which can be read as an acknowledgement of 137's performance on the whiteboard as a response to his recent chat query. Because no chat message was posted after Jason's request in line 16, and the only shared actions were 137's work on the whiteboard, Jason's chat posting can be read as a response to the ongoing drawing activity on the whiteboard. As it is made evident in his posting, Jason is treating the evolving drawing on the shared diagram as a response to his earlier query for highlighting the hexagonal array on the whiteboard: The question/answer adjacency pair is spread across the two interaction spaces in an unproblematic way.

Following provisional acknowledgement of 137's drawing actions on the whiteboard, Jason posts a claim in line 19. This posting is built as a declarative: "so it has at least 6 triangles," with a question mark appended to the end. The use of "so" in this posting invites readers to treat what follows in the posting as a consequence of the prior actions of 137. In this way, Jason is (a) proposing a defeasible extension of his understanding of the sense of 137's actions and (b) inviting others to endorse or correct this provisional claim about the hexagonal array by presenting this as a query using the question mark.

In line 20, Jason provides further specificity to what he is indexing with the term "it" in line 19 by highlighting a region on the grid with the referencing tool of the VMT system. The textual part of the posting makes it evident that the highlighted region is an instance of the object mentioned in line 19. Moreover, the six triangles highlighted by the explicit reference recognizably make up a hexagon shape altogether. Hence, Jason's explicit reference seems to be pointing to a particular stage (indexed by "at least") of the hexagonal array that the team is oriented to (see Figure 7).

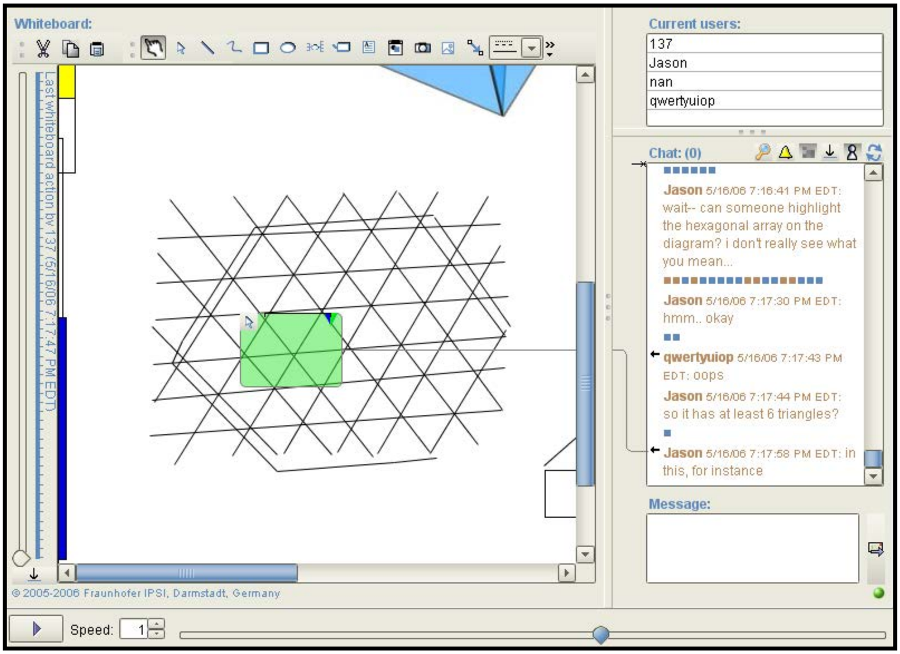


Figure 7. Use of the referencing tool to point to a stage of the hexagonal array.

In other words, having witnessed the production of the hexagonal shape on the whiteboard as a response to his earlier query, Jason displays his competence by demonstrating his recognition of the hexagonal pattern implicated in 137's graphical illustration. 137's drawing actions highlight a particular stage of a growing pattern made of triangles—stage N=3, as we will see in Figure 9. However, recognizing the stick-pattern implicated in 137's highlighting actions requires other members to project how the displayed example can be grown and/or shrunk to produce other stages of the hexagonal array. Thus, Jason's description of the shape of the "hexagonal array" at a different stage—N=1—is a public display of his newly achieved comprehension of the significance of the math object in the whiteboard and the achievement of "indexical symmetry" among the parties involved with respect to this math object (see Stahl, 2009b, chap.14).

Although Jason explicitly endorsed 137's drawing as an adequate illustration, the small boxes in the chat stream that appear after Jason's acknowledgement in line 17 show that 137 is still oriented to and operating on the whiteboard. In line 21, 137 solicits other members' help regarding how he can change the color of an object on the board, which opens a side sequence about a specific feature of the whiteboard system. Based on the description he got, 137 finishes marking the hexagon by coloring all its edges with blue, and he posts "that hexagon" in line 25. This can be read as a chat reference to the whiteboard shape enclosed by the blue contour, and as a response to other members' earlier requests for clarification.

In this excerpt, we have observed two referential methods enacted by participants to bring relevant graphical objects on the whiteboard to other group members' attention. In the first case, 137 *marked the drawing* with a different color to identify the contour of a hexagonal shape. As evidenced in other members' responses, this was designed to make the hexagonal array embedded in a grid of triangles visible to others. Jason demonstrated another method by using the explicit referencing tool to support his *textual description* of the first stage of the pattern. Both mechanisms play a key role in directing other members' attention to features of the shared *visual field* in particular ways. This kind of deictic usage isolates components of the shared drawing and constitutes them as relevant objects to be attended to for the purposes at hand. As we shall see, these guided shifts in visual focus of the group have strategic importance for the group's mathematical work. Hence, such referential work establishes a fundamental *relationship between the narrative and mathematical terminology used in text chat and the animated graphical constructions produced on the whiteboard*. The shared sense of the textual terms and the inscriptions co-evolve through the referential linkages established as the interaction sequentially unfolds in both interaction spaces.

In Log 3, the group tentatively proposes a major mathematical insight—that a hexagon can be viewed as six symmetric triangular areas. It is a visual achievement. It emerges from a visual inspection by Jason of 137's graphical diagram, based on Qwertyuiop's method of visually representing hexagons as patterns of triangularly intersecting lines. By literally focusing his eyes on a smallest hexagon in the larger array and counting the number of triangles visible within a hexagonal border, Jason discovers that there are at least six triangles at the initial stage of a hexagon with one unit on each side. We will see how the group visualizes the generalization of this picture to other stages. However, it is already interesting to note that Jason not only observes the composition of a small hexagon out of six triangles, but he conveys this to the rest of the group in both media. He posts chat line 19 and then references from chat line 20 to a visually highlighted view in the whiteboard, so that his *visual* understanding can be shared by the group as well as his *narrative* description in his claim. The next step for the group will be to formulate a *symbolic* mathematical expression of this claim.

Whiteboard Visualizations, Chat Narratives and Wiki Symbolisms

The excerpt in Log 4 immediately follows Log 3. The way 137 uses both interaction spaces in this episode highlights another important aspect of collaborative problem-solving work in an environment like VMT. Because participants can contribute to only one of the interaction spaces at a time, they cannot narrate their whiteboard actions simultaneously with chat postings, as can be done with talking about a whiteboard in a face-to-face setting. However, as we will observe in 137's use of the whiteboard in the following excerpt, participants can achieve a similar interactional organization by coordinating their actions in such a way that whiteboard actions can be seen as part of an exposition performed in chat.

Log 4

27	7:20:02	Jason	so... should we try to find a formula i guess
28	7:20:22	Jason	input: side length; output: # triangles
29	7:20:39	Qwertyuiop	It might be easier to see it as the 6 smaller triangles.
30	7:20:48	137	Like this? <*REF to line 29*>
	7:20:53	137	<*137 draws a red line*>
	7:20:57	137	<*137 draws a red line*>
	7:21:00	137	<*137 draws a red line*>
31	7:21:02	Qwertyuiop	Yes
32	7:21:03	Jason	Yup
	7:21:03	137	<*137 moves the second red line*>
	7:21:05	137	<*137 moves the second red line again. It is positioned on the grid now*>
33	7:21:29	Qwertyuiop	Side length is the same...
34	7:22:06	Jason	Yeah

Jason brings the prior activity of locating the hexagonal array on the shared drawing to a close with his so-prefaced posting in line 27, where he invokes the task of finding a formula that was mentioned by 137 earlier. Jason provides further specificity to the formula he is referring to in the next line (i.e., given the side length as input the formula should return the number of triangles as output). In line 29, Qwertyuiop takes up Jason's proposal by suggesting the team consider the hexagonal array as six smaller triangles to potentially simplify the task at hand. In the next line, 137 posts a question phrased as "like this?" which is addressed to Qwertyuiop's prior posting, as indicated by the use of the referential arrow. Next, we observe the appearance of three red lines on the shared diagram, which are all added by 137. Here, 137 demonstrates a particular way of splitting the hexagon into six parts: The image on the left of Figure 8 corresponds to the sequence of three whiteboard actions represented as three boxes in the chat excerpt. After 137 adds the third line whose intersection with the previously drawn red lines recognizably produces six triangular regions on the shared representation, Qwertyuiop and Jason both endorse 137's demonstration of a particular way of splitting up the hexagonal shape.

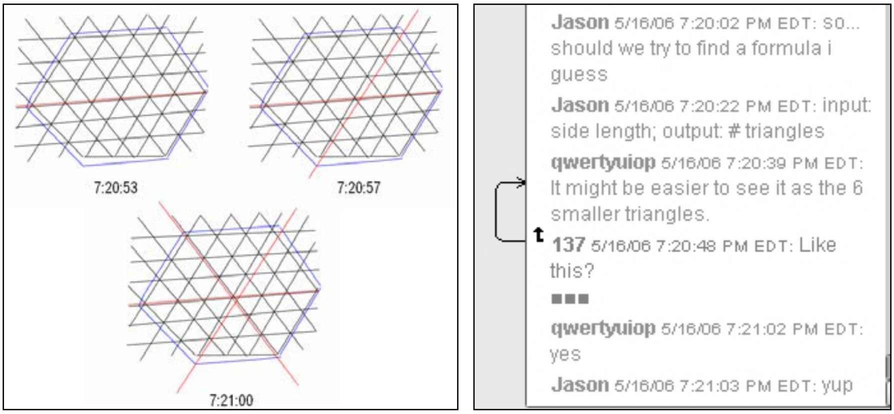


Figure 8. 137 splits the hexagon into six parts.

One important aspect of this organization is directing other members' attention to the projected whiteboard activity as a relevant step in the sequentially unfolding exposition in chat. For instance, the deictic term "this" in 137's chat line 30 refers to something yet to be produced, and thereby projects that there is more to follow the current posting, possibly in the other interaction space. Moreover, the use of the referential link and the term "like" together inform others that what is about to be done should be read in relation to the message to which 137 is responding. Finally, 137's use of a different color marks the newly added lines as recognizably distinct from what is already there as the background, and hence, noticeable as a demonstration of what is implicated in recent chat postings.

Again, the progress in understanding the mathematics of the problem is propelled through visual means. In response to Jason's proposal of finding a formula, Qwertyuiop suggests that "it might be easier to see it" in a certain way. Jason's proposed approach might be difficult to pursue because no one has suggested a concrete approach to constructing a formula that would meet the general criteria of producing an output result for any input variable value. By contrast, the group has been working successfully in the visual medium of the whiteboard drawing and has been literally able to "see" important characteristics of the math object that they have co-constructed out of intersecting lines. Jason has pointed out that at least six triangles are involved (in the smallest hexagon). So, Qwertyuiop proposes building on this insight. 137 asks if the way to see the general case in terms of the six small triangles as proposed by Qwertyuiop can be visualized by intersecting the hexagon array with three intersecting lines to distinguish the six regions of the array. He does this through a visual construction, simply referenced from the chat with his "Like this?" post.

By staring at the final version of the array (stage 3 in Figure 8), all members of the group can see the hexagon divided into six equal parts at each stage of the hexagonal pattern. Near the intersection of the red lines, they can see a single small triangle nestled in each of the six regions. As will be evidenced in Log 5, within the larger hexagon delimited by the blue lines, they can see a set of 1+3+5=9 small triangles in each of the six larger triangular regions. Similarly, midway between stage N=1 and stage N=3, one can visually observe 1+3=4 small triangles in each region. The new view, scaffolded by 137's red lines, entails *visual reasoning* that leads to mathematical deductions. As soon as Qwertyuiop and Jason see 137's construction, they both concur with it as the easier way to see the mathematical pattern of triangles in the hexagonal array. The visual reasoning supported by whiteboard and narrated textually in the chat will lead in the next episode to symbolic reasoning for posting in the wiki.

A first glance at the chat logs might suggest that the group is narrating their problem-solving process in the chat and illustrating what they mean by "napkin" drawings in the whiteboard, to use Dillenbourg & Traum's (2006) metaphor. However, a second look reveals that the most significant insight and sharing is occurring in the whiteboard, more along the lines of a visual "model" metaphor. Perhaps the best way to describe what is going on is to say that the group is very carefully coordinating their work in the dual space as a whole to achieve a shared progression of understanding of the pattern problem. This is accomplished with an efficiency and effectiveness that could not be achieved in either a purely textual chat system or a purely graphical whiteboard. Although in this view the chat and whiteboard both function as symmetric parts of a coordinated whole in which chat references drawing and drawing illustrates chat, it is important to differentiate their roles as well.

Using Representations of Specific Instances as a Resource for Generalization

Immediately following the previous excerpt, the team moves on to figuring out a general formula to compute the number of triangles in a hexagonal pattern. In line 34 of Log 5, Jason relates the particular partitioning of the hexagon illustrated on the whiteboard to the problem at hand by stating that the number ("#") of triangles in the hexagon will equal 6 times ("x6") the number of triangles enclosed in each partition. In the next posting, 137 seems to be indexing one of the six partitions with the phrase "each one." Hence, this posting can be read as a proposal about the number of triangles included in a partition. The sequence of numbers in the expression "1+3+5" calls others to look at a partition in a particular way. While 137 could have simply said here that there are nine triangles in each partition, he instead organizes the numbers in summation form and offers more than an aggregated result. His expression also demonstrates a systematic method for counting the triangles. In other words, his construction is designed to highlight a particular orderliness in the organization of triangles that form a partition. Moreover, the sequence includes increasing consecutive odd numbers, which implicitly informs a certain progression for the growth of the shape under consideration.

Log 5

34	7:22:13	Jason	so it'll just be x6 for # triangles in the hexagon
35	7:22:19	137	Each one has 1+3+5 triangles.
36	7:22:23	Jason	but then we're assuming just regular hexagons
37	7:22:29	Qwertyuiop	the “each polygon corrisponds to 2 sides” thing we did last time doesn't work for triangles
38	7:23:17	137	It equals 1+3+...+(n+n-1) because of the “rows”?
39	7:24:00	Qwertyuiop	yes- 1st row is 1, 2nd row is 3...
40	7:24:49	137	And there are n terms so... n(2n/2)
41	7:25:07	137	or n^2 <*REF to line 40*>
42	7:25:17	Jason	Yeah
43	7:25:21	Jason	then multiply by 6
44	7:25:31	137	To get 6n^2 <*REF to line 43*>

About a minute after his most recent posting, 137 offers an extended version of his sequence as a query in line 38. The relationship between the sequence for the special case and this one is made explicit through the repetition of the first two terms. In the new version the "…" notation is used to substitute a series of numbers following the second term up to a generic value represented by "n+n-1," which can be recognized as a standard expression for the n^{th} odd number. Hence, this representation is designed to stand for something more general than the one derived from the specific instance illustrated on the whiteboard. 137 attributes this generalization to the concept of "rows," and solicits other members' assessment regarding the validity of his version (by ending with a question mark). 137's use of the term "rows" seems to serve as a pedagogic device that attempts to locate the numbers in the sequence on the n^{th} stage of the hexagonal pattern (see Figure 9 for an analyst's illustration of the generalized hexagonal pattern). For stages 1, 2, and 3, the hexagonal shape has 6*(1) = 6, 6*(1+3) = 24, 6*(1+3+5) = 54 triangles, respectively.

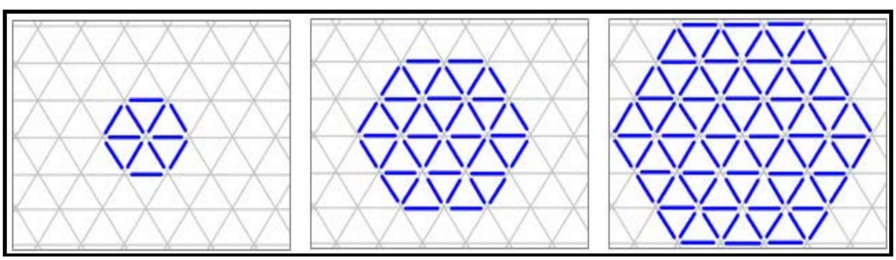


Figure 9. A reconstruction of the first three iterations of the geometric pattern.

Qwertyuiop's endorsement of 137's proposal comes in line 39. He also demonstrates a row-by-row iteration on a hexagon, where each number in the sequence corresponds to a row of triangles in a partition. In other words, Qwertyuiop elaborates on 137's statement in line 38 of the chat by displaying his understanding of the relationship between the rows and the sequence of odd numbers. Although he does not explicitly reference it here, Qwertyuiop may be viewing the figure in the whiteboard to see the successive rows. The figure is, of course, also available to 137 and Jason to help them follow Qwertyuiop's chat posting and check it.

Then 137 proposes an expression for the sum of the first n odd numbers in line 40.[4] Jason agrees with the proposed expression and suggests that it should be multiplied by 6 next. In the following line, 137 grammatically completes Jason's posting with the resulting expression. In short, by virtue of the agreements and the co-construction work of Jason and 137, the team demonstrates its endorsement of the conclusion that the number of triangles would equal $6n^2$ for a hexagonal array made of triangles. As the group collaboratively discovered, when n equals the stage number (as "input" to the formula), the number of triangles is given by the expression $6n^2$.

The way team members orient themselves to the shared drawing in this episode illustrates that the drawings on the whiteboard have a figurative role in addition to their concrete appearance as illustrations of specific cases. The particular cases captured by concrete, tangible marks on the whiteboard are often used as a resource to investigate and talk about general properties of the mathematical objects indexed by them.

Another important aspect of the team's achievement of a general expression in this episode is the way they transformed a particular way of *counting* the triangles in one of the partitions (i.e., a geometric observation) into an algebraic mode of investigation. This shift from a visual method led the team members to recognize that a particular sequence of numbers can be associated with the way the partition grows in subsequent iterations. The shift to this symbolic mode of engagement, which heavily uses the shared drawing as a resource, allowed the team to go further in the task of generalizing the pattern of growth by invoking algebraic resources. In other words, the team made use of multiple realizations (graphical and linguistic) of the math object (the hexagonal array) distributed across the dual-interaction space to co-construct a general formula for the task at hand.

---

[4] 137 makes use of Gauss's method for summing this kind of series, adding the first and last term and multiplying by half of the number of terms: $(1 + n + n - 1)*n/2=2n*n/2=n^2$. This method was used by the group and shared in previous sessions involving the stair pattern that is still visible in the whiteboard.

Chat Versus Whiteboard Contributions as Persistent Referential Resources

In all of the excerpts we have considered so far, the shared drawing has been used as a resource within a sequence of related but recognizably distinct activities. For instance, the group has oriented itself to the following activities: (1) drawing a grid of triangles, (2) formulating a problem that relates a hexagonal array to a grid of triangles, (3) highlighting a particular hexagon on the grid, (4) illustrating a particular way to split the shape into six smaller pieces, and (5) devising a systematic method to count the number of triangles within one of the six pieces. As the group oriented to different aspects of their shared task, the shared diagram was modified on the whiteboard and annotated in chat accordingly. Yet, although it had been modified and annotated along the way, the availability of this shared drawing on the screen and the way participants organize their discussion around it highlights its persistent characteristic as an ongoing referential resource. In contrast, none of the chat postings in prior excerpts were attributed a similar referential status by the participants. As we have seen, in each episode the postings responded or referred either to recently posted chat messages or to the visual objects in the shared space.

The textual chat postings and the graphical objects produced on the whiteboard differ in terms of the way they are used as referential resources by the participants. The content of the whiteboard is persistently available for reference and manipulation, whereas the chat content is visually available for reference for a relatively shorter period. This is due to the linear growth of chat content, which replaces previous messages with the most recent contributions inserted at the bottom of the chat window. Although one can make explicit references to older postings by using the scroll-bar feature, the limited size of the chat window affords a referential locality between postings that are visually (and hence temporally) close to each other.

By contrast, objects drawn in the whiteboard tend to remain there for a long time. They are often only erased or moved out of view when space is needed for drawings related to a new topic. While they may be modified, elaborated, or moved around, whiteboard objects may remain visible for an entire hour-long session or even across sessions. Like the chat, the whiteboard has a history scrollbar, so that any past state of the drawing can be made visible again—although in practice students rarely use this feature. Although both media technically offer a persistent record of their contents, the visual locality of the whiteboard—the fact that graphical objects tend to stay available for reference from the more fleeting chat—qualifies it as the more persistent medium as an interactional resource. This notion of persistence does not imply that the shared sense of whiteboard objects is fixed once they are registered to the shared visual field. As they continue to serve as referential resources during the course of the problem-solving effort, the sense of whiteboard objects may become increasingly evident and shared, or their role may be modified as participants make use of them for varying purposes.

Implications for CSCL Chat Interaction Analysis

In this case study, we investigated how a group of three upper-middle-school students put the features of an online environment with dual-interaction spaces into use as they collaboratively worked on a math problem they themselves came up with. Our analysis has revealed important insights regarding the affordances of systems with dual-interaction spaces. First, we observed that the whiteboard can make visible to everyone the animated evolution of a geometric construction, displaying the *visual reasoning* process manifested in drawing actions. Second, whiteboard and chat contents differ in terms of *mutability* of their contents, due to the object-oriented design of the whiteboard that allows modification and annotation of past contributions. Third, the media differ in terms of the *persistence* of their contents: Whiteboard objects remain in the shared visual field until they are removed, whereas chat content gradually scrolls off as new postings are produced. Although contents of both spaces are persistently available for reference, due to linear progression of the chat window, chat postings are likely to refer to visually (and hence temporally) close chat messages and to graphical whiteboard objects. Finally, the whiteboard objects *index* a horizon of past and future activities as they serve as an interactional resource through the course of recognizably distinct but related episodes of chat discussion.

Our analysis of this team's joint work has also revealed methods for the organization of collaborative work, through which group members co-construct mathematical meaning sedimented in semiotic objects distributed across the dual- interaction spaces of the VMT environment. We observed that bringing relevant math artifacts referenced by indexical terms such as "hexagonal array" to other members' attention often requires a coordinated sequence of actions across the two interaction spaces. Participants use explicit and verbal references to guide each other about how a new contribution should be read in relation to prior contents. Indexical terms stated in chat referring to the visible production of shared objects are instrumental in the reification of those terms as meaningful mathematical objects for the participants. Verbal references to co-constructed objects are often used as a resource to index complicated and abstract mathematical concepts in the process of co-constructing new ones. Finally, different representational affordances of the dual-interaction spaces allow groups to develop multiple realizations of the math artifacts to which they are oriented. Shared graphical inscriptions and chat postings are used together as semiotic resources in mutually elaborating ways. Methods of coordinating group interaction across the media spaces also interrelate the mathematical significances of the multiple realizations.

Overall, we observed that actions performed in both interaction spaces constitute an evolving historical context for the joint work of the group. What gets done now informs the relevant actions to be performed next, and the significance of what was done previously can be modified depending on the circumstances of the ongoing activity. As the interaction unfolds sequentially, the sense of previously posted whiteboard objects and chat statements may become evident and/or refined. In this way, the group's joint problem space is maintained.

Through the sequential coordination of chat postings and whiteboard inscriptions, the group successfully solved their mathematical challenge, to find a formula for the number of small triangles in a hexagonal array of any given side-length. Their interaction was guided by a sequence of proposals and responses carried out textually in the chat medium. However, the sense of the terms and relationships narrated in the chat were largely instantiated, shared, and investigated through observation of visible features of graphical inscriptions in the whiteboard medium. The mathematical object that was visually co-constructed in the whiteboard was named and described in words within the chat. Finally, a symbolic expression was developed by the group, grounded in the graphic that evolved in the whiteboard and discussed in the terminology that emerged in the chat. The symbolic mathematical result was then posted to the wiki, a third medium within the VMT environment. The wiki is intended for sharing group findings with other groups as part of a permanent archive of work by virtual math teams.

Our case study in this paper demonstrates that it is possible to analyze how math problem solving—and presumably other cognitive achievements—can be carried out by small groups of students. The students can define and refine their own problems to pursue; they can invent their own methods of working; they can use unrestricted vocabulary; they can coordinate work in multiple media, taking advantage of different affordances. Careful attention to the sequentiality of references and responses is necessary to reveal *how* the group coordinated its work and how that work was driven by the reactions of the group members' actions to each other. Only by focusing on the sequentiality of the actions can one see how the visual, narrative, and symbolic build on each other as well as how the actions of the individual students respond to each other. Through these actions, the students co-construct math objects, personal understanding, group agreement, and mathematical results that cannot be attributed to any one individual, but that emerge from the interaction as complexly sequenced.

This analysis illustrates a promising approach for CSCL research to investigate aspects of group cognition that are beyond the reach of alternative methods that systematically ignore the full sequentiality of their data.

The Group as the Unit of Analysis

For methodological reasons, quantitative approaches—such as those reviewed in the next section—generally (a) constrain (scaffold) subject behaviors, (b) filter (code) the data in terms of operationalized variables, and (c) aggregate (count) the coded data.

These acts of standardization and reduction of the data eliminate the possibility of observing the details and enacted processes of unique, situated, indexical, sequential, group interaction (Stahl, 2006, chap. 10). An alternative form of interaction analysis is needed to explore the organization of interaction that can take place in CSCL settings.

In this paper, we focused on small-group interactions mediated by a multimodal interaction space. Our study differs from similar work in CSCL by our focus on groups larger than dyads whose members are situated outside a controlled lab environment, and by our use of open-ended math tasks where students are encouraged to come up with their own problems. Moreover, we do not impose any deliberate restrictions on the ways students access the features of our online environment or on what they can say. Our main goal is to investigate how small groups of students construe and make use of the "available features" of the VMT online environment to discuss mathematics with peers from different schools outside their classroom setting. In other words, we are interested in studying interactional achievements of small groups in complex computer mediations "in the wild" (Hutchins, 1996).

Our interest in studying the use of an online environment with multiple interaction spaces in a more naturalistic use scenario raises serious methodological challenges. In an early VMT study where we conducted a content analysis of collaborative problem-solving activities mediated by a standard text-chat tool in a similar scenario of use, we observed that groups larger than dyads exhibit complex interactional patterns that are difficult to categorize based on a theory-informed coding scheme with a fixed/predetermined unit of analysis (Stahl, 2009b, chap. 20). In particular, we observed numerous cases where participants post their messages in multiple chat turns, deal with contributions seemingly out of sequence, and sustain conversations across multiple threads that made it problematic to segment the data into fixed analytic units for categorization. Moreover, coming to agreement on a code assignment for a unit that is defined a priori (e.g., a chat line) turned out to be heavily dependent upon how the unit can be read in relation to resources available to participants (e.g., the problem description) and to prior units (Stahl, 2009b, chap. 22). In other words, the sense of a unit not only depends on the semantic import of its constituent elements, but also on the occasion in which it is situated (Heritage, 1984). This often makes it possible to apply multiple categories to a given unit and threatens the comparability of cases that are labeled with the same category. More importantly, once the data is reduced to codes and the assignments are aggregated, the complex sequential relationships among the units are largely lost. Hence, the coding approach's attempt to enforce a category to each fixed unit without any consideration to how users sequentially organize their actions in the environment proved to be too restrictive to adequately capture the interactional complexity of chat (Stahl, 2009b, chap. 23). Moreover, the inclusion of a shared drawing area in our online environment made the use of a standard coding schema even harder due to increased possibilities for interaction. The open-ended nature of the tasks we use in our study makes it especially challenging to model certain types of actions and to compare them against ideal solutions.

The issue of unit of analysis has theoretical implications. In text chat, it is tempting to take a single posting as the unit to be analyzed and coded, because a participant defined this as a unit by posting it as a message and because the chat software displays it as a visual unit. However, this tends to lead the analyst to treat the posting as a message from the posting individual—that is, as an expression of a thought in the poster's mind, which must then be interpreted in the minds of the post readers. Conversation analysis has argued for the importance of *interactions* among participants as forming more meaningful units for analysis. These consist of sequences of multiple utterances by different speakers; the individual utterances take each other into account. For instance, in a question/answer "adjacency pair," the question elicits an answer and the answer responds to the question. To take a pair of postings such as a question/answer pair as the analytic unit is to treat the interaction within the group as primary. It focuses the analysis at the level of the group rather than the individual. As mentioned, in online text chat, responses are often separated from their referents, so the analysis is more complicated. In general, we find that the important thing is to trace as many references as possible between chat postings or whiteboard actions in order to analyze the interaction of the group as it unfolds (Stahl, 2009b, chap. 26). As seen in our case study, it is through the co-construction of a rich nexus of such references that the group weaves its joint problem space.

Analysis at the group unit focuses on the co-construction, maintenance, and progressive refinement of the joint problem space. This is a distinctive analytic task that takes as its data only what is shared by the group. Whatever may go on in the physical, mental, or cultural backgrounds of the individual participants is irrelevant unless it is brought into the group discourse. Because the students know nothing about the gender, age, ethnicity, accent, appearance, location, personality, opinions, grades, or skills of the other participants other than what is mentioned or displayed in the chat interaction, these "factors" from the individual and societal levels can be bracketed out of the group analysis. Survey and interview data is unnecessary; individual learning trajectories are not plotted. The VMT Project has been designed to make available to the analyst precisely what was shared by the student group, and nothing else.

Relatedly, the notion of common ground (see section on grounding below) as an abstract placeholder for registered cumulative facts or pre-established meanings has been critiqued in the CSCL literature for treating meaning as a fixed/denotative entity transcendental to the meaning-making activities of inquirers (Koschmann, 2002). The common ground that supports mutual understanding in group cognition or group problem solving is a matter of semantic references that unfold sequentially in the momentary situation of dialog, not a matter of comparing mental contents (Stahl, 2006, pp. 353-356). Committing to a reference-repair model (Clark & Marshall, 1981) for meaning making falls short of taking into account the dynamic, constitutive nature of meaning-making interactions that foster the process of inquiry (Koschmann et al., 2001).

As we saw in the preceding case study, the understanding of the mathematical structure of the hexagon area did not occur as a mental model of one of the students that was subsequently externalized in the chat and whiteboard and communicated to the other students. It emerged in the discourse media in a way that we could witness as analysts. It consisted of the layering of inscriptions (textual and graphical) that referenced one another. The referential network of group meaning can be observed in the way that deictic and indexical expressions are resolved. The three students each contribute to the progressive development of the shared meaning by responding appropriately to the ongoing state of the discourse. This is a matter of linguistic skill—including ability in discussing mathematical matters—not of articulating mental representations. It is surprising from a rationalist perspective how poor students are at explaining (Stahl, 2009b, chap. 26), reproducing (Koschmann & LeBaron, 2003), or even recalling (Stahl, 2009b, chap. 6) what they did in the group when they are no longer situated in the moment.

Given these analytical and theoretical issues, we opted for an alternative to the approaches reviewed below that involve modeling of actions and correct solution paths or treating shared understanding as alignment of preexisting individual representations and opinions. In this paper, we built on our previous work on referencing math objects in a system with chat and a whiteboard (Stahl, 2009b, chap. 17); we presented a "micro-ethnographic" (Streeck & Mehus, 2003) case study using interaction analysis (Jordan & Henderson, 1995). We focused on the *sequence of actions* in which the group co-constructs and makes use of *semiotic resources* (Goodwin, 2000) distributed across dual-interaction spaces to *do* collaborative problem-solving work. In particular, we focused on the joint organization of activities that produce graphical drawings on the shared whiteboard and the ways those drawings are used as resources by actors as they collaboratively work on an open-ended math task. Through detailed analysis at the group unit of analysis, we investigated how actions performed in one workspace inform the actions performed in the other and how the group coordinates its actions across both interaction spaces.

Other Approaches in CSCL to Analyzing Multimodal Interaction

Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text chat and a shared graphical workspace—have been widely used to support collaborative learning activities of small groups (Dillenbourg & Traum, 2006; Jermann, 2002; Mühlpfordt & Wessner, 2005; Soller & Lesgold, 2003; Suthers et al., 2001). The way such systems are designed as a juxtaposition of several technologically independent online communication tools carries important interactional consequences for the users. Engaging in forms of joint activity in such online environments requires group members to use the technological features available to them in methodical ways to make their actions across multiple spaces intelligible to each other and to sustain their joint problem-solving work.

In this section we summarize our review (Çakir, 2009) of previous studies in the CSCL research literature that focus on the interactions mediated by systems with multimodal interaction spaces to support collaborative work online. Our review is not meant to be exhaustive, but representative of the more advanced analytical approaches employed. We have selected sophisticated analyses, which go well beyond the standard coding-and-counting genre of CSCL quantitative reports, in which utterances are sorted according to a fixed coding scheme and then statistics are derived from the count of utterances in each category. Unlike the simple coding-and-counting studies, the approaches we review attempt to analyze some of the structure of the semantic and temporal relationships among chat utterances and workspace inscriptions in an effort to get at the fabric of common ground in dual-interaction online environments.

The communicative processes mediated by multimodal interaction spaces have attracted increasing analytical interest in the CSCL community. A workshop held at CSCL 2005 specifically highlighted the need for more systematic ways to investigate the unique affordances of such online environments (Dillenbourg, 2005). Previous CSCL studies that focus on the interactions mediated by systems with two or more interaction spaces can be broadly categorized under: (1) prescriptive approaches based on models of interaction and (2) descriptive approaches based on content analysis of user actions.

(1) The *modeling approach* builds on the content-coding approach by devising models of categorized user actions performed across multimodal interaction spaces, for example:

(a) Soller & Lesgold's (2003) use of hidden Markov models (HMM) and
(b) Avouris et al.'s (2003) object-oriented collaboration analysis framework (OCAF).

In these studies, the online environment is tailored to a specific problem-solving situation so that researchers can partially automate the coding process by narrowing the possibilities for user actions to a well-defined set of categories. The specificity of the problem-solving situation also allows researchers to produce models of idealized solution cases. Such ideal cases are then used as a baseline to make automated assessments of group work and learning outcomes.

(2) The *descriptive approach* informed by content analysis also involves categorization of user actions mediated by multimodal interaction spaces, applying a theoretically informed coding scheme. Categorized interaction logs are then subjected to statistical analysis to investigate various aspects of collaborative work such as:

(c) The correlation between planning moves performed in chat and the success of subsequent manipulations performed in a shared workspace (Jermann, 2002; Jermann & Dillenbourg, 2005),
(d) The relationship between grounding and problem-solving processes across multiple interaction spaces (Dillenbourg & Traum, 2006),
(e) A similar approach based on cultural-historical activity theory (Baker et al., 1999), and
(f) The referential uses of graphical representations in a shared workspace in the absence of explicit gestural deixis (Suthers, Girardeau, & Hundhausen, 2003).

These studies all focus on the group processes of collaboration, rather than treating it as a mere experimental condition for comparing the individuals in the groups. Also, they employ a content-coding approach to categorize actions occurring in multiple interaction spaces. In most cases, representational features like sentence openers or nodes corresponding to specific ontological entities are implemented in the interface to guide/constrain the possibilities for interaction. Such features are also used to aid the categorization of user actions. The categorization schemes are applied to recorded logs and subjected to statistical analysis to elicit interaction patterns.

The analytic thrust of these studies is to arrive at quantitative results through statistical comparisons of aggregated data. To accomplish this, they generally have to restrict student actions in order to control variables in their studies and to facilitate the coding of student utterances within a fixed ontology. We fear that this unduly restricts the interaction, which must be flexible enough to allow students to invent unanticipated behaviors. The restrictions of laboratory settings make problematic experimental validity and generalization of results to real-world contexts. Even more seriously, the aggregation of data—grouping utterances by types or codes rather than maintaining their sequentiality—ignores the complexity of the relations among the utterances and actions. According to our analysis, the temporal and semiotic relations are essential to understanding, sharing, and coordinating meaning, problem solving, and cognition. While quantitative approaches can be effective in testing model-based hypotheses, they seem less appropriate both for exploring the problem of interactional organization and for investigating interactional methods, which we feel are central to CSCL theory.

Despite the accomplishments of these studies, we find that their approaches introduce systematic limitations. Interactional analysis is impossible because coherent excerpts from recorded interactions are excluded from the analysis itself. (Excerpts are only used anecdotally, outside of the analysis, to introduce the features of the system to the reader, to illustrate the categorization schemes employed, or to motivate speculative discussion). Moreover, most studies like these involve dyads working on specific problem-solving contexts through highly structured interfaces in controlled lab studies in an effort to manage the complexity of collaboration. The meanings attributed by the researchers to such features of the interface need to be discovered/unpacked by the participants as they put them into use in interaction—and this critical process is necessarily ignored by the methodology. Finally, most of these papers are informed by the psycholinguistic theory of common ground, and are unable to critique it systematically. By contrast—as we shall see in the following section—our analysis of the joint organization of interaction in the case study positions us to understand how the group grounds its shared understanding in interactional terms at the group level.

Grounding through Interactional Organization

The coordination of visual and linguistic methods (across the whiteboard and chat workspaces) plays an important role in the establishment of common ground through the co-construction of references between items in the different media within the VMT environment. Particularly in mathematics—with its geometric/algebraic dual nature—symbolic terms are often grounded in visual presence and associated visual practices, such as counting or collecting multiple units into a single referent (Goodwin, 1994; Healy & Hoyles, 1999; Livingston, 2006; Sfard, 2008; Wittgenstein, 1944/1956). The visually present can be replaced by linguistic references to objects that are no longer in the visual field, but that can be understood based on prior experience supported by some mediating object such as a name—see the discussion of mediated memory and of the power of names in thought by Vygotsky (1930/1978; 1934/1986). A more extended analysis of the co-construction of mathematical artifacts by virtual math teams, the complementarity of their visual, semantic, and symbolic aspects, their reliance on pre-mathematical practices and processes of reification into concepts are beyond the scope of this paper and require comparison of multiple case studies (see Çakir, 2009). However, for this paper it is important to understand something of how the interactional organization that we have observed here functions to ground the group's understanding of their math object (the hexagonal array) as a shared group achievement.

As implied in the OCAF study (Avouris et al., 2003) mentioned in the previous section, investigating grounding and problem-solving processes in online dual-interaction environments like VMT requires close attention to the relationships among actions performed in multiple interaction spaces. Our case study illustrates some of the practical challenges involved with producing mathematical models that aim to exhaustively capture such relationships. For instance, the hexagonal array that was co-constructed by the team draws upon a triangular grid that is formed by three sets of parallel lines that intersect with each other in a particular way. In other words, these objects are layered on top of each other by the participants to produce a shape recognizable as a hexagon. Despite this combinatoric challenge, a modeling approach can still attempt to capture all possible geometric relationships among these graphical objects in a bottom-up fashion. However, when all chat messages referring to the whiteboard objects are added to the mix, the resulting model may obscure rather than reveal the details of the interactional organization through which group members discuss more complicated mathematical objects by treating a collection of atomic actions as a single entity. Terminology co-constructed in the chat-and-whiteboard environment—like "hexagonal array"—can refer to complexly defined math objects. What is interesting about the student knowledge building is how they aggregate elements and reify them into higher order, more powerful units (Sfard, 2008). A model should mirror this rather than to simply represent the elements as isolated.

The challenges involved with the modeling approach are not limited to finding efficient ways to capture all relationships among actions and identifying meaningful clusters of objects. The figurative uses of the graphical objects present the most daunting challenge for such an undertaking. For instance, the team members in our case study used the term "hexagonal array" to refer to a mathematical object implicated in the witnessed production of prior drawing actions. As we have seen in the way the team used this term during their session, "hexagonal array" does not simply refer to a readily available whiteboard illustration. Instead it is used as a *gloss* (Garfinkel & Sacks, 1970) to talk about an imagined pattern that grows infinitely and takes the shape illustrated on the whiteboard only at a particular stage. In the absence of a fixed set of ontological elements and constraints on types of actions a user can perform, modeling approaches that aim to capture emergent relationships among semiotic objects distributed across multiple interaction spaces need to adequately deal with the retrospective and prospective uses of language in interaction. Rather than relying upon a generic approach to modeling imposed by the researchers, our ethnographic approach aims to discover the unique "model"—or, better, the specific meaning—that was constructed *by the group* in its particular situation.

In another study discussed earlier, Dillenbourg & Traum (2006) offer the napkin and mockup models in their effort to characterize the relationship between whiteboard and chat spaces. In short, these models seem to describe two use scenarios where one interaction space is subordinated to the other during an entire problem-solving session. The complex relationships between the actions performed across both interaction spaces in our case made it difficult for us to describe the interactions we have observed by committing to only one of these models, as Dillenbourg & Traum did in their study. Instead, we have observed that in the context of an open-ended math task, groups may invoke either type of organization, depending upon the contingencies of their ongoing problem-solving work. For instance, during long episodes of drawing actions where a model of some aspect of the shared task is being co-constructed on the whiteboard (as in our first excerpt), the chat area often serves as an auxiliary medium to coordinate the drawing actions, which seems to conform to the mockup model. In contrast, when a strategy to address the shared task is being discussed in chat (as in the excerpt where the group considered splitting the hexagon into six regions), the whiteboard may be mainly used to quickly illustrate the textual descriptions with annotations or rough sketches, in accordance with the napkin model. Depending on the circumstances of ongoing interaction, participants may switch from one type of organization to another from moment to moment. Therefore, instead of ascribing mockup and napkin models to entire problem-solving sessions, we argue that it would be more fruitful to use these terms as glosses or descriptive categories for types of interactional organizations that group members may invoke during specific episodes of their interaction.

Another provocative observation made by Dillenbourg & Traum is that the whiteboard serves as a kind of shared external memory where group members keep a record of agreed-upon facts. In their study, the dyads were reported to post text notes on the whiteboard to keep track of the information they had discovered about a murder-mystery task. This seems to have led the authors to characterize the whiteboard as a placeholder and/or a shared working memory for the group, where agreed-upon facts or "contributions" in Clark's sense are persistently stored and spatially organized. As Dillenbourg & Traum observed, the scale of what is shared in the course of collaborative problem solving becomes an important issue when a theory operating at the utterance level like contribution theory (Clark & Marshall, 1981) is used as an analytic resource to study grounding processes that span a longer period of time. Dillenbourg & Traum seem to have used the notion of persistence to extend common ground across time to address this limitation. In particular, they argued that the whiteboard grounds the solution to the problem itself rather than the contributions made by each utterance. In other words, the whiteboard is metaphorically treated as a physical manifestation of the common ground. We certainly agree with this broadening of the conceptualization of common ground, although we do not see the whiteboard as just a metaphor or externalization of a mental phenomenon. Rather, *common ground is established in the discourse spaces* of text chat and graphical whiteboard. Their differential forms of persistence provide a continuing resource for sharing, modifying, and remembering the group meaning of joint artifacts and products of group cognition.

In our case study, we have observed that the whiteboard does not simply serve as a kind of shared external memory where the group keeps a record of agreed-upon facts, opinions, hypotheses, or conclusions. The shared visible communication media are places where the group does its work, where it cognizes. Ideas, concepts, meanings, and so forth can *subsequently* be taken up by individuals into their personal memories as resources for future social or mental interactions. There is no need to reduce group meaning to identical individual mental contents or to hypothesize a mysterious "group mind" as the location of common ground—the location is the discourse medium, with all its particular affordances and modes of access.

In our sessions, the whiteboard was primarily used to draw and annotate graphical illustrations of geometric shapes, although users occasionally posted textboxes on the whiteboard to note formulas they had found (see Figure 2 above). While the whiteboard mainly supported visual reasoning—and textual discussion or symbolic manipulation occurred chiefly in the chat stream—actions were carefully, systematically coordinated across the media and integrated within an interactionally organized group-cognitive process. As we have illustrated in our analysis, the fact that there were inscriptions posted on the whiteboard did not necessarily mean that all members immediately shared the same sense of those graphical objects. The group members did considerable interactional work to achieve a shared sense of those objects that was adequate for the purposes at hand. For instance, the crosshatched lines that Qwertyuiop originally drew became increasingly meaningful for the group as it was visually outlined and segmented and as it was discussed in the chat and expressed symbolically.

Hence, the whiteboard objects have a different epistemic status in our case study than in Dillenbourg & Traum's experiment. Moreover, the participants did not deem all the contents of the whiteboard relevant to the ongoing discussion. For instance, Figure 2 above shows a snapshot of the entire whiteboard as the team was discussing the hexagonal pattern problem. The figure shows that there are additional objects in the shared scene like a blue hypercube and a 3-D staircase, which are remnants of the group's prior problem-solving work. Finally, the sense of previously posted whiteboard objects may be modified or become evident as a result of current actions (Suchman, 1990).

In other words, group members can not only reuse or reproduce drawings, but they can also make subsequent sense of those drawings or discard the ones that are not deemed relevant anymore. Therefore, the technologically extended notion of common ground as a placeholder for a worked-out solution suffers from the same issues stated in Koschmann & LeBaron's (2003) critique of Clark's theory. As an abstract construct transcendental to the meaning-making practices of participants, the notion of common ground obscures rather than explains the ways the whiteboard is used as a resource for collaborative problem solving.

Instead of using an extended version of common ground as an analytical resource, we frame our analysis using the notion of "indexical ground of deictic reference," which is a notion we appropriated from linguistic anthropology (Hanks, 1992). In face-to-face interaction, human action is built through the sequential organization of not only talk but also coordinated use of the features of the local scene that are made relevant via bodily orientations, gesture, eye gaze, and so forth. In other words, "human action is built through simultaneous deployment of a range of quite different kinds of semiotic resources" (Goodwin, 2000, p. 1489). Indexical terms and referential deixis play a fundamental role in the way these semiotic resources are interwoven in interaction into a coherent whole.

Indexical terms are generally defined as expressions whose interpretation requires identification of some element of the context in which it was uttered, such as who made the utterance, to whom it was addressed, when and where the utterance was made (Levinson, 1983). Because the sense of indexical terms depends on the context in which they are uttered, indexicality is necessarily a relational phenomenon. Indexical references facilitate the mutually constitutive relationship between language and context (Hanks, 1996). The basic communicative function of indexical-referentials is "to individuate or single out objects of reference or address in terms of their relation to the current interactive context in which the utterance occurs" (Hanks, 1992, p. 47).

The specific sense of referential terms such as *this*, *that*, *now*, *here* is defined locally by interlocutors against a shared indexical ground. Conversely, the linguistic labels assigned to highlighted features of the local scene shapes the indexical ground. Hence, the indexical ground is not an abstract placeholder for a fixed set of registered contributions. Rather, it signifies an emergently coherent field of action that encodes an interactionally achieved set of background understandings, orientations, and perspectives that make references intelligible to interlocutors (Zemel et al., 2008).

Despite the limitations of online environments for supporting multimodality of embodied interaction, participants make substantial use of their everyday interactional competencies as they appropriate the features of such environments to engage with other users. For instance, Suthers et al.'s (2003) study reports that deictic uses of representational proxies play an important role in the interactional organization of online problem-solving sessions mediated by the Belvedere system. The authors report that participants in the online case devised mechanisms that compensate for the lack of gestural deixis with alternative means, such as using verbal deixis to refer to the most recently added text nodes and visual manipulation of nodes to direct their partner's attention to a particular node in the shared argument map.

In contrast to the Belvedere system, VMT offers participants additional resources such as an explicit referencing mechanism, a more generic workspace that allows producing and annotating drawings, and an awareness feature that produces a sense of sequentiality by embedding indicators for drawing actions in the sequence of chat postings. Our case study shows that despite the online situation's lack of the familiar resources of embodied interaction, team members can still achieve a sense of shared access to the meaningful objects displayed in the dual-interaction spaces of the VMT environment. Our analysis indicates that coherence among multiple modalities of an online environment like VMT is achieved through group members' development and application of shared methods for using the features of the system to coordinate their actions in the interface.

Through coordinated use of indexical-referential terms and highlighting actions, team members help each other to literally "see" the objects implicated in the shared visual field (Goodwin, 1994) and to encode them with locally specified terminology for subsequent use. They demonstrate how to "read" graphical as well as textual objects through the way the objects are built up sequentially and are spatially arranged in relation to each other through sequences of actions. The deictic references that link chat messages to features of graphical inscriptions and to prior chat content are instrumental in the sequential achievement of indexical symmetry, intersubjectivity, or common ground.

Sequential Analysis of the Joint Organization of Interaction

To sum up, the focus of our ethnomethodological inquiry is directed toward documenting how a virtual team achieved intersubjectivity and coherence among their actions in an online CSCL environment with multiple interaction spaces. We looked at the moment-to-moment details of the practices through which participants organize their chat utterances and whiteboard actions as a coherent whole in interaction—a process that is central to CSCL. We observed that referential practices enacted by the users are essential, particularly in the coordinated use of multimodalities afforded by environments like VMT. The referential uses of available features are instrumental not only in allocating other members' attention to specific parts of the interface where relevant actions are being performed, but also in the achievement of reciprocity (intersubjectivity, common ground, shared understanding, group cognition) among actions in the multiple interaction spaces, and hence, a sense of sequential organization across the spaces.

In our case study, we have seen the establishment of an indexical ground of deictic references co-constructed by the group members as an underlying support for the creation and maintenance of their joint problem space. We have seen that nexus of references created interactionally as group members propose, question, repair, respond, illustrate, make visible, supply symbols, name, and so forth. In the VMT dual-media environment, the differential persistence, visibility, and mutability of the media are consequential for the interaction. Group members develop methods of coordinating chat and drawing activities to combine visual and conceptual reasoning by the group and to co-construct and maintain an evolving shared indexical ground of their discourse.

In this paper, we have reconceptualized the problem of common ground from an issue of sharing mental representations to a practical matter of being able to jointly relate semiotic objects to their indexed referents. The references do not reside in the minds of particular actors, but have been crafted into the presentation of the chat postings and drawing inscriptions through the details of wording and sequential presentation. The references are present in the data as affordances for *understanding* by group participants as well as by analysts (Stahl, 2006, chap. 17). The *meaning* is there in the visual presentation of the communication objects and in the network of interrelated references (Stahl, 2007), rather than in mental re-presentations of them. The understanding of the references is a matter of normally tacit social practice, rather than of rationalist explicit deduction. The references can be explicated by analysis, but only if the structure of sequentiality and indexicality is preserved in the data analysis and only if the skill of situated human understanding is applied.

In our case study of an 18-minute excerpt taken from a four-hour group chat, three students construct a diagram of lines, triangles, and hexagons, propose a math pattern problem, analyze the structure of their diagram, and derive an algebraic formula to solve their problem. They propose their own creative problem about mathematical properties; gradually construct a complex mathematical object; explore related patterns with visual, narrative, and symbolic means; express wonder; gain mathematical insight; and appreciate their achievement. They do this by coordinating their whiteboard and chat activities in a synchronous online environment. Their accomplishment is precisely the kind of educational math experience recommended by mathematicians (Livingston, 2006; Lockhart, 2008; Moss & Beatty, 2006). It was not a mental achievement of an individual, but a group accomplishment carried out in computer-supported discourse. By analyzing the sequentiality and indexicality of their interactions, we explicated several mechanisms of the group cognition by which the students coordinated the group meaning of their discourse and maintained an effective joint problem space.

The coordination of visual and textual realizations of the mathematical objects that the students co-construct provides a grounding of the algebraic formulas the students jointly derive using the line drawings that they inspect visually together. As the students individualize this experience of group cognition, they can develop the deep understanding of mathematical phenomena that comes from seeing the connections among multiple realizations (Sfard, 2008; Stahl, 2008). Our case study does not by any means predict that all students can accomplish similar results under specific conditions, but merely demonstrates that such group cognition is possible within a synchronous CSCL setting and that a fine-grained sequential analysis of interaction can study how it is collaboratively accomplished.

Acknowledgment The reviews coordinated by Dan Suthers helped us to structure this paper more clearly. Some of the larger methodological, technological, and pedagogical issues the reviewers raised are addressed in (Stahl, 2009b), which lists the VMT research team members. This paper is a result of the team's group cognition. Access to the complete data using the VMT Replayer is available by emailing the authors.

References

Avouris, N., Dimitracopoulou, A., & Komis, V. (2003). On analysis of collaborative problem solving: An object-oriented approach. *Computers in Human Behavior, 19*, 147–167.

Baker, M., Hansen, T., Joiner, R., & Traum, D. (1999). The role of grounding in collaborative learning tasks. In P. Dillenbourg (Ed.), *Collaborative learning: Cognitive and computational approaches* (pp. 31–63). Oxford, UK: Pergamon.

Çakir, M. P. (2009). *How online small groups co-construct mathematical artifacts to do collaborative problem solving.* Unpublished Dissertation, Ph.D., College of Information Science and Technology, Drexel University, Philadelphia, PA, USA.

Clark, H., & Brennan, S. (1991). Grounding in communication. In L. Resnick, J. Levine, & S. Teasley (Eds.), *Perspectives on socially-shared cognition* (pp. 127–149). Washington, DC: APA.

Clark, H. H., & Marshall, C. (1981). Definite reference and mutual knowledge. In A. K. Joshi, B. Weber, & I. A. Sag (Eds.), *Elements of discourse understanding* (pp. 10–63). New York, NY: Cambridge University Press.

Coulon, A. (1995). *Ethnomethodology*. Thousand Oaks, CA: Sage.

Dillenbourg, P. (2005). Dual-interaction spaces. In T. Koschmann, D. D. Suthers, & T.-W. Chan (Eds.), *Computer-supported collaborative learning 2005: The next ten years! (Proceedings of CSCL 2005).* Taipei, Taiwan: Mahwah, NJ: Lawrence Erlbaum Associates.

Dillenbourg, P., Baker, M., Blaye, A., & O'Malley, C. (1996). The evolution of research on collaborative learning. In P. Reimann, & H. Spada (Eds.), *Learning in humans and machines: Towards an interdisciplinary learning science* (pp. 189–211). Oxford, UK: Elsevier.

Dillenbourg, P., & Traum, D. (2006). Sharing solutions: Persistence and grounding in multimodal collaborative problem solving. *Journal of the Learning Sciences, 15*(1), 121–151.

Garcia, A., & Jacobs, J. B. (1998). The interactional organization of computer mediated communication in the college classroom. *Qualitative Sociology, 21*(3), 299–317.

Garcia, A., & Jacobs, J. B. (1999). The eyes of the beholder: Understanding the turn-taking system in quasi-synchronous computer-mediated communication. *Research on Language and Social Interaction, 34*(4), 337–367.

Garfinkel, H., & Sacks, H. (1970). On formal structures of practical actions. In J. Mckinney, & E. Tiryakian (Eds.), *Theoretical sociology: Perspectives and developments* (pp. 337–366). New York, NY: Appleton-Century-Crofts.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*(3), 606–633.

Goodwin, C. (2000). Action and embodiment within situated human interaction. *Journal of Pragmatics, 32*, 1489–1522.

Goodwin, C., & Heritage, J. (1990). Conversation analysis. *Annual Review of Anthropology, 19*, 283–307.
Hanks, W. (1992). The indexical ground of deictic reference. In C. Goodwin, & A. Duranti (Eds.), *Rethinking context: Language as an interactive phenomenon*. Cambridge, UK: Cambridge University Press.
Hanks, W. (1996). *Language and communicative practices*. Boulder, CO: Westview.
Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers. *Mathematical Thinking and Learning, 1*(1), 59–84.
Heritage, J. (1984). *Garfinkel and ethnomethodology*. Cambridge, UK: Polity Press.
Hutchins, E. (1996). *Cognition in the wild*. Cambridge, MA: MIT Press.
Jermann, P. (2002). *Task and interaction regulation in controlling a traffic simulation.* Paper presented at the Computer support for collaborative learning: Foundations for a CSCL community. Proceedings of CSCL 2002, Boulder, CO. Proceedings pp. 601–602.
Jermann, P., & Dillenbourg, P. (2005). Planning congruence in dual spaces. In T. Koschmann, D. D. Suthers, & T.-W. Chan (Eds.), *Computer-supported collaborative learning 2005: The next ten years! (Proceedings of CSCL 2005).* Taipei, Taiwan: Mahwah, NJ: Lawrence Erlbaum Associates.
Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *Journal of the Learning Sciences, 4*(1), 39–103. Retrieved from http://lrs.ed.uiuc.edu/students/c-merkel/document4.HTM.
Koschmann, T. (2002). Dewey's contribution to the foundations of CSCL research. In G. Stahl (Ed.), *Computer support for collaborative learning: Foundations for a CSCL community: Proceedings of CSCL 2002* (pp. 17–22). Boulder, CO: Lawrence Erlbaum Associates.
Koschmann, T., & LeBaron, C. (2003). *Reconsidering common ground: Examining clark's contribution theory in the operating room.* Paper presented at the European Computer-Supported Cooperative Work (ECSCW '03), Helsinki, Finland. Proceedings pp. 81–98.
Koschmann, T., LeBaron, C., Goodwin, C., & Feltovich, P. J. (2001). Dissecting common ground: Examining an instance of reference repair. In J. D. Moore, & K. Stenning (Eds.), *Proceedings of the twenty-third annual conference of the cognitive science society* (pp. 516–521). Mahwah, NJ: Lawrence Erlbaum Associates.
Koschmann, T., Stahl, G., & Zemel, A. (2007). The video analyst's manifesto (or the implications of Garfinkel's policies for the development of a program of video analytic research within the learning sciences). In R. Goldman, R. Pea, B. Barron, & S. Derry (Eds.), *Video research in the learning sciences* (pp. 133–144). Mahway, NJ: Lawrence Erlbaum Associates. Retrieved from http://GerryStahl.net/publications/journals/manifesto.pdf.
Levinson, S. (1983). *Pragmatics*. Cambridge, UK: Cambridge University Press.
Livingston, E. (2006). Ethnomethodological studies of mediated interaction and mundane expertise. *The Sociological Review, 54*(3).

Lockhart, P. (2008). Lockhart's lament. *MAA Online, 2008*(March). Retrieved from http://www.maa.org/devlin/devlin_03_08.html.

Moss, J., & Beatty, R. A. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 1*(4), 441–466.

Mühlpfordt, M., & Stahl, G. (2007). The integration of synchronous communication across dual interaction spaces. In C. Chinn, G. Erkens, & S. Puntambekar (Eds.), *The proceedings of CSCL 2007: Of mice, minds, and society (CSCL 2007)*. New Brunswick, NJ. Retrieved from http://GerryStahl.net/vmtwiki/martin.pdf.

Mühlpfordt, M., & Wessner, M. (2005). Explicit referencing in chat supports collaborative learning. In T. Koschmann, D. D. Suthers, & T.-W. Chan (Eds.), *Computer-supported collaborative learning 2005: The next ten years! (Proceedings of CSCL 2005)* (pp. 460–469). Taipei, Taiwan: Mahwah, NJ: Lawrence Erlbaum Associates.

O'Neill, J., & Martin, D. (2003). *Text chat in action*. Paper presented at the ACM Conference on Groupware (GROUP 2003), Sanibel Island, FL.

Psathas, G. (1995). *Conversation analysis: The study of talk-in-interaction*. Thousand Oaks, CA: Sage.

Roschelle, J., & Teasley, S. (1995). The construction of shared knowledge in collaborative problem solving. In C. O'Malley (Ed.), *Computer-supported collaborative learning* (pp. 69–197). Berlin, Germany: Springer Verlag.

Sacks, H. (1962/1995). *Lectures on conversation*. Oxford, UK: Blackwell.

Sarmiento, J., & Stahl, G. (2008). *Extending the joint problem space: Time and sequence as essential features of knowledge building*. Paper presented at the International Conference of the Learning Sciences (ICLS 2008), Utrecht, Netherlands. Retrieved from http://GerryStahl.net/pub/icls2008johann.pdf.

Sarmiento-Klapper, J. W. (2009). *Bridging mechanisms in team-based online problem solving: continuity in building collaborative knowledge*. Unpublished Dissertation, Ph.D., College of Information Science and Technology, Drexel University, Philadelphia, PA, USA.

Scardamalia, M., & Bereiter, C. (1996). Computer support for knowledge-building communities. In T. Koschmann (Ed.), *CSCL: Theory and practice of an emerging paradigm* (pp. 249–268). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schegloff, E., & Sacks, H. (1973). Opening up closings. *Semiotica, 8*, 289–327.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge, UK: Cambridge University Press.

Soller, A., & Lesgold, A. (2003). *A computational approach to analyzing online knowledge sharing interaction*. Paper presented at the 11th International Conference on Artificial Intelligence in Education, AI-ED 2003, Sydney, Australia. Proceedings pp. 253–260: Amsterdam: IOS Press.

Stahl, G. (2006). *Group cognition: Computer support for building collaborative knowledge*. Cambridge, MA: MIT Press. Retrieved from http://GerryStahl.net/mit/.

Stahl, G. (2007). *Meaning making in CSCL: Conditions and preconditions for cognitive processes by groups.* Paper presented at the international conference on Computer-Supported Collaborative Learning (CSCL '07), New Brunswick, NJ: ISLS. Retrieved from http://GerryStahl.net/pub/cscl07.pdf.

Stahl, G. (2008). Book review: Exploring thinking as communicating in CSCL. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 3*(3), 361–368.

Stahl, G. (2009a). *For a science of group interaction.* Paper presented at the GROUP 2009, Sanibel Island, FL.

Stahl, G. (Ed.). (2009b). *Studying virtual math teams.* New York, NY: Springer. Computer-supported collaborative learning book series, vol 11 Retrieved from http://GerryStahl.net/vmt/book.

Stahl, G., Koschmann, T., & Suthers, D. (2006). Computer-supported collaborative learning: An historical perspective. In R. K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 409–426). Cambridge, UK: Cambridge University Press. Retrieved from http://GerryStahl.net/cscl/CSCL_English.pdf in English, http://GerryStahl.net/cscl/CSCL_Chinese_simplified.pdf in simplified Chinese, http://GerryStahl.net/cscl/CSCL_Chinese_traditional.pdf in traditional Chinese, http://GerryStahl.net/cscl/CSCL_Spanish.pdf in Spanish, http://GerryStahl.net/cscl/CSCL_Portuguese.pdf in Portuguese, http://GerryStahl.net/cscl/CSCL_German.pdf in German, http://GerryStahl.net/cscl/CSCL_Romanian.pdf in Romanian.

Streeck, J., & Mehus, S. (2003). Microethnography: The study of practices. In K. F. R. Sanders (Ed.), *Handbook of language and social interaction.* Mahway, NJ: Erlbaum Associates.

Suchman, L. A. (1990). Representing practice in cognitive science. In M. Lynch, Woolgar, S. (Ed.), *Representation in scientific practice.* Cambridge, MA: MIT Press.

Suthers, D., Connelly, J., Lesgold, A., Paolucci, M., Toth, E., Toth, J., et al. (2001). Representational and advisory guidance for students learning scientific inquiry. In K. D. Forbus, & P. J. Feltovich (Eds.), *Smart machines in education: The coming revolution in educational technology* (pp. 7–35). Menlo Park: AAAI Press.

Suthers, D., Girardeau, L., & Hundhausen, C. (2003). Deictic roles of external representations in face-to-face and online collaboration. In B. Wasson, S. Ludvigsen, & U. Hoppe (Eds.), *Designing for change in networked learning environments, Proceedings of the international conference on computer support for collaborative learning 2003* (pp. 173–182). Dordrecht: Kluwer Academic Publishers.

Teasley, S. D., & Roschelle, J. (1993). Constructing a joint problem space: The computer as a tool for sharing knowledge. In S. P. Lajoie, & S. J. Derry (Eds.), *Computers as cognitive tools* (pp. 229–258). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

ten Have, P. (1999). *Doing conversation analysis: A practical guide.* Thousand Oaks, CA: Sage.

Vygotsky, L. (1930/1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Vygotsky, L. (1934/1986). *Thought and language*. Cambridge, MA: MIT Press.
Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Associates.
Wittgenstein, L. (1944/1956). *Remarks on the foundations of mathematics*. Cambridge, MA: MIT Press.
Zemel, A., Koschmann, T., LeBaron, C., & Feltovich, F. (2008). "What are we missing?" Usability's indexical ground. *Computer Supported Cooperative Work, 17*, 63–85.

3. The Integration of Mathematics Discourse, Graphical Reasoning and Symbolic Expression by a Virtual Math Team

Murat Perit Çakır and Gerry Stahl

Abstract: Learning mathematics involves mastering specific forms of social practice. In this chapter, we describe socially situated, interactional processes involved with collaborative learning of mathematics online. We provide a group-cognitive account of mathematical understanding in an empirical case study of an online collaborative learning environment called Virtual Math Teams. The chapter looks closely at how an online small group of mathematics students coordinates their collaborative problem solving using chat, shared drawings and mathematics symbols. Our analysis highlights the methodic ways group members enact the affordances of their situation (a) to display their reasoning to each other by co-constructing shared mathematical artifacts and (b) to coordinate their actions across multiple interaction spaces to relate their narrative, graphical and symbolic contributions while they are working on open-ended mathematics problems. In particular, we identify key roles of referential and representational practices in the co-construction of deep mathematical understanding at the group level, which is achieved through methodic uses of the environment's features to coordinate narrative, graphical and symbolic resources.

Mathematical Practices

Developing pedagogies and instructional tools to support learning mathematics with understanding is a major goal in Mathematics Education (CCSSI, 2011; NCTM, 2000). A common theme among various characterizations of mathematical understanding in the mathematics education literature involves *constructing relationships* among mathematical facts and procedures (Hiebert & Wearne, 1996). In particular, recognition of connections among *multiple realizations* of a mathematics concept encapsulated in various inscriptional forms is considered as evidence of deep understanding of that subject matter (Kaput, 1998; Sfard, 2008; Healy & Hoyles, 1999). For instance, the concept of function in the modern mathematics curriculum is introduced through its graphical, narrative, tabular and symbolic realizations. Hence, a deep understanding of the function concept is ascribed to a learner to the extent he/she can demonstrate how seemingly different graphical, narrative and symbolic forms are interrelated as realizations of each other in specific problem-solving circumstances that require the use of functions. On the other hand, students who demonstrate difficulties in realizing such connections are considered to perceive actions associated with distinct forms as isolated sets of skills, and hence are said to have a shallow understanding of the subject matter (Carpenter & Lehrer, 1999).

Reflecting on one's own actions and *communicating*/articulating mathematical rationale are considered as important activities through which students realize connections among seemingly isolated facts and procedures in mathematics education theory (Sfard, 2002; Hiebert et al., 1996). Such activities are claimed to help learners notice broader structural links among underlying concepts, reorganize their thoughts around these structures, and hence develop their understanding of mathematics (Carpenter & Lehrer, 1999; Skemp, 1976). Consequently, collaborative learning in peer-group settings is receiving increasing interest in mathematics education practice due to its potential for promoting student participation and creating a natural setting where students can explain their reasoning and benefit from each others' perspectives (Barron, 2003).

Representational capabilities offered by Information and Communication Technologies (ICT) provide important affordances for exploring connections among different realizations of mathematical objects. Dynamic geometry applications like Cabri, Geometer's Sketchpad, GeoGebra (Goldenberg & Cuoco, 1998); algebra applications such as Casyospee (Lagrange, 2005), or statistical modeling and exploratory data analysis tools like TinkerPlots (Konold, 2007) provide representational capabilities and virtual manipulatives that surpass what can be done with conventional methods of producing mathematical inscriptions in the classroom (Olive, 1998). In addition to this, widespread popularity of social networking and instant messaging technologies among the so-called Net Generation requires designers of educational technology to think about innovative ways for engaging the new generation of students with mathematical activity (Lenhart et al., 2007). Therefore, bringing the representational capabilities of existing mathematical packages together with communicational affordances of social-networking/messenger software can potentially support the kinds of interactions that foster deeper understanding of mathematics. Computer-Supported Collaborative Learning (CSCL) is a research paradigm in the field of

Instructional Technology that investigates how such opportunities can be realized through carefully designed learning environments that support collective meaning-making practices in computer-mediated settings (Stahl, Koschmann, & Suthers, 2006).

Multimodal interaction spaces—which typically bring together two or more synchronous online communication technologies such as text-chat and a shared graphical workspace—have been widely employed in CSCL research and in commercial collaboration suites such as Elluminate and Blackboard-Wimba to support collaborative-learning activities of small groups online (Dillenbourg & Traum, 2006; Suthers et al., 2001). The way such systems are designed as a juxtaposition of several technologically independent online communication tools not only brings various *affordances* (i.e., possibilities for and/or constraints on actions), but also carries important interactional consequences for the users (Cakir, Zemel & Stahl, 2009; Suthers, 2006; Dohn 2009). Providing access to a rich set of modalities for action allows users to demonstrate their reasoning in multiple semiotic forms. However, the achievement of connections that foster the kind of mathematical understanding desired by mathematics educators is conditioned upon team members' success in devising shared methods for coordinated use of these resources (Mühlpfordt & Stahl, 2007).

Although CSCL environments with multimodal interaction spaces offer rich possibilities for the creation, manipulation, and sharing of mathematical artifacts online, the *interactional organization of mathematical meaning-making activities* in such online environments is a relatively unexplored area in CSCL and in mathematics education. In an effort to address this gap, we have designed an online environment with multiple interaction spaces called Virtual Math Teams (VMT), which allows users to exchange textual postings as well as share graphical contributions online (Stahl, 2009). The VMT environment also provides additional resources, such as explicit referencing and special awareness markers, to help users coordinate their actions across multiple spaces. Of special interest to researchers, this environment includes a Replayer tool to replay a chat session as it unfolded in real time and inspect how students organize their joint activity to achieve the kinds of connections indicative of deep understanding of mathematics (Stahl, 2011).

In this chapter we focus on the interactional practices through which VMT participants achieve the kinds of connections across multiple semiotic modalities that are indicative of deep mathematical understanding. In particular, the chapter will look closely at how an online small group of mathematics students coordinated their collaborative problem solving using digital text, drawings and symbols. We take the mathematics-education practitioners' account of what constitutes deep learning of mathematics as a starting point, but instead of treating understanding as a mental state of the individual learner that is typically inferred by outcome measures, we argue that deep mathematical understanding can be located in the practices of collective multimodal reasoning displayed by groups of students through the sequential and spatial organization of their actions (Stahl, 2006). In an effort to study the practices of multimodal reasoning online, we employ an ethnomethodological case-study approach and investigate the methods through which small groups of students achieve *joint attention* to particular mathematical features of their representations in order to ground their co-construction of shared mathematical meaning (Sarmiento & Stahl, 2008, Stahl, et al., 2011). Our analysis of the excerpts presented below has identified key roles of *referential and representational practices* in the co-construction of deep mathematical understanding at the group level, which is elaborated further in the discussion section.

Data Collection & Methodology

The excerpts analyzed in this chapter are obtained from a problem-solving session of a team of three upper-middle-school students who participated in the VMT Spring Fest 2006. This event brought together several teams from the US, Singapore and Scotland to collaborate on an open-ended mathematics task on combinatorial patterns. Students were recruited anonymously through their teachers. Members of the teams generally did not know each other before the first session. Neither they nor we knew anything about each other (e.g., age or gender) except chat screen names and information that may have been communicated during the sessions. Each group participated in four sessions during a two-week period, and each session lasted over an hour. Each session was moderated by a Math Forum member; the facilitators' task was to help the teams when they experienced technical difficulties, not to participate in the problem-solving work.

During their first session, all the teams were asked to work on a particular pattern of squares made up of sticks (see Figure 1). For the remaining three sessions the teams were asked to come up with their own stick patterns, describe the patterns they observed as mathematical formulae, and share their observations with other teams through a wiki page.

This task was chosen because of the possibilities it afforded for many different solution approaches ranging from simple counting procedures to more advanced methods, such as the use of recursive functions and exploring the arithmetic properties of various number sequences. Moreover, the task had both algebraic and geometric aspects, which would potentially allow us to observe how participants put many features of the VMT software system into use. The open-ended nature of the activity stemmed from the need to agree upon a new shape made by sticks. This required groups to engage in a different kind of problem-solving activity as compared to traditional situations where questions are given in advance and there is a single "correct" answer—presumably already known by a teacher. We used a traditional problem to seed the activity and then left it up to each group to decide the kinds of shapes they found interesting and worth exploring further (Moss & Beatty, 2006; Watson & Mason, 2005).

(1) 4 sticks, 1 square

(2) 10 sticks, 3 squares

(3) 18 sticks, 6 squares

N	Sticks	Squares
1	4	1
2	10	3
3	18	6
4	?	?
5	?	?
6	?	?
...	...	...
N	?	?

Session I

1. Draw the pattern for N=4, N=5, and N=6 in the whiteboard. Discuss as a group: How does the graphic pattern grow?
2. Fill in the cells of the table for sticks and squares in rows N=4, N=5, and N=6. Once you agree on these results, post them on the VMT Wiki
3. Can your group see a pattern of growth for the number of sticks and squares? When you are ready, post your ideas about the pattern of growth on the VMT Wiki.

Sessions II and III

1. Discuss the feedback that you received about your previous session.
2. **WHAT IF?** Mathematicians do not just solve other people's problems - they also explore little worlds of patterns that they define and find interesting. Think about other mathematical problems related to the problem with the sticks. For instance, consider other arrangements of squares in addition to the triangle arrangement (diamond, cross, etc.). **What if** instead of squares you use other polygons like triangles, hexagons, etc.? Which polygons work well for building patterns like this? How about 3-D figures, like cubes with edges, sides and cubes? What are the different methods (induction, series, recursion, graphing, tables, etc.) you can use to analyze these different patterns?

3. Go to the VMT Wiki and share the most interesting math problems that your group chose to work on.

Figure 1: Task description for VMT Spring Fest 2006

The VMT system that hosted these sessions has two main interactive components that conform to the typical layout of systems with dual-interaction spaces: a shared whiteboard that provides basic drawing features on the left and a chat window on the right. The online environment has features to help users relate the actions happening across dual-interaction spaces. One of the unique features of the VMT chat system is the *referencing* support mechanism (Mühlpfordt & Stahl, 2007), which allows users to visually connect their chat postings to previous postings or to objects on the whiteboard via arrows (e.g., Figure 7 below illustrates a *message-to-message* reference, whereas Figure 6 shows a *message-to-whiteboard* reference). The referential links attached to a message are displayed until a new message is posted. Messages including referential links are marked with an arrow icon in the chat window. A user can see where such a message is pointing at any time by clicking on it.

Studying the collective meaning-making practices enacted by the users of CSCL systems requires a close analysis of the process of collaboration itself (Stahl, Koschmann & Suthers, 2006; Koschmann, Stahl & Zemel, 2007). In an effort to investigate the organization of interactions across the dual-interaction spaces of the VMT environment, we consider the *small group* as the unit of analysis (Stahl, 2006), and we appropriate methods of Ethnomethodology (EM) (Garfinkel, 1967; Livingston, 1987) and Conversation Analysis (CA) (Sacks, 1962/1995; ten Have, 1999) to conduct case studies of online group interaction. Our work is informed by EM/CA studies of interaction mediated by online text-chat (Garcia & Jacobs, 1998; O'Neill & Martin, 2003), although the availability of a shared drawing area and explicit support for deictic references in our online environment, as well as our focus on mathematical practice significantly differentiate our study from theirs.

The goal of ethnomethodological conversation analysis is to describe the commonsense understandings and procedures group members use to organize their conduct in particular interactional settings. Commonsense understandings and procedures are subjected to analytical scrutiny because they "enable actors to recognize and act on their real world circumstances, grasp the intentions and motivations of others, and achieve mutual understandings" (Goodwin & Heritage, 1990, p. 285). Group members' shared competencies in organizing their conduct not only allow them to produce their own actions, but also to interpret the actions of others (Garfinkel & Sacks, 1970). Since members enact these understandings and/or procedures in their visually displayed situated actions, researchers can discover them through detailed analysis of members' sequentially organized conduct (Schegloff & Sacks, 1973).

We conducted numerous VMT Project data sessions, where we subjected our analysis of VMT data to intersubjective agreement by conducting CA data sessions (Jordan & Henderson, 1995; ten Have, 1999). During the data sessions we used the VMT Replayer tool, which allows us to replay a VMT chat session as it unfolded in real time based on the timestamps of actions recorded in the log file. The order of actions—chat postings, whiteboard actions, awareness messages—we observe with the Replayer as researchers exactly matches the order of actions originally observed by the users. This property of the Replayer allowed us to study the sequential unfolding of events during the entire chat session. In short, the VMT environment provided us as researchers a perspicuous setting in which the mathematical meaning-making process is *made visible* as it is "observably and accountably embedded in collaborative activity" (Koschmann, 2001, p. 19).

Setting Up the Mathematical Analysis

In the following excerpts we will observe a team of three upper-middle-school students in action, who used "Qwertyuiop", "137" and "Jason" as login screen names. Prior to the session containing these excerpts, this team completed two chat sessions where they explored similar stick patterns together. In the current session, team members will be working on a new stick pattern that they co-constructed and named as the "hexagonal pattern", whose first three stages are illustrated in Figure 2. Details of this co-construction process was analyzed and published elsewhere (Cakir, Zemel & Stahl, 2009; Cakir, 2009), so we will skip the part where the group constituted this pattern as a shared problem and figured out a method to count the number of *triangles* enclosed in its n^{th} stage. In the excerpts presented below, the team will be working on devising a formula for characterizing the number of *sticks* that will be needed to construct the hexagonal pattern in general (i.e., in its n^{th} stage). Our main analytic goal is to identify the practices or group methods team members enacted to achieve a shared understanding of the problem at hand by co-constructing and acting on the mathematical artifacts in graphical, narrative and symbolic forms.

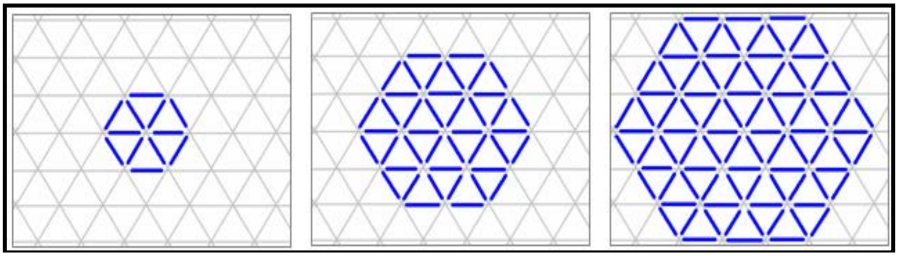


Figure 2: Hexagonal stick pattern co-constructed by this team

Excerpt 1: Constitution of a New Math Task

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
742	19:24:39	19:25:48	Qwertyuiop	an idea: Find the number of a certain set of colinear sides (there are 3 sets) and multiply the result by 3	
743	19:25:55	19:26:03	Jason	i did--apparently it didn't work for him	Message No. 740
744	19:26:05	19:26:13	Jason	or his internet could be down, as he's not even on IM right now	
745	19:26:10	19:26:13	Nan	i see. thanks!	Message No. 743
		19:26:23 - 19:26:33		***137 produces two green lines on the diagonals of the hexagon and two green arrows as displayed in Figure 3***	
746	19:26:20	19:26:36	137	As in those?	Message No. 742
747	19:26:46	19:27:05	Qwertyuiop	no-in one triangle. I'll draw it...	Message No. 746
		19:27:10 - 19:28:08		***Qwertyuiop repositions some of the existing green lines on a particular section of the hexagon (see Figure 4 below)***	
748	19:28:09	19:28:10	Qwertyuiop	Those	
		19:28:13 - 19:28:19		***137 makes the green lines thicker (see Figure 4 below)***	
749		19:28:28	Qwertyuiop	find those, and then multiply by 3	
750	19:28:48	19:28:50	137	The rows?	
751	19:29:01	19:30:01	Qwertyuiop	The green lines are all colinear. There are 3 identical sets of colinear lines in that	

				triangle. Find the number of sides in one set, then multiply by 3 for all the other sets.	
752	19:30:20	19:30:23	137	Ah. I see.	

This excerpt illustrates a number of rich referencing methods: special terms, graphical practices, VMT tools, etc. Excerpt 1[5] opens with Qwertyuiop's announcement of "**an idea**"[6] in line 742. He suggests the team find the number of a set of objects he calls "**collinear sides**" and multiply that number by 3. The statement in parenthesis elaborates further that there are 3 such sets. The use of the term "**sides**" makes it evident that this statement is about the problem of finding the number of sticks to construct a given stage, rather than the problem of finding the number of triangles that make up a hexagon that has been recently discussed by the team[7]. Thus, Qwertyuiop seems to be proposing to his teammates a way to approach the problem of counting the number of sticks needed to construct the hexagonal shape in general.

[5] The referential links used by the students to connect their messages to previous messages are displayed in the right-most column in the excerpts. For instance, line 745 includes Message #742 in the right-most column. This indicates that message 745 was linked to 742 by its contributor (i.e. Nan in this case). References to whiteboard objects are also marked in this column. Whiteboard drawing actions are described in bold-italics to separate them from chat messages. Note that chat postings and whiteboard drawings often interleave each other.

[6] Phrases quoted from chat messages are printed in bold to highlight the terms used by the participants.

[7] There is a parallel conversation unfolding in chat at this moment between the facilitator (Nan) and Jason about an administrative matter. Lines 740, 743, 744, and 745 are omitted from the analysis to keep the focus on the math problem solving.

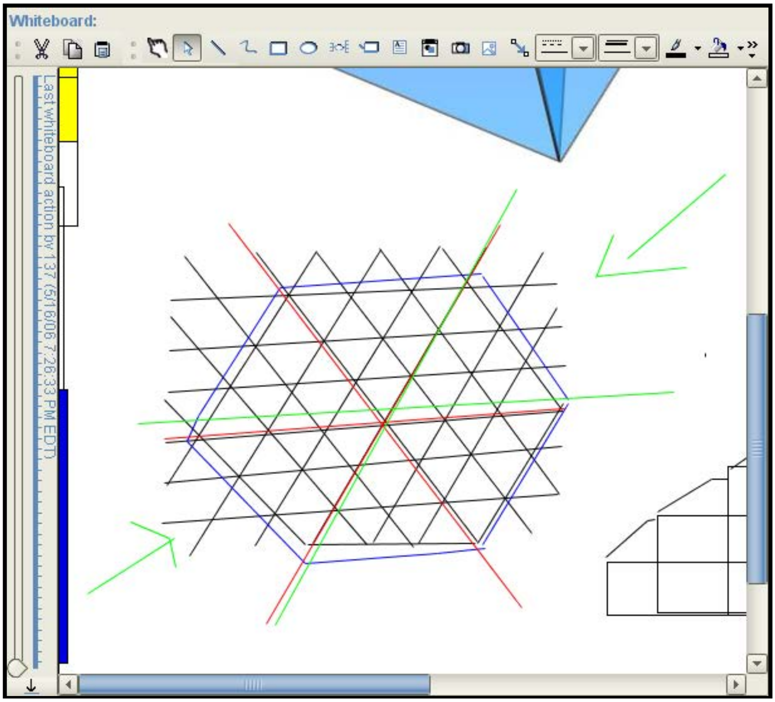


Figure 3: Green lines and arrows produced by 137.

A minute after this posting, 137 begins to type at 19:26:20. While the awareness marker continues to display that 137 is typing, he adds two green lines on the hexagon that intersect each other and two green arrows (see Figure 3). The arrows are positioned outside the hexagon and their tips are mutually pointing at each other through a projected diagonal axis. Shortly after his last drawing move, 137 completes his typing action by posting the message "**as in those?**" in line 746, which is explicitly linked to Qwertyuiop's previous posting with a referential arrow. The plural[8] deictic term "**those**" in this posting instructs others to attend to some objects beyond the chat statement itself, possibly located in the other interaction space. The way the drawing actions are embedded as part of the typing activity suggests that they are designed to be seen as part of a single turn or exposition. Hence, the deictic term

[8] 137's referential work involves multiple objects in this instance. Although the referencing tool of VMT can be used to highlight more than one area on the whiteboard, this possibility was not mentioned during the tutorial and hence was not available to the users. Although the explicit referencing tool of the system seemed to be inadequate to fulfill this complicated referential move, 137 achieves a similar referential display by temporally coordinating his moves across both interaction spaces and by using the plural deictic term "those" to index his recent moves.

"**those**" can be read as a reference to the objects pointed to by the recently added green arrows[9]. Moreover, the use of the term "**as**" and the referential link together suggest that these drawings are related to Qwertyuiop's proposal in line 746. Therefore, based on the evidence listed above, 137 proposes a provisional *graphical representation* of what was described in narrative form by Qwertyuiop earlier and calls for an assessment of its adequacy.

In line 747 Qwertyuiop posts a message linked to 137's message with the referential arrow, which indicates that he is responding to 137's recent proposal. The use of "**no**" at the beginning expresses disagreement and the following phrase "**in one triangle**" gives further specificity to where the relevant relationship should be located. The next sentence "**I will draw it…**" in the same posting informs other members that he will continue his elaboration on the whiteboard. The use of ellipsis "**...**" also marks the incomplete status of this posting, which informs others that his subsequent drawings should be seen as related to this thread.

Following this line, Qwertyuiop begins to *reposition* some of the green lines that 137 drew earlier. He forms three green horizontal lines within one of the six triangular partitions (see the snapshot on the left in Figure 4). Then in line 748, he posts the deictic term "**those**" that can be read as a reference to the recently added lines. Immediately following Qwertyuiop's statement, 137 modifies the recently added lines by increasing their *thickness* (see the snapshot on the right in Figure 4). These moves make the new lines more visible. In line 749, Qwertyuiop continues his exposition by stating that what has been marked (indexed by "**those**") is what needs to be found and then multiplied by 3.

137's posting "**the rows?**" follows shortly after in line 750. The term "**rows**" has been previously used by this team to describe a method to systematically count the triangles located in one of the 6 regions of the hexagonal array earlier. By invoking this term here again, 137's posting proposes a relationship between what is highlighted on the drawing and a term the team has previously used to articulate a method of counting. The question mark appended invites others to make an assessment of the inferred relationship.

[9] We have observed that students use "those" (or "that") in chat to reference items already existing in the whiteboard, but "these" (or "this") to reference items that they are about to add to the whiteboard.

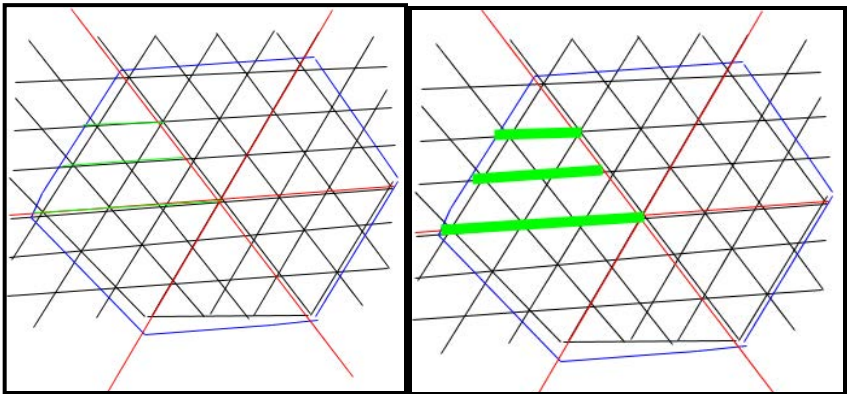


Figure 4: Qwertyuiop repositions the green lines on the left. Shortly after, 137 increases their thickness.

A minute after 137's question, Qwertyuiop posts a further elaboration. The first sentence states that the lines marked with green on the drawing are collinear to each other. The way he uses the term "**collinear**" here in relation to recently highlighted sticks indicates that this term is a reference to sticks that are aligned with respect to each other along a grid line. The second sentence asserts that there are "**3 identical sets of collinear lines**" (presumably located within the larger triangular partition, since the green lines are carefully placed in such a partition). Finally, the last sentence states that one needs to find the number of sides (i.e., sticks) in one set and multiply that number by "**3"** (to find the total number of sticks in one partition). Although Qwertyuiop does not explicitly state it here, the way he places the green lines indicates that he is oriented to one of the 6 larger partitions to perform the counting operation he has just described. Following Qwertyuiop's elaboration, 137 posts "Ah. I see." in line 752. This is a token of *cognitive change* (Heritage, 2002), where the person who made the utterance announces that she/he can see something he has not been able to see earlier. Yet, it is still ambiguous what is understood or seen since no display of understanding is produced by the recipients yet.

Excerpt 2: Co-construction of a method for counting sticks

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
752	19:30:20	19:30:23	137	Ah. I see.	
		19:30:48 - 19:30:58		***137 drew an elongated hexagon in orange***	
753	19:31:00	19:31:07	137	Wait. Wouldn't that not work for that one?	
754	19:31:11	19:31:12	Jason	Yeah	

755	19:31:12	19:31:15	Jason	beacuse that's irregular	
756	19:31:09	19:31:17	137	Or are we still only talking regular ones?	
757	19:31:20	19:31:22	137	About	
758	19:30:38	19:31:24	Qwertyuiop	side length 1 = 1, side length 2 = 3, side length 3 = 6...	
		19:31:45 - 19:32:15		***137 removes the orange hexagon***	
759	19:32:32	19:32:50	137	Shouldn't side length 2 be fore?	Message No. 758
760	19:32:52	19:32:53	137	*four	
761	19:33:06	19:33:10	Qwertyuiop	I count 3.	Message No. 759
762	19:33:20	19:33:25	137	Oh. Sry.	
763	19:33:24	19:33:30	Qwertyuiop	It's this triangle.	Reference to whiteboard (see Figure 6)
764	19:33:44	19:33:45	137	We	
765	19:33:47	19:33:54	Qwertyuiop	I don't see the pattern yet...	Message No. 758
766	19:33:50	19:34:01	137	We're ignoring the bottom one?	
		19:34:10 – 19:34:18		***137 first moves the longest green line, adds an orange line segment, moves the longest line back to its original position (see Figure 7)***	
767	19:34:11	19:34:29	Qwertyuiop	no, 3 is only for side length 2.	Message No. 766

About 18 seconds after 137's last posting, Qwertyuiop begins typing, but he does not post anything in chat for a while. After 10 seconds elapsed since Qwertyuiop started typing, 137 begins to produce a drawing on the whiteboard. In about 10 seconds, 137 produces a smaller hexagonal shape with orange color on the triangular grid. The new elongated hexagonal shape is placed on the right side of the recently added green lines, possibly to avoid overlap (see Figure 5). Once the hexagon is completed, 137 posts a chat message in line 753. The message starts with "**wait**"[10] which can be read as an

[10] The token "wait" is used frequently in math problem-solving chats to suspend ongoing activity of the group and solicit attention to something problematic for the participant who uttered it. This token

attempt to suspend the ongoing activity. The remaining part of the message states that the aforementioned approach may not work for a case indexed by the deictic term "**that one**". Since 137 has just recently produced an addition to the shared drawing, his message can be read in reference to the orange hexagon. Moreover, since the referred case is part of a message designed to suspend ongoing activity for bringing a potential problem to others' attention, the recently produced drawing seems to be presented as a *counterexample* to the current approach for counting the sticks.

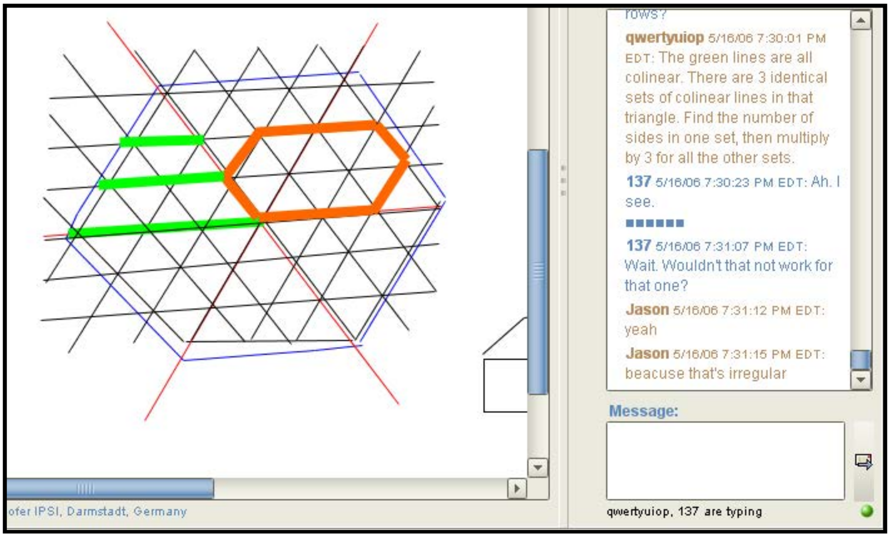


Figure 5: 137 adds an elongated hexagon in orange.

In the next line Jason posts the affirmative token "yes". Since it follows 137's remark sequentially, the affirmation can be read as a response to 137. Jason's following posting provides an account for the agreement by associating "**irregularity**" with an object indexed by the deictic term "**that**". When these two postings are read together in response to 137's message, the deictic term can be interpreted as a reference to the orange hexagon. In short, Jason seems to be stating that the strategy under consideration would not work for the *orange hexagon* because it is "**irregular**". In the meantime, 137 is still typing the statement that will appear in line 756, which asks whether the hexagon under consideration is still assumed to be regular. This question mitigates the prior problematization offered by the same author since it leaves the possibility that the proposed strategy by Qwertyuiop may still work for the regular case.

may be used as a preface to request explanation (e.g., wait a minute, I am not following, catch me up) or to critique a result or an approach as exemplified in this excerpt.

In line 758, Qwertyuiop posts a chat message stating "**side length 1 = 1, side length 2 = 3, side length 3 = 6...**" It took about a minute for him to compose this message after he was first seen as typing at 19:30:38. The way the commas are used to separate the contents of the statement and the ellipsis placed at the end indicate that this posting should be read as an open-ended, ordered list. Within each list item the term "**side length**" is repeated. "Side length" has been used by this team during a prior session as a way to refer to different stages of a growing stick-pattern. In the hexagonal case the pattern has 6 sides at its boundary and counting by side-length means figuring out how many sticks would be needed to construct a given side as the pattern grows step by step. Note that this method of indexing stages assumes a stick-pattern that grows symmetrically. So a side length equal to 1, 2 or 3 corresponds to the first, second or third stage of the hexagonal stick pattern, respectively. When the statement is read in isolation, it is not clear what the numbers on the right of the equals sign may mean, yet when this posting is read together with Qwertyuiop's previous posting where he described what needs to be found, these numbers seem to index the number of sticks within a set of collinear lines as the hexagonal array grows.

After Qwertyuiop's message, 137 removes the orange lines he has drawn earlier to produce an irregular hexagon. By erasing the irregular hexagon example, 137 seems to be taking Qwertyuiop's recent posting as a response to his earlier question posted in line 756, where he asked whether they were still considering regular hexagons or not. Although Qwertyuiop did not explicitly respond to this question, his message in line 758 (especially his use of the term "side length" which implicitly assumes such a regularity) seems to be seen as a continuation of the line of reasoning presented in his earlier postings. In other words, Qwertyuiop's sustained orientation to the symmetric case is taken as a response to the critique raised by 137.

In line 759, 137 posts a message explicitly linked to Qwertyuiop's most recent posting. It begins with the negative token "**Shouldn't**", which expresses disagreement. The subsequent "**side length 2**" indexes the problematic item and "**be fore**" offers a repair for that item. Moreover, the posting is phrased as a question to solicit a response from the intended recipient. 137's next posting in line 760 repairs his own statement with a repair notation peculiar to online chat environments. The asterisk at the beginning instructs readers to attend to the posting as a correction (usually to the most recent posting of the same author). In this case, due to its syntactic similarity to the word in the repair statement, "**fore**" seems to be the token that is supposed to be read as "**four**."

In his reply in line 761, Qwertyuiop insists that his counting yields "**three**" for the problematized case. In the next posting 137's "oh" marks the previous response as surprising or unexpected. The subsequent "**sry**—short for "sorry"—can be read as backing down. In line 763, Qwertyuiop posts a message that states "**it's this triangle**" and explicitly points at a region on the shared drawing. The explicit reference and the deictic terms again require the interlocutors to attend to something beyond the text involved in the posting. In short, the sequential unfolding of the recent postings suggests that this posting is designed to bring the relevant triangle in which the counting operation is done for the problematic case (indexed by side length 2) to other members' attention (see Figure 6).

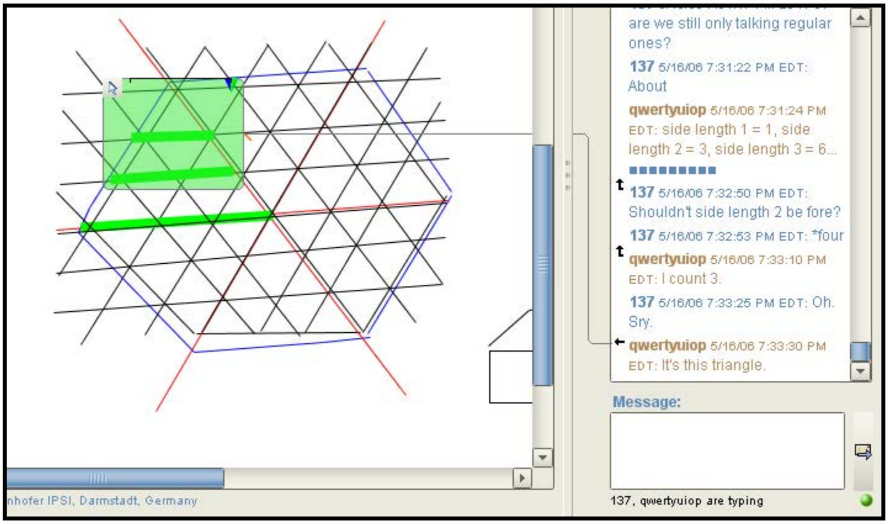


Figure 6: Qwertyuiop points to the triangle which contains the sticks to be counted for the stage indexed by side length=2. The green lines enclosed by the reference correspond to 1+2=3 sticks.

In line 765, Qwertyuiop posts another message explicitly pointing to his earlier proposal for the first few values he obtained through his method of counting, where he states that he has not been able to "**see a pattern yet**." Hence, this statement explicitly specifies "**the pattern**" as what is missing or needed in this circumstance. The message not only brings a *prospective indexical*[11] (Goodwin, 1996), "**the pattern,**" into the ongoing discussion as a problem-solving objective, but also invites other members of the team to join the search for that pattern.

In the next line, 137 posts a question that brings other members' attention to something potentially ignored so far. The term "**bottom one**" when used with "**ignore**" indexes something excluded or left out. Nine seconds after his posting, 137 performs some drawing work on the whiteboard. He moves the longest green line to the right first, then he adds a short line segment with orange color, and then he moves

[11] Goodwin (1996) proposes the term prospective indexicals for those terms whose sense is not yet available to the participants when it is uttered, but will be discovered subsequently as the interaction unfolds. Recipients need to attend to the subsequent events to see what constitutes a "pattern" in this circumstance.

the same green line back to its original location (see Figure 7). These moves make 137's orientation to a particular part of the drawing explicit. When read together with his previous question, the orange line could be seen as a graphical illustration for the left-out part previously referred as the "bottom one". When read as a response to Qwertyuiop's recent exposition in lines 761 and 763, the "bottom one" seems to be a reference to the part of the drawing that was not enclosed by Qwertyuiop's explicit reference.

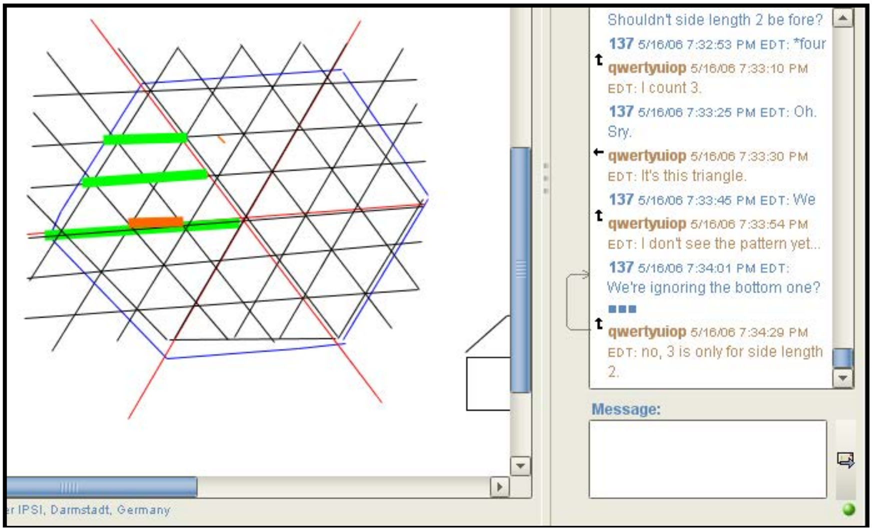


Figure 7: 137 adds an orange segment to the drawing.

The next posting by Qwertyuiop, which appears in line 767, is explicitly linked to 137's question in the previous line. The message begins with "no" which marks the author's disagreement with the linked content, and the subsequent part of the message provides an account for the disagreement by stating that the value 3 is only relevant to the case indexed by "**side length 2**".

The sequence of exchanges between 137 and Qwertyuiop in this excerpt indicates that there is a misalignment within the group about the procedure used for counting the number of sticks. This misalignment is made evident through explicit problematizations and disagreements. The way the members make use of both spaces as they interact with each other makes it increasingly clear for them (a) where the relevant pieces indexed by the terms like "collinear" and "triangle" are located, and (b) how they are used in the counting process. Nevertheless, the misalignment between the counting procedures suggested in 137's and Qwertyuiop's contributions have not been resolved yet.

Excerpt 3: Collective noticing of a pattern of growth

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
765	19:33:47	19:33:54	Qwertyuiop	I don't see the pattern yet...	Message No. 758
766	19:33:50	19:34:01	137	We're ignoring the bottom one?	
		19:34:10-19:34:18		***137 first moves the longest green line, adds an orange line segment, moves the longest line back to its original position***	
767	19:34:11	19:34:29	Qwertyuiop	No, 3 is only for side length 2.	Message No. 766
768	19:34:36	19:34:52	137	And I think the'y;re all triangular numbers.	Message No. 765
		19:35:03-19:35:16		***137's changes the color of the longest green line to red, and then to green again***	
769	19:35:06	19:35:17	Qwertyuiop	"triangular numbers"?	Message No. 768
		19:35:27-19:35:36		***137's draws a red hexagon on the diagram (Figure 8)***	
770	19:35:28	19:35:37	Jason	You mean like 1, 3, 7, ...	
771	19:35:39	19:35:39	Jason	?	
772	19:35:48	19:35:59	137	Like 1,3,6,10,15,21,28.	Message No. 770
773	19:35:51	19:36:02	Qwertyuiop	The sequence is 1, 3, 6...	Message No. 770
774	19:36:02	19:36:30	137	Numbers that can be expressed as n(n+1)/2, where n is an integer.	
775	19:36:44	19:36:45	Qwertyuiop	Ah	
776	19:37:09	19:37:18	137	So are we ignoring the bottom orange line for now?	Message No. 766

In line 768, 137 posts a message linked to Qwertyuiop's posting in line 765. The preface "And" and the explicit reference together differentiate this contribution from the ongoing discussion about a piece that was potentially excluded from the second stage. Note that Qwertyuiop's message in line 765 refers further back to an older posting where he proposed a sequence of numbers for the first 3 stages "side length 1 = 1, side length 2 = 3, side length 3 = 6..." When 137's message is read in relation to these two prior messages, the phrase "**they are all**" seems to be a reference to this sequence of numbers. Therefore, the message can be read as an uptake of the issue of finding a pattern that fits this sequence. Moreover, by proposing the term "**triangular numbers**" as a possible characterization for the sequence, 137 offers further specificity to the prospective indexical, the "**pattern**", which was initially brought up by Qwertyuiop.

Following his proposal, 137 changes the color of the longest green line segment at the bottom to red and then to green again. In the meantime Qwertyuiop is typing what will appear in line 769, which can be read as a question soliciting further elaboration of the newly contributed term "triangular numbers." 137 continues to act on the whiteboard and he adds a red hexagon to the shared drawing (see Figure 8). Since the hexagon is located on the section referenced by Qwertyuiop several times earlier and shares an edge with the recently problematized orange section, this drawing action can be treated as a move related to the discussion of the ignored piece.

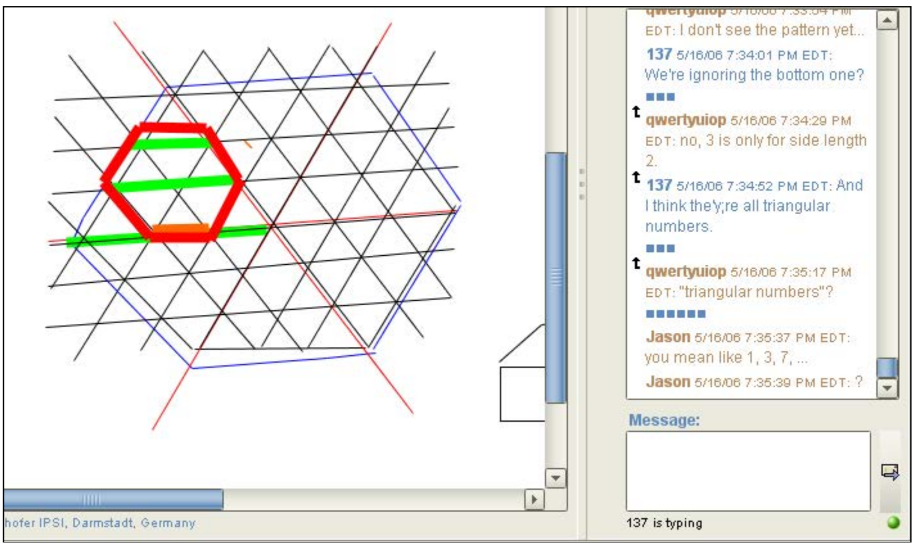


Figure 8: 137 adds a red hexagon inside the partition the team has been oriented to.

Jason joins the discussion thread about triangular numbers by offering a list of numbers in line 770. The term "**like**" is used here again to relate a mathematical term to what it may be indexing. This posting alone can be read as an assertion, but the question mark Jason posts immediately after in the next line mitigates it to a statement soliciting others' assessment. At roughly the same time, 137 posts a substantially longer sequence of numbers, and immediately after Qwertyuiop points out the difference between 137's sequence and what Jason offered as a list of triangular numbers. In line 774, 137 elaborates his definition further by offering an algebraic characterization of triangular numbers as integers that can be expressed with the formula "**n(n+1)/2**".

In short, the sequence resulting from Qwertyuiop's counting work based on his notion of "**collinearity**" has led the team to notice a relationship between that sequence and a mathematical object called "**triangular numbers**". The latter symbolic definition offered by 137 for triangular numbers in response to the ongoing search for a pattern has established a relationship between geometrically motivated counting work and an algebraic/symbolic representation stated in generic form as **n(n+1)/2**.

Excerpt 4: Resolution of referential ambiguity via visual proof

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
776	19:37:09	19:37:18	137	So are we ignoring the bottom orange line for now?	Message No. 766
777	19:37:32	19:37:36	Qwertyuiop	"green"?	Message No. 776
778	19:37:44	19:37:48	137	THe short orange segment.	
		19:37:59-19:38:02		***137 changes the color of the green lines enclosed by the red hexagon to blue (see Figure 9)***	
779	19:37:49	19:38:05	137	PArallel to the blue lines.	
780	19:37:58	19:38:05	Qwertyuiop	I don't think so...	
781	19:38:20	19:38:26	137	Wait, we are counting sticks right now, right?	Message No. 780
782	19:38:35	19:38:48	Qwertyuiop	yes-one of the colinear ets of sticks	
783	19:38:55	19:39:08	Qwertyuiop	oops-"sets" not " ets"	
784	19:39:22	19:39:42	137	So we are trying to find the total number of sticks in a given regular hexagon?	Message No. 782
785	19:39:50	19:40:18	Qwertyuiop	not yet-we are finding one of the three sets, then multiplying by 3	Message No. 784

786	19:40:25	19:40:40	Qwertyuiop	that will give the number in the whol triangle	
787	19:40:34	19:40:51	137	Then shouldn't we also count the bottom line?	Message No. 785
788	19:40:52	19:41:01	Jason	are you taking into account the fact that some of the sticks will overlap	Message No. 786
789	19:41:25	19:41:41	137	Then number of sticks needed for the hexagon, right?	Message No. 786
790	19:41:16	19:42:22	Qwertyuiop	Yes. The blue and green/orange lines make up on of the three colinear sets of sides in the triangle. Each set is identical and doesn't overlap with the other sets.	Message No. 788
791	19:42:50	19:42:50	Jason	Ok	
792	19:43:03	19:43:11	Jason	this would be true for hexagons of any size right>	
793	19:43:09	19:43:13	Qwertyuiop	triangle, so far	Message No. 789
794	19:43:25	19:43:25	137	Oh.	
795	19:43:25	19:43:26	Qwertyuiop	this one	Reference to whiteboard (see Figure 10)
796	19:43:42	19:43:52	137	Yes, but they will overlap...	
797	19:43:59	19:44:13	137	Eventually when you multiply by 6 to get it for the whole figure.	
798	19:44:01	19:44:30	Qwertyuiop	no, the sets are not collinear with eachother. I'll draw it...	Message No. 796
		19:44:35-19:44:56		***Qwertyuiop moves the small hexagon in red and blue lines out of the grid (see Figure 11)***	
799		19:44:59	137		Message No. 798
		19:44:59-19:45:17		***Qwertyuiop repositions and resizes the red lines on the grid***	
		19:45:20		***Qwertyuiop continues adjusting the red lines***	

		19:45:23-19:45:37		***Qwertyuiop continues adjusting the red lines***	
		19:45:41-19:46:16		***Qwertyuiop adds purple lines (see Figure 12)***	
800	19:46:22	19:46:34	137	Oh. I see.	
801	19:46:22	19:46:52	Qwertyuiop	Those are the 3 sets. One is red, one is green, one is purple.	
		19:47:07 -19:47:11	137	***137 starts to make green lines thicker***	
802	19:47:04	19:47:12	Jason	wait--- i don't see the green/purple ones	
		19:47:17 -19:47:33	137	***137 makes the purple lines thicker (see Figure 13 below)***	
803	19:47:18	19:47:40	Qwertyuiop	so we find a function for that sequence and multiply by 3	Message No. 774

In line 776, 137 posts a message which is explicitly linked to his prior message in line 766 where he mentioned a potentially ignored piece indexed by the phrase "**the bottom one**". The use of "**So**" at the beginning can be read as an attempt to differentiate this message from the recently unfolding discussion about triangular numbers. The subsequent part of the message brings other team members' attention to a potentially ignored piece indexed by the phrase "**the bottom orange line**". 137 used the phrase "**the bottom one**" earlier, but this time he makes use of *color referencing* as an additional resource to provide further specificity to what he is referencing. At this moment a *red hexagon* and a *short orange segment* are visible on the shared drawing space, which are layered on top of the triangular grid (see Figure 8). The way 137 orients to the new state of the drawing indicates that his earlier drawing actions (marked in the prior excerpt before line 770) seem to be performed in preparation for this posting. Hence, this posting can be read as an attempt to *re-initiate a prior thread* about a potentially ignored piece in the counting work, which is distributed over both interaction spaces.

Qwertyuiop's message in the next line involves **"green"** in quotes, ends with a question mark, and is explicitly linked to 137's last message in line 776. The quotation marks seem to give significance to an object indexed by the color reference. Note that there are 3 green lines on the shared drawing at the moment (see Figure 8). The use of the color reference and the explicit link suggest that this message is posted in response to 137's question in line 776. When it is read in this way, Qwertyuiop seems to be asking if the relevant line located at the bottom should have been the green one instead.

Following Qwertyuiop's posting, 137 provides further specificity to the problematized object by first stating that it is "**the short orange segment**" in line 778. Next, 137 modifies the two green lines inside the red hexagon by changing their color to blue (see Figure 9). Then, he posts another message in line 779 that refers to a particular location on the whiteboard that is "**parallel**" to the recently added "**blue lines**". Thus, 137's recent actions suggest that the object indexed by his phrase, "**short bottom orange line**" segment, is the one parallel to the blue lines.

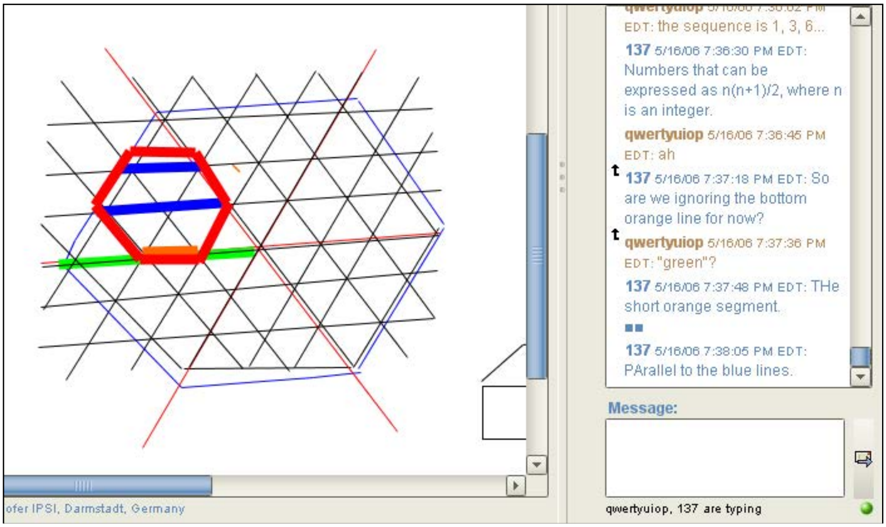


Figure 9: 137 changes the color of the green lines inside the red hexagon to blue

In line 780, Qwertyuiop states his disagreement. Since the message appears shortly after 137's point that the orange segment is left out of the computation, Qwertyuiop seems to be disagreeing with the remark that there is a missing piece in the counting method. In the next line, 137 posts a question prefaced with "**wait**" that calls for suspending the ongoing activity and asks if one can still characterize what the team ("**we**") is currently doing as "**counting the sticks**". The posting is explicitly linked to Qwertyuiop's last message. By posting a question about the ongoing group process following a sustained disagreement with his peer, 137 is making it explicit that there is a misalignment within the team with respect to the task at hand. Hence, this exchange marks a breakdown in interaction that needs to be attended to before the team can proceed any further.

In the next line, Qwertyuiop takes up this question by providing his account of the ongoing process as counting "**one of the collinear sets of sticks**." Next, 137 posts another question explicitly linked to Qwertyuiop's answer, which gives further specificity to 137's earlier characterization of the counting work undertaken by the team (i.e., counting the sticks for the "**whole hexagon**"). Qwertyuiop's response to this question states that the focus is not on the whole hexagon yet, but on what he is referring to as "**one of the three sets**", which would then be followed by a multiplication by 3. In the next line Qwertyuiop continues his explanation that this will give them the number of sticks for "**the whole triangle**", which can be read as a reference to one of the six triangular partitions that altogether form the hexagon.

In line 787, 137 posts a message explicitly linked to the first part of Qwertyuiop's explanation. The posting is phrased as a question problematizing again that the bottom line should also be included in the counting operation described by Qwertyuiop. Next, Jason joins the discussion by posting a question linked to the latter half of Qwertyuiop's explanation in line 786, which asks him if he has taken into account "**the fact that some of the sticks will overlap**". The way Jason phrases his posting brings "**overlap**" as an issue that needs to be addressed by the counting method under discussion.

In line 789, 137 posts a chat message with a referential link to Qwertyuiop's last posting in line 786. This message seems to extend the order of computations described in Qwertyuiop's exposition by anticipating the next step of the computation, namely calculating the number of sticks needed for the hexagon once the step mentioned in 786 is achieved. In other words, 137 displays that he is able to follow the order of computations suggested by his peer to address the task at hand.

In line 788 Qwertyuiop responds to the overlapping sticks issue raised by Jason. He makes reference to the blue and green/orange lines to describe one of the three collinear sets of sides within the triangular partition (since the shared image has remained unchanged, this message can be read in reference to the state displayed in Figure 9). He further asserts that each set is identical and does not overlap. In the next line Jason concurs, and then asks if this should hold for hexagons of any size.

Following Jason's messages, Qwertyuiop posts a message linked to 137's earlier question in line 789. Qwertyuiop stresses again that the focus has been on the "**triangle**" so far. His next posting in line 795 includes a referential arrow to the shared diagram and a deictic term "**this one**" that together provide further specificity to which part of the hexagon he was referring to with the indexical term "**triangle**" (see Figure 10).

In lines 796 and 797, 137 first accepts what Qwertyuiop has asserted, but points to a potential issue that will be faced when the result will be multiplied by 6 to extend the counting operation to the whole hexagon. Before 137 posts his elaboration in line 797, Qwertyuiop begins typing a response to 137's first remark that appears in line 798. In that message Qwertyuiop expresses his disagreement and asserts that "**the sets are not collinear with each other**". Hence, this posting shows that Qwertyuiop has treated 137's use of the pronoun "**they**" in line 796 as a reference to the notion of collinear sets. In the latter part of his posting, Qwertyuiop announces that he will draw what he is talking about, so this section of the message projects that a related drawing action will follow his statement shortly.

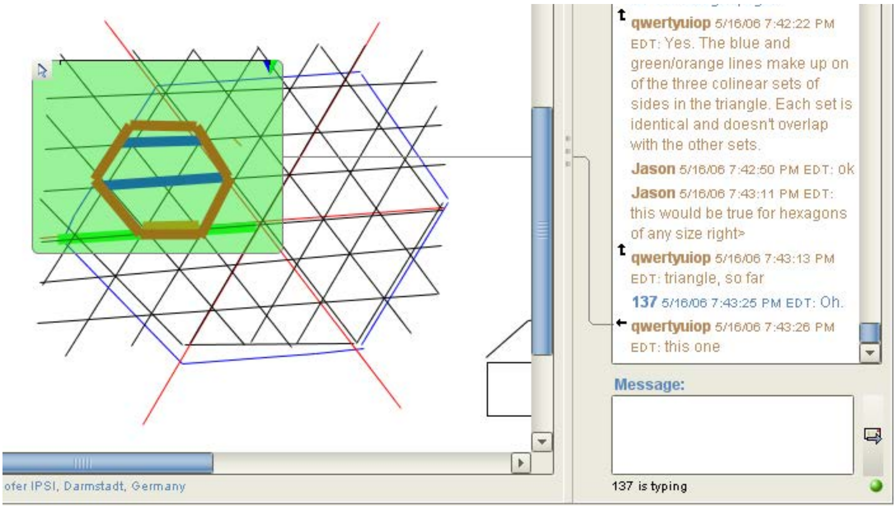


Figure 10: Qwertyuiop highlights the triangle by using the referencing tool.

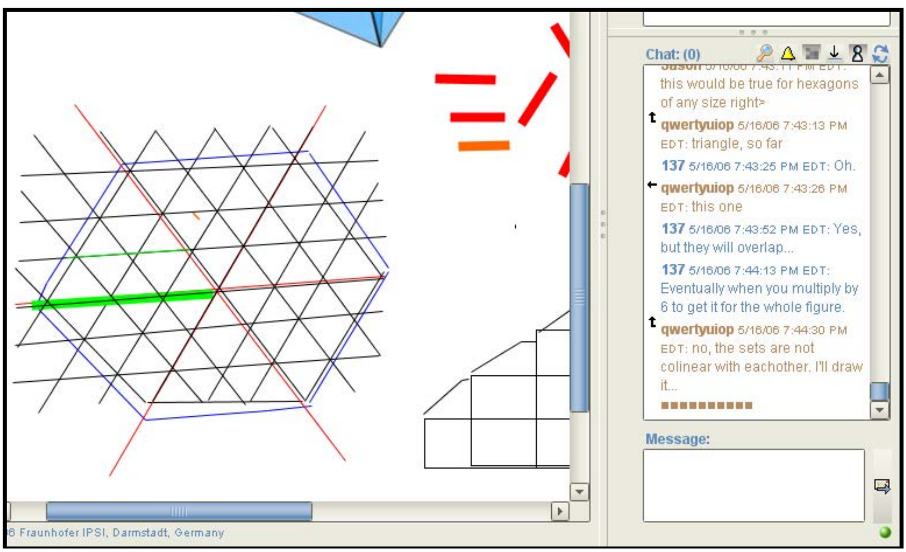


Figure 11: Qwertyuiop moves the lines added by 137 away.

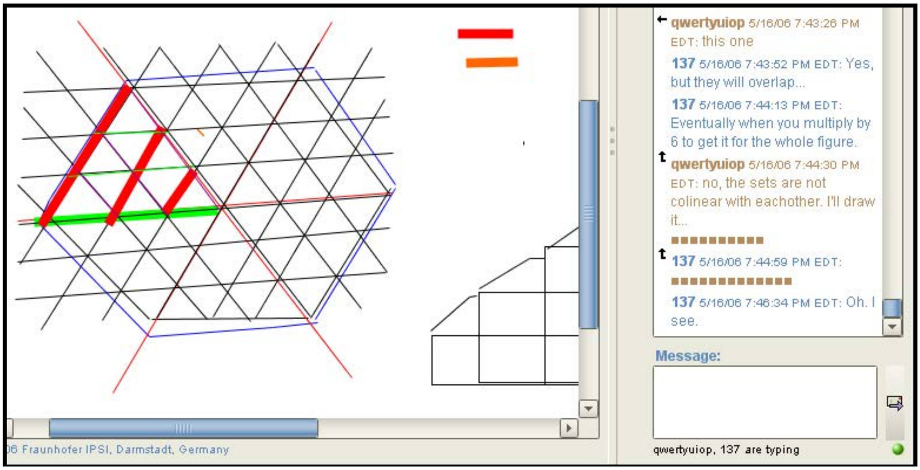


Figure 12: Qwertyuiop repositions the red lines to mark a part of the larger triangle. Then he adds two horizontal lines in green, parallel to the existing green line. Finally, he adds 3 more lines in purple. Since Qwertyuiop uses a thinner brush to draw the green and purple lines, they are difficult to see.

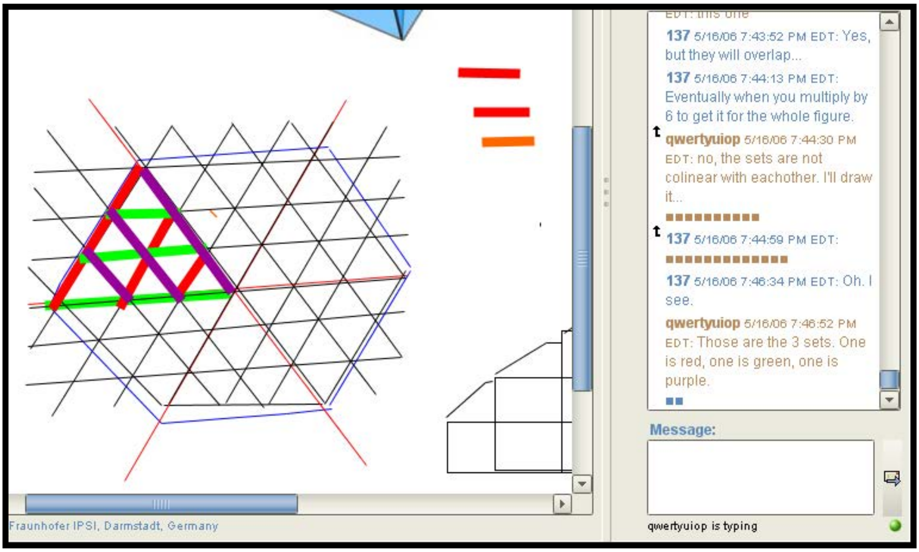


Figure 7: 137 increases the thickness of the newly added green and purple lines. The final state of the diagram presents a visual proof that 3 sets of collinear lines do not overlap with each other.

Figures 11 and 12 display snapshots from Qwertyuiop's drawing actions following his last posting. First he moves the red and orange lines to the side, and then he repositions the red lines to highlight 3 segments that are parallel to each other. Next, he adds 2 green lines parallel to the remaining green line. Finally, he adds 3 purple lines to cover the remaining sticks in that triangular section. The green and purple lines are drawn with a thin brush (see Figure 12).

Once the drawing reaches the stage in Figure 12, 137 posts "**oh I see**" in line 800, which can be read in response to Qwertyuiop's recent drawing work. Qwertyuiop's graphical illustration seemed to have helped 137 to notice something he had not been able to see earlier. Next, Qwertyuiop posts a message that refers to the lines he has recently drawn with the plural deictic term "**those**". The message provides further specificity to the mathematics object "**3 sets**" by locating each set on the diagram through the use of color references "**red**", "**green**" and "**purple**". In other words, Qwertyuiop has provided a visual realization of the phrase "**3 sets of collinear sides**" he coined earlier, which has been treated as problematic by his teammates.

In line 802, Jason states that he cannot see the green/purple lines, which were marked with a thin brush by Qwertyuiop. In response 137 makes these new additions more visible by increasing their thickness (see Figure 13). The final state of the diagram presents a *visual proof* that 3 sets of collinear lines marked with green, purple, and red do not indeed overlap with each other.

In line 803, Qwertyuiop provides further specificity as to what needs to be found given the visual realization of the collinear sides recently produced on the whiteboard. His message is explicitly linked to an old message posted by 137 several lines ago (line 774 in Excerpt 3) that provides a formulaic realization for triangular numbers previously associated with the pattern of growth of collinear sides. Hence, Qwertyuiop's statement, "**find a function for that sequence and multiply by 3**", can be read as a proposal for a strategy to find the number of sticks required to build a triangular partition. In particular, Qwertyuiop is pointing (narratively) to a candidate (symbolic) algebraic realization of what he has just demonstrated with (graphical) visual resources on the whiteboard. This is the culmination of a sublte and complex collaborative process in which mathematical discourse, graphical reasoning and symbolic expression were tightly integrated by the group.

To sum up, in this episode the team has achieved a sense of *common ground* (Clark & Brennan, 1991), *intersubjectivity* (Stahl, et al., 2011) or *indexical symmetry*[12] (Hanks, 1992;

[12] Hanks proposes the notion of indexical symmetry to characterize the degree to which the interactants share, or fail to share, a common framework relative to some field of interaction on which reference can be made. In particular, "…the more interactants share, the more congruent, reciprocal and transposable their perspectives, the more symmetric is the interactive field. The greater the differences that divide them, the more asymmetric the field." (Hanks, 2000, p. 8.). These excerpts show that mathematical terms are inherently

2000) with respect to the term "**set of collinear sides**" and its projected application towards solving the task at hand. The challenges voiced by 137 and Jason through the course of the episode solicited further elaboration from Qwertyuiop regarding how **collinear sides** can be located in the shared diagram and how they can be used to devise a method to count the number of sticks. In particular, in this excerpt the team members worked out the overall organization of their joint problem-solving work by discussing what they are trying to find, how they should locate the objects relevant to the task, and how they should order some of the steps that have been proposed so far to arrive at a solution. For instance, Qwertyuiop's initial proposal including the indexical term "**collinear sets**" focuses on one of the triangular regions. Yet, the focus on a triangular region was left implicit, which seemed to have led 137 to treat Qwertyuiop's proposal as applied to the whole hexagon. Through their discussion across both interaction spaces the team has incrementally achieved a *shared understanding* in terms of how a triangular region is decomposed into 3 sets of collinear, non-overlapping sides, and how that can be used to systematically count the number of sticks in that region. The visual practices have been encapsulated in linguistic terms in ways that become shared within the small group through their interactions, which integrate graphical and narrative actions. The graphical moves are strategically motivated to decompose a complicated pattern into visually obvious sub-patterns, with an eye to subsequently constructing a symbolic representation of the pattern. The elaboration of a mathematical vocabulary allows the group to reference the elements of their analysis in order to establish a shared view of the graphical constructions, to make proposals about the patterns to each other and to index past established results.

Concluding the Mathematical Analysis

The group is now ready to return to the symbolic work. In line 818[13], Qwertyuiop resumes the discussion about the shared task by proposing a formula "**f(n) = 2n-1**" where he declares n to be the "**side length**" (see Excerpt 5). It is not evident from the text itself what the formula is standing for. Yet, the message is explicitly linked to an older posting (line 772) where 137 posted the statement "**Like 1,3,6,10,15,21,28**" as part of a prior discussion on triangular numbers (see Excerpt 3). Hence, when this

indexical. Establishing a shared understanding of such indexical terms require collaborators to establish a reciprocity of perspectives towards the reasoning practices displayed/embodied in the organization of the texts and inscriptions in the shared scene (Zemel & Cakir, 2009).

[13] A brief administrative episode including the facilitator took place between excerpts 4 and 5, which is omitted in an effort to keep the focus of our analysis on problem solving.

message is read in reference to line 772, it can be treated as a proposal to generalize the values derived from Qwertyuiop's geometrically informed counting method with a formula stated in symbolic form.

Excerpt 5: Re-initiating the discussion of the algebraic formula

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
818	19:51:11	19:52:19	qwertyuiop	what about: f(n)=2n-1 where n is side length	Message No. 772
		19:52:28	137	***137 changes the layout of the last straight line by making it a dashed line.***	
819	19:52:55	19:53:03	137	I don't think that works.	Message No. 818
820	19:53:07	19:53:18	137	Howbout just n(n+1)/2	
821	19:53:37	19:53:41	Jason	for # sticks?	
822	19:53:38	19:53:48	qwertyuiop	that's number of sides for one set	Message No. 820
823	19:53:50	19:53:51	qwertyuiop	?	
824	19:53:57	19:53:59	Jason	oh ok nvm	
825	19:54:26	19:54:29	137	Ya.	Message No. 822
826	19:54:36	19:54:58	qwertyuiop	then x3 is 3(n(n+1)/2)	Message No. 820
827	19:55:04	19:55:07	qwertyuiop	simplified to...	Message No. 826
828	19:55:11	19:55:37	qwertyuiop	(n(n+1)1.5	
829	19:55:34	19:55:44	137	On second thought, shouldn't we use n(n-1) for these:	Message No. 826
		19:55:50 - 19:55:55	137	***137 changes the color of two dashed lines into orange (see Figure 13 below)***	
830	19:55:31	19:55:55	Nan	just a kind reminder: Jason mentioned that he needs to leave at 7p central time sharp	

137 rejects Qwertyuiop's proposal in line 819 and then makes a counter proposal in the next line. As we saw in Excerpt 3, the sequence of numbers resulting from Qwertyuiop's counting method was previously associated with a math artifact called

triangular numbers by 137. The counter proposal includes the same expression 137 provided earlier when he gave a definition of triangular numbers as "**integers that can be represented as n(n+1)/2**" (see line 774). Jason joins the discussion in line 821 by asking if the proposed formula is for the number ("**#**") of sticks. Although Jason does not specify which object (e.g., the whole hexagon) he is associating the formula with, his posting can be read as an attempt to solicit further elaboration with regards to what the recently proposed formulas are about.

Qwertyuiop's posting in the next line states that the object indexed by the deictic term "that" corresponds to the "number of sides for one set". Note that Qwertyuiop's message is explicitly linked to 137's counterproposal in line 820, so the deictic term "**that**" can be read as a reference to the expression "**n(n+1)/2**" included in 137's posting. Moreover, the message sequentially follows Jason's question. Hence, Qwertyuiop seems to be responding to Jason's query by pointing out which object the recently proposed formulas are about. The question mark Qwertyuiop posts in the next line mitigates his previous statement into a question. This can be read as a move to solicit the remaining member's (i.e. 137) assessment of the association Qwertyuiop has just offered. By making his reading of 137's formula explicit, Qwertyuiop also indicates that he concurs with the alternative expression proposed by his peer. Jason's next posting in line 824 indicates that he is now following his peers' reasoning, which comes just before 137's confirmation linked to Qwertyuiop's claim in 822. Therefore, at this point it seems to be evident for all members in the group that the algebraic expression n(n+1)/2 is associated with one of the "collinear sets of sticks" within a triangular section.

In line 826, Qwertyuiop posts a message linked back to 137's proposal in 820. The use of "**then**" at the beginning suggests that this message is a consequence or follow up of the message he is referring to. "**x3**" can be read as a reference to multiplication by 3, where the remaining part of the message provides the expression yielded by this operation. In other words, Qwertyuiop seems to be proposing the next step in the computation, given the expression for the number of sticks for a single "**set**". In the next two lines he further simplifies this expression by evaluating 3/2 to 1.5.

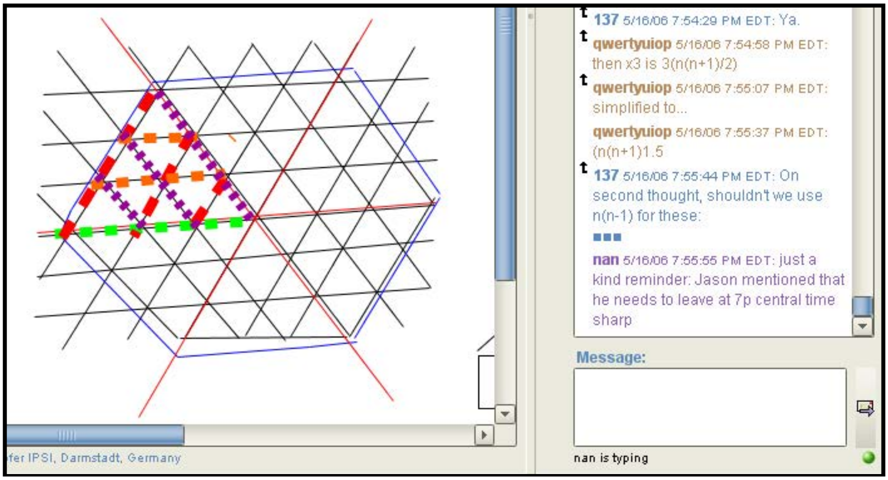


Figure 14: 137 highlights 2 horizontal lines in orange following his proposal at 7:55:44 (line 829).

In line 829, 137 posts a message phrased as a question. The posting begins with "on second thought" which indicates that the author is about to change a position he took prior with respect to the matter at hand. The rest of the statement is phrased as a question and it is addressed to the whole team as indicated by the use of the first person plural pronoun "we". The question part associates the expression "**n(n-1)**" with the deictic term "**these**" which is yet to be specified[14]. The posting ends with "**:**" which projects that more content will likely follow this message subsequently. Next, 137 begins to act on the whiteboard by changing the color of two horizontal lines from green to orange (see Figure 14). The temporal unfolding of these actions suggests that the sticks highlighted in orange are somehow associated with the expression n(n-1). In other words, 137's recent actions can be seen as a move for adjusting the index values in the generalized formula.

In this episode, the team achieves an important transition from a geometrically motivated counting procedure applied on "**one of the collinear sets**" to a symbolic formula generalizing the procedure to a set of any given sidelength. The generality is achieved through one member's noticing that the sequence of numbers derived from the counting procedure corresponds to "**triangular numbers**", which seems to be a familiar concept at least for the member who proposed it. The formula that was provided as part of the definition of triangular numbers is then applied to the relevant

[14] See footnote to line 746 on the use of "these" and "those". The consistency of the usage of these terms for forward and backward references from the narrative chat to the graphical whiteboard suggests an established syntax of the relationships bridging those interaction spaces within the temporal structure of the multi-modal discourse.

portion of the pattern at hand to achieve the transition from geometric to algebraic mode of reasoning, mediated by the narrative concept of "triangular numbers".

Excerpt 6: Co-reflection on what the team has achieved so far

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
841	19:58:23	19:58:25	qwertyuiop	Back to this?	Message No. 829
842	19:58:32	19:58:34	137	Ya	
843	19:58:39	19:58:49	qwertyuiop	why not n(n-1)?	Message No. 829
844	19:58:39	19:58:50	Jason	you guys pretty much have the formula for this hexagon problem...	
845	19:58:57	19:59:28	qwertyuiop	We almost have it for the triangle. I don't know about the hexagon.	Message No. 844
846	19:59:35	19:59:50	Jason	well that's just multiplied by a certain number for a hexagon, provided that it is regular	Message No. 845
847	19:59:58	20:00:14	qwertyuiop	but the sides of the triangles making up the hexagon overlap	Message No. 846
848	19:59:52	20:00:18	Jason	well i have to leave now; sorry for not participating as much as i wanted to, it's a pretty busy night for me with school and extracurricular stuff	

At the end of excerpt 5 an administrative discussion was initiated by the facilitator about Jason's departure from the chat session[15]. Some of this exchange is left out since it involved a brief chat about the schedule of the next session. However, while Jason was saying farewell to his peers, an exchange related to the task at hand occurred which is captured in Excerpt 6. This episode begins with Qwertyuiop's attempt to reinitiate the problem-solving work by making a reference to an older message posted in line 829 by 137. Following 137's acknowledgement in line 842, Qwertyuiop posts

[15] The session was scheduled to end at 7 pm, yet the students were allowed to continue if they wished to do so. In this case Jason informed the facilitators in advance that he had to leave at 7 pm Central (the log is displayed in US Eastern time).

a question linked to line 829 which indicates that he is oriented to the expression 137 proposed in that message.

About a second later, Jason posts a message stating that the formula for the hexagon problem is pretty much done. Jason's use of the phrase "you guys" ascribes this achievement to the remaining members of the team. In line 845, Qwertyuiop posts a message explicitly linked to Jason's last comment. The first sentence "**We almost have it for the triangle**" provides an alternative account of what has been achieved so far. In his second sentence, Qwertyuiop declares that he does not know about the hexagon yet. Hence, these postings make it evident how Qwertyuiop is treating what the team has accomplished so far.

In line 846, Jason posts a message linked to Qwertyuiop's latest remark. In his response Jason states that getting the formula for the hexagon requires a simple multiplicative step provided that the hexagon is regular. Qwertyuiop's response (as indicated by the referential arrow) follows next, where he brings in how the issue of overlap will play out when they move from the large triangles to the whole hexagon. This is followed by Jason's exiting remark where he apologizes for not being able to participate as much as he wanted.

In this excerpt, team members explicitly commented on how they characterize their collective achievement. In other words, these postings can be read as a joint reflection on what has been done so far. Another interesting aspect of this short exchange is the apparent shift in the positions with respect to the issue of overlapping sticks in the counting procedure. Jason was the person who raised the issue of overlap for the first time in excerpt 3, yet his most recent characterization of the team's work seems to dismiss overlap as a relevant matter. Surprisingly, Qwertyuiop, who was the person previously critiqued by Jason for possibly ignoring the issue of overlapping sticks, explains now why it is a relevant matter that needs to be attended to, before the number of sticks in one triangle is multiplied by a certain number as Jason suggested in 846. In excerpt 3, Qwertyuiop argued that overlaps would not be an issue in his counting work, but that assertion seems to be applied only to the triangular section he was oriented to at that time. His most recent posting displays his awareness with regards to when the overlapping sticks will become an issue, i.e. when they move from the triangular partition to the whole hexagon. These remarks also specify what has not been accomplished yet, and hence suggest the team to find a way to address overlaps as an issue to consider next.

Excerpt 7: Overcoming the problem of overlapping sticks

Chat Index	Time Start Typing	Time of Posting	Author	Content	Refers to
853		20:01:07		*Jason leaves the room*	

854	20:01:19	20:01:31	137	Anyways, if we multiply the orange by 3, we get the:	
855	20:01:14	20:01:34	Nan	do two of you want to continue working for a bit or stop here?	
		20:01:42 - 20:01:48	137	***137 begins to add blue lines on top of the triangular grid***	
856	20:01:40	20:01:44	Nan	i guess that's the answer	Message No. 854
857	20:01:47	20:01:48	Nan	go ahead	
		20:01:49 – 20:01:53	137	***137 continues to add blue lines. The resulting shape is displayed in Figure 15***	
858	20:01:57	20:02:14	137	So then we add 12n for:	
859	20:01:28	20:02:15	qwertyuiop	actually, this doesn't complicate it that much. The overlaps can be accounted for with "-6n"	Message No. 847
		20:02:32 – 20:02:52	137	***137 adds pink contours to the shared drawing, The resulting shape is displayed in Figure 16***	
860	20:02:54	20:02:55	137	Oh.	Message No. 859
861	20:02:56	20:03:07	137	I like addition more than subtraction.	

Excerpt 7 follows Jason's departure[16]. In line 854, 137 re-initiates the problem-solving work by proposing to multiply by 3 what is indexed by "**the orange**". Figure 15 shows the state of the shared drawing at the moment, where there are two dashed orange lines covering a portion of the hexagon. The remaining part of the message announces the outcome of the suggested operation, but no result is provided yet. The message ends with a colon "**:**" indicating that more content is about to follow subsequently. Next, 137 performs a series of drawing actions where he highlights a set of sticks on the triangular grid with blue lines (see Figure 16). These actions are done within a section of the shared drawing that has been empty. Based on the way these actions sequentially unfold and the way the drawing was set up in chat, one can read these actions as the ***visual outcome*** of the operation described in text in line 854. In short, multiplying the number of orange dashed lines by 3 seems to yield the number of

[16] The facilitator opens the possibility to end the session in line 855. The facilitator takes the sustained orientation of the remaining team members to the problem as an affirmative answer and lets the team continue their work.

sticks highlighted in blue, which is an elaborate mathematical move spanning across textual and graphical modalities.

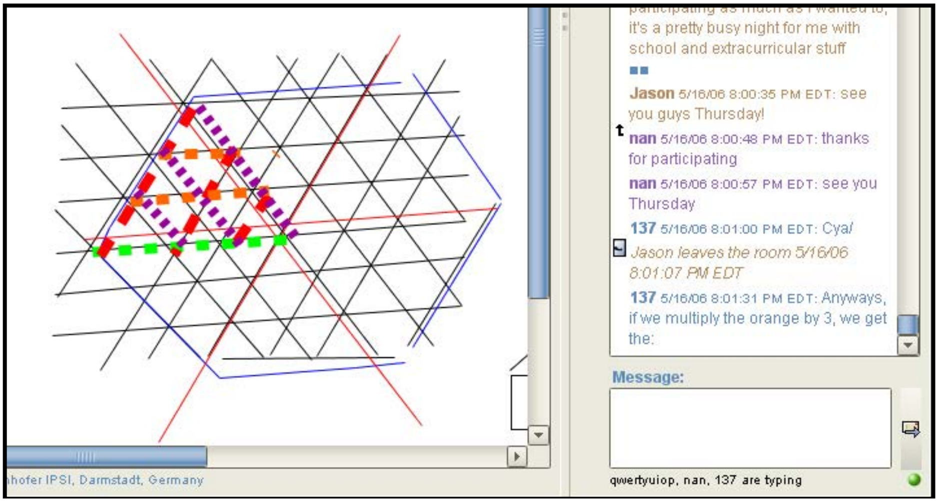


Figure 15: The state of the whiteboard when 137 began his exposition at 8:01:31 (line 854)

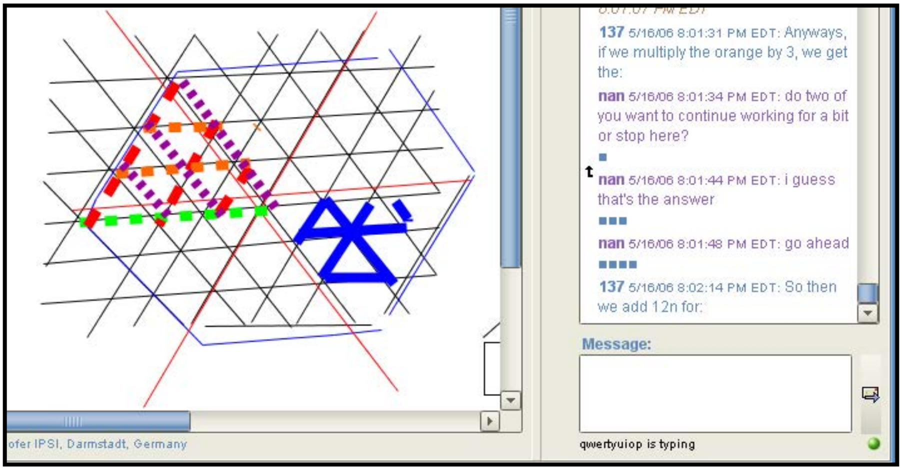


Figure 16: 137's drawing that followed his posting at 8:01:31 (i.e. line 854). The triangles added in blue follow the chat posting that proposes the multiplication of what is marked with orange by 3.

137 posts another message in line 858 which announces adding "**12n**" as the next step in his ongoing exposition. The message ends with "**for:**" which is consistent with his prior use of the colon to project that more elaboration will follow, possibly in the other interaction space. Next, 137 begins to add pink lines to the shared drawing, which covers the boundaries and the diagonals of the hexagonal array (see Figure 17). The sequential continuity of 137's actions suggests that the lines marked with pink provide a geometric ***realization*** of what is indexed by the symbolic expression "**12n**" on the particular instance represented by the shared drawing.

While 137 was composing his message, Qwertyuiop was busy typing the message that will appear in line 859. The message appears 1 second after 137's posting and just before he begins adding the pink lines. Hence, the temporal unfolding of actions suggests that these two messages were produced in parallel. In this posting Qwertyuiop makes a reference to an older message where he mentioned the problem of overlapping sticks among the 6 triangular regions. The current message announces that this may not be a big complication. The next sentence in the same post states that the overlaps can be accounted for with the expression "**-6n**". 137's response (as suggested by his use of the explicit reference) to Qwertyuiop's proposal comes after he is done with marking the pink lines on the whiteboard. The "**oh**" in line 861 makes 137's noticing of Qwertyuiop's proposal. In his next posting, 137 states that he prefers addition rather than subtraction. The contrast made between addition and subtraction suggests that 137 is treating his and Qwertyuiop's methods as distinct but related approaches to the task at hand.

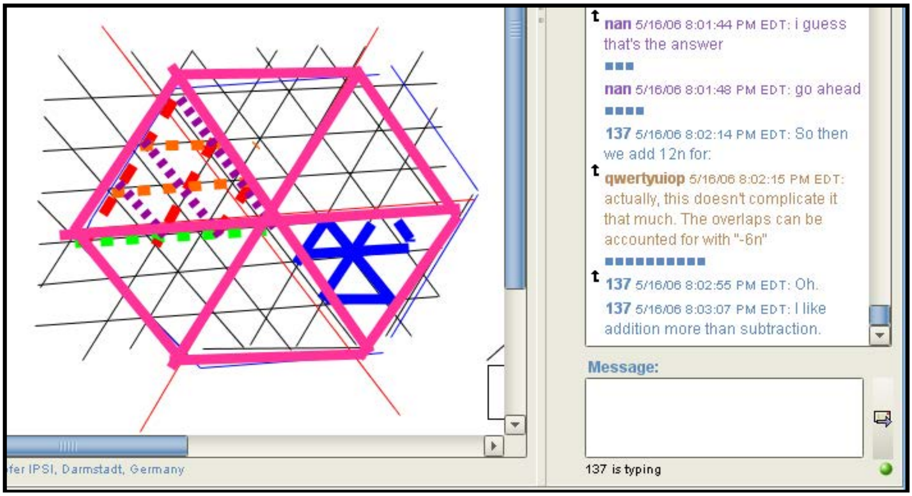


Figure 17: 137's posting "So then we add 12n for:" is followed by his drawing work where he adds the pink lines. Again the temporal sequencing suggests that the pink lines show visually which sticks will be covered when the proposed computation is performed (i.e., "adding 12n")

What 137 is referring to as an "**additive**" approach can be observed through his prior actions distributed across both interaction spaces. 137's approach begins with a method to cover a specific portion of one of the six partitions of the hexagon. This is referred as "**multiplying the orange by three**" and the outcome of this operation is marked in blue. In other words, the orange lines seem to be used as a way to index a single side of a total of 1+2 = 3 triangles (or n(n-1)/2 in general) inside one of the 6 partitions. Hence, multiplying this value by 3 covers the 3 blue triangles enclosed in a partition. Moreover, none of these triangles share a stick with the diagonals and the boundary of the hexagon, so the sticks highlighted in pink are added to cover the missing sticks. In short, 137's reasoning for the additive approach is evidenced in his drawing actions as well as in the way he coordinated his chat postings with the drawings.

The other approach referred to as "**subtraction**" by 137 has been discussed by the team for a while. This approach starts with counting the sticks for one of the six partitions of the hexagon. A partition is further split into "**3 collinear sets**" of sticks that do not "**overlap"** with each other. The number of sticks covered by a single set turned out to be equivalent to a "**triangular number"**. Nevertheless, since this approach covers all the sticks forming a partition and partitions share a boundary with their neighbors, when this value is multiplied by 6 to cover the whole hexagon, the sticks at the boundaries (i.e., at the diagonals) would be counted twice. This is referred to by the team as the **overlap** problem. Qwertyuiop's latest proposal provides the expression that needs to be subtracted from the general formula to make sure all sticks at the internal boundaries are counted exactly once. In contrast, the additive approach does not need subtraction since it splits the shape in such a way that each stick is counted exactly once.

The main point we would like to make about this excerpt is that 137's approach takes the previously demonstrated approaches and their critiques as resources. He offers a new approach informed by previous discussion in an effort to address the practical issues witnessed (e.g., overlaps, adjusting the index in the expression for triangular numbers, etc.). Hence, 137's additive approach is firmly situated within the ongoing discussion. In other words, 137's reasoning has been *socially shaped*; it is not a pure cognitive accomplishment of an individual mind working in isolation from others.

Excerpt 8: Derivation of the formula for the number of sticks

Chat Index	**Time Start Typing**	**Time of Posting**	**Author**	**Content**	**Refers to**
862	20:03:11	20:03:16	Qwertyuiop	do you see why that works	Message No. 859
863	20:03:18	20:03:18	Qwertyuiop	?	
864	20:03:12	20:03:29	137	So: 9n(n+1)-6n.	
865	20:03:41	20:03:45	Qwertyuiop	9, not 3?	
866	20:04:13	20:04:14	137	?	Message No. 865
867	20:04:18	20:04:35	Qwertyuiop	you have "9n(n..."	
868	20:04:37	20:04:47	Qwertyuiop	not "3n(n..."?	
869	20:04:51	20:05:00	137	But we need to multiply by 6 then divide by 2	Message No. 868
870	20:05:10	20:05:22	Qwertyuiop	x6 and /2 for what?	Message No. 869
871	20:05:44	20:05:47	137	FOr each triangle	
872	20:05:48	20:06:02	137	and /2 because it's part of the equation.	
873	20:06:03	20:06:06	137	of n(n+1)/2	
874	20:05:36	20:06:20	Qwertyuiop	it's x3 for the 3 colinear sets, then x6 for 6 triangles in a hexagon... where's the 9 and 2?	

875	20:06:28	20:06:28	Qwertyuiop	Oh	Message No. 872
876	20:06:35	20:06:38	137	So 18/2.	
877	20:06:42	20:06:50	137	A.K.A. 9	
878	20:06:48	20:07:08	Qwertyuiop	(n(n+1)/2)x3x6	Message No. 873
879	20:07:14	20:07:15	137	Yeah.	
880	20:07:20	20:07:27	Qwertyuiop	Which can be simplified...	
881	20:07:42	20:07:46	137	To 9n(n+1)	Message No. 880
882	20:08:01	20:08:04	Qwertyuiop	that's it?	Message No. 881
883	20:08:10	20:08:12	137	-6n.	
884	20:08:17	20:08:24	137	So 9n(n+1)-6n	
885	20:08:20	20:08:34	Qwertyuiop	i'll put it with the other formulas...	

Excerpt 8 immediately follows the prior one. It begins with Qwertyuiop's question addressed to 137, which asks if he could see why subtracting 6n would work. In the meantime, 137 seems to be busy typing the message that will appear in line 864. The use of "**So**" suggests that this message is stated as a consequence of what has been discussed so far. The colon is followed by the formula **"9n(n+1)-6n**", which involves the term "**-6n**" in it. By using the term "**-6n**", 137 makes his orientation to Qwertyuiop's proposal explicit. Moreover, the sequential build up suggests that the proposed expression stands for the formula for the number of sticks for the hexagonal array. In these ways—through the details of its contextual situating—the symbolic expression is tied to the on-going discourse, including the graphical features.

Qwertyuiop's next posting in line 864 problematizes the appearance of **9** in the proposed formula and asks if **3** should have appeared there instead. Next, 137 posts a question mark linked to Qwertyuiop's question, which can be read as a request for more elaboration. Qwertyuiop elaborates in the next two lines by posting the part of the formula that is problematic for him and then by suggesting a repair for that part. His elaboration ends with a question mark that can be seen as an attempt to solicit his peer's feedback. 137's reply in line 869 states that the steps of the computation should also include multiplication by 6 and division by 2. In response Qwertyuiop asks for what part of the pattern those operations need to be done. 137's reply spans 3 lines, where he first states "**for each triangle**" and then mentions that "**/2**" comes from the equation n(n+1)/2. Hence the sequential organization of these messages suggest that 137 associates multiplication by 6 with the triangles (i.e., the larger triangular partitions) and "**/2**" with the equation for triangular numbers.

In the meantime, Qwertyuiop has been typing what will appear in line 874. The first sentence associates each multiplication operation with a specific section of the hexagonal pattern, namely "**x3**" for the 3 "**collinear sets**" within a triangular partition and "**x6**" for the 6 triangular partitions of the hexagon. The next sentence in that posting problematizes again the appearance of 9 and 2 in the steps of the calculation. Eight seconds later, Qwertyuiop posts "**oh**" in response to 137's remark about the equation in line 872, which indicates that the referenced message has led him to notice something new. This is followed by 137's demonstration of the derivation of 9 from the numbers previously mentioned. Meanwhile, Qwertyuiop is composing an expression that brings all the items they have just talked about together in symbolic form, which appears in line 878 in response to line 873 where 137 reminded him about the equation n(n+1)/2. 137 expresses his agreement in the next line. Next, they simplify the expression and add"**-6n**" to derive the final formula for the number of sticks.

In short, the episode following 137's proposal shows that Qwertyuiop had trouble understanding how 137 derived the formula he reported in line 864. 137 seems to have gone ahead with putting together all the different pieces of the problem that have been discussed so far to produce the final formula. Note that the additive approach 137 was describing earlier included a step summarizing the pink boundary as 12n, which also includes the diagonals causing the overlap issue. The commonality between the two lines of reasoning may have informed 137's quick recognition of the algebraic implication of Qwertyuiop's subtraction move as an alternative to his approach.

Qwertyuiop's problematizations of some of the terms that appear in the proposed formula have led 137 to reveal more details of his algebraic derivation. This exchange has revealed how each algebraic move is based on the corresponding concept the team had developed earlier (e.g., n(n+1)/2 sticks to cover a collinear set, multiply by 3 to cover 3 collinear sets making up a triangular partition, multiply by 6 to cover the hexagon, subtract 6n to remove those sticks at the internal boundaries that are counted twice). 137's contributions in this and the previous excerpts demonstrate that he can competently associate the narrative descriptions and visual representations with symbolic formulas. Qwertyuiop's initial trouble and its resolution in the last excerpt provided us further evidence with regards to how participants made use of the narrative/geometric resources to co-construct a generalized symbolic formula addressing the problem at hand. In short, the team members complemented each other's skills as they incorporated geometric and algebraic insights proposed by different members into a solution for the task at hand during the course of their one hour long chat session.

Discussion

In this section we discuss the findings of our case study regarding the affordances of a multimodal CSCL environment for joint mathematical meaning making online and the interactional organization of mathematics discourse.

Visibility of the Production Process

Our first observation is related to the mathematical affordances of the drawing area. As we have seen in Excerpts 1, 2, 4, and 7, the construction of most shared diagrams includes multiple steps (e.g., addition of several lines). Moreover, the object-oriented design of the whiteboard allows users to re-organize its content by adding new objects and by moving, annotating, deleting and reproducing existing ones. Hence, the sequencing of drawing actions that produce and/or modify these diagrams is available for other members to observe. In other words, the whiteboard affords an *animated evolution* of the shared space, which makes the *reasoning process visually* manifest in drawing actions available for other members to observe. For instance, the sequence of drawing actions that led to the drawing displayed in Figure 13 (Excerpt 4) allowed the team members to locate what was indexed by the term "set of 3 collinear sides." The drawing also served as a visual proof for the argument that those three sets do not share any sticks (i.e., they do not overlap). Finally, Figures 16 and 17 show cases where a textually described algebraic operation was subsequently animated on the whiteboard. Such *demonstrable tweaks* make the mathematical details of the construction work visible and relevant to observers, and hence serve as a vital resource for joint mathematical sense making.

Persistent Presence of Contributions

In the VMT online environment, contributions have a *persistent* presence that allows participants to revisit a prior posting or reorganize a shared drawing to orient themselves to shared artifacts in new ways. One important consequence of persistence is illustrated by Qwertyuiop in Excerpts 4 and 5 (lines 803 and 818) and by 137 in Excerpt 3 (line 776), where they used the explicit referencing tool to point to a previous chat posting in an effort to re-initiate a past topic or thread. When combined with the referential arrows, the persistent availability of the chat messages affords re-initiation of past conversations and the management of multiple threads (e.g., the discussion on a missing stick and the formula for triangular numbers that unfolded in parallel in Excerpts 2 and 3 illustrates how users manage multiple threads).

One important consequence of quasi-synchronous interactions mediated by a persistent display of text messages is that participants are not subjected to the same set of physical constraints underlying the turn-taking apparatus associated with talk in face-to-face settings. In natural conversations, speakers take turns due to the practical intelligibility issues involved with overlapping speech. In contrast, the persistent availability of the text messages affords simultaneous production of contributions, and hence provides more possibilities for participation. This may introduce intelligibility issues referred to as chat confusion (Fuks, Pimentel & de Lucena, 2006) or phantom adjacency pairs (Garcia & Jacobs, 1998), when simultaneously produced messages can be mistakenly treated in relation to each other. However, as we have seen in the excerpts analyzed above, participants routinely provide enough specificity to their contributions (e.g., by using the referential tool or specific tokens) and orient to the temporal/linear order in which messages appear on the screen to avoid such issues of intelligibility. Finally, when coupled with resources such as the explicit referencing tool and repetition of specific terms (e.g., "sidelength"), the persistency of chat messages also allows participants to make a previous discussion relevant to the current discussion. For instance, in line 818 in Excerpt 5, Qwertyuiop re-oriented the current discussion to the issue of devising a formula for the sequence of numbers that was stated back in line 772 by using the explicit referencing tool. Likewise, in line 841 in Excerpt 6 Qwertyuiop proposed that the team re-initiate a discussion on a point stated 13 lines above with his message "go back to this" coupled with an explicit referential link.

The possibility of engaging activities across multiple threads spanning both chat and whiteboard spaces is an important affordance of online environments like VMT due to the opportunities it brings in for more people to contribute to the ongoing discussion. For instance, in Excerpt 4 we have seen that 137 was engaged in two simultaneous threads where (a) he drew a line segment that was potentially ignored by the method of computation described by Qwertyuiop, and (b) he contributed to the simultaneously unfolding discussion about characterizing the pattern implicated by the numbers offered by Qwertyuiop as triangular numbers. Although the management of multiple threads across spaces can create confusion, the resolution of ambiguities and the intertwining of perspectives can lead to germination/fertilization of mathematical ideas across threads. This point is well demonstrated by how the aforementioned threads led to Qwertyuiop's visual proof, which (a) located visually what the term "3 sets of collinear lines" meant, (b) established that the sets do not overlap with each other, and (c) highlighted the association between the cardinality of a single set and a triangular number.

Finally, there is a subtle but important difference between the chat and whiteboard features in terms of the degree of persistence of their contents. As a session progresses chat postings gradually scroll away, but whiteboard drawings stay on the whiteboard until they are erased. For instance, in all the excerpts we have seen above, the particular illustration of the hexagonal pattern continued to serve as an interactional resource as team members illustrated and offered different ideas. Several chat postings presume the availability of such a persistent resource on display so that others can make sense of the contribution (e.g., indexical terms such as "the orange", "3 sets", etc.). Such persistently available artifacts provided the background against which new contributions were interpreted and made sense of.

Methods for Referencing Relevant Artifacts in the Shared Visual Field

Bringing relevant mathematical artifacts to other members' attention requires a coordinated sequence of actions performed in both the chat and whiteboard spaces (Stahl et al., 2011). In the excerpts above we have observed several referential methods enacted by participants to bring relevant graphical objects on the whiteboard to other group members' attention. In Excerpt 1, 137 *marked the drawing* with a different color to identify what he thought collinear sides meant in reference to the shared drawing. Qwertyuiop also used the same approach when he highlighted the collinear sides in the shared drawing with different colors in Excerpts 1 and 3. *Color coding* was another method used by members to draw others' attention to specific parts of the drawing (e.g., "the orange", "the green times 3"). Finally, members used the explicit referencing tool to support their *textual descriptions*. For instance, Qwertyuiop used the explicit referencing tool in Excerpts 2 and 4 to direct his teammates' attention to the relevant section of the hexagon where he was performing his counting work. In all these cases, chat messages included either an explicit reference or a deictic term such as "this", "that", or "the green", which are designed to inform other members of the group that they need to attend to some features beyond the textual statement itself to make sense of the chat message.

These referential mechanisms play a key role in directing other members' attention to features of the shared *visual field* in particular ways. This kind of deictic usage isolates components of the shared drawing and constitutes them as relevant objects to be attended to for the purposes at hand. Hence, such referential work establishes a fundamental *relationship between the narrative and mathematical terminology used in text chat and the animated graphical constructions produced on the whiteboard*. The shared sense of the textual terms and the inscriptions co-evolve through the referential linkages established as the interaction sequentially unfolds in the dual-interaction space.

Deictic uses of text messages and drawings presume the availability of a *shared indexical ground* (Hanks, 1992) where the referential action can be seen as the *figure* oriented towards some part of the shared *background*. In other words, referential moves are not performed in isolation; they rely on a part/whole relationship between the referential action (i.e., figure) and a shared visual ground. For example, the color markings of collinear lines in Excerpt 4 worked as a referential action, because they were performed on top of an existing graphical artifact, namely the triangular grid. Even the design of the explicit referential tool, which attaches a semi-transparent green rectangle to a chat message, reflects this visual relationship between the figure (i.e., the green rectangle) and the background, which guides other members' attention to a particular location in the shared visual field. As virtual teams collaboratively explore their problem and co-construct shared artifacts, they collectively constitute a shared problem space with increasing complexity (Sarmeinto & Stahl, 2008). By enacting referential practices, participants isolate features of the shared scene, assign specific terminology to them, and guide other members' perception of the ongoing activity to achieve a shared mathematical vision.

Coordination of Whiteboard Visualizations and Chat Narratives

The previous section focused on single actions that refer to some feature of the shared scene for its intelligibility. We argued that such actions involve a part/whole relationship that presumes the availability of a shared visual ground for their mutual intelligibility. In addition to this, such actions are also embedded within broader sequences of actions that establish their relevance. In other words, messages that establish a referential link between narrative and graphical resources routinely respond to practical matters made relevant or projected by prior actions. Thus, such actions are also tied to the *context* set by the sequentially unfolding discussion.

When the scope of analysis is broadened to sequences of actions that include messages with referential links, one can observe an important affordance of online environments with multiple interaction spaces: Since one can contribute to only one of the interaction spaces at a time, a participant cannot narrate his/her whiteboard actions with simultaneous chat postings, as can be done with talk in a face-to-face setting. However, as we have observed in 137's performance in Excerpts 1 and 7, participants can achieve a similar interactional organization by temporally coordinating their actions in such a way that whiteboard actions can be seen as part of an exposition performed in chat.

For instance, in Excerpt 1, Qwertyuiop's drawing activity was prefaced by his chat posting "I'll draw it". The posting was in response to a recent graphical illustration proposed by 137. Hence, the pronoun "it" included in the preface was not pointing to an existing drawing or to a prior posting. Instead, it *projected* a subsequent action to be performed next by the same author. In contrast, prior to Qwertyuiop's actions in Excerpt 1, 137 produced his drawings before he was seen as typing by others. Although the sequence of the chat and whiteboard actions are the opposite in this case (i.e., the referential move was made after the drawing was finished), 137 achieves a similar temporal organization through his use of deictic terms (e.g., "those", "that", "it"), referential arrows, and tokens of similarity such as "like" and "as". Therefore, these instances suggest that, although they can be ordered in different ways, the sequential organization and temporal proximity of actions are consequential for the treatment of a set of drawing actions in relation to a narrative account produced in chat.

In face-to-face settings, locational deictic terms such as "this" and "those" are used to point out contextual elements beyond the lexical content of the uttered statement, and they are often accompanied by co-occurring pointing gestures and body movements displaying the speaker's orientation towards what is being referred to in the vicinity (Hanks, 1992; Goodwin, 2000). As demonstrated by the actual cases of use in the excerpts analyzed above, a similar organization presents an interactional challenge for the participants in an online setting with dual interaction spaces like VMT. However, as participants demonstrated in these excerpts, a functionally comparable interactional organization can be achieved online through the use of available features so that chat messages can be seen as related to shared drawings that are either on display ("those") or in production ("these"). The sequential organization of actions, explicit referencing, and the temporal proximity of actions across both spaces together guide other members' attention so that they can treat such discrete actions as a coherent whole addressed to a particular prior message or to a thread of discussion unfolding at that moment.

Another important aspect of such achievements from a mathematics education perspective is that it shows us how *saming*[17] (Sfard, 2008) among narrative and graphical accounts or realizations can be done as an interactional achievement across dual-interaction spaces. This phenomenon is demonstrated in various episodes such as (a) Qwertyuiop's demonstration of collinear set of lines on the shared diagram in Excerpt 4, and (b) 137's exposition in Excerpt 7, where he showed the geometric implication of his proposal in narrative form by performing a drawing immediately after his chat message. The referential links, the temporal proximity of actions, the awareness indicators for those actions, and the persistent availability of both prior messages and the recently added drawings all work together as a semiotic system that allows group members to make connections among different realizations of the mathematical artifacts that they have co-constructed. Therefore, referential practices across modalities are consequential for the collective achievement of deep understanding of mathematics, which is characterized in mathematics education theory as establishing relationships between different realizations of mathematical ideas encapsulated in graphical, narrative or symbolic forms.

Past and Future Relevancies Implied by Shared Mathematical Artifacts

The objects on the whiteboard and their visually shared production index a horizon of past and future activities. The indexical terms in many proposals made in the analyzed excerpts (like "hexagonal array", "collinear lines", "rows") not only rely on

[17] Sfard (2008) describes saming as the process of "…assigning one signifier (giving one name) to a number of things previously not considered as being the same" (p. 302).

the availability of the whiteboard objects to propose a relevant activity to pursue next, but also *reflexively* modify their sense by using linguistic and semantic resources to label or gloss the whiteboard object and its production. This allows actors to orient in particular ways to the whiteboard objects and the procedures of their co-construction—providing a basis for subsequent coordinated joint activity.

This suggests that shared representations are not simply manifestations or externalizations of mental schemas as they are commonly treated in cognitive models of problem-solving processes. Instead, our case studies suggest that shared representations are used as resources to interactionally organize the ways actors participate in collaborative problem-solving activities. As we have seen in this case study, once produced as shared mathematical artifacts, drawings can be mobilized and acted upon as resources for collective reasoning as different members continue to engage with them. Shared meanings of those artifacts are contingently shaped by these engagements, which are performed against the background of a shared visual space including other artifacts and prior chat messages (i.e., against a shared *indexical ground*). This does not mean that the achievement of shared understanding implies that each member has to develop and maintain mental contents that are isomorphic to each other's, which is often referred as registering shared facts to a "common ground" in psycholinguistics (Clark & Brennan, 1991). Instead, shared understanding is a practical achievement of participants that is made visible through their reciprocal engagements with shared mathematical artifacts.

The way team members oriented themselves to the shared drawing while they were exploring various properties of the hexagonal array showed that the drawings on the whiteboard have a figurative role in addition to their concrete appearance as illustrations of specific cases. In other words, the particular cases captured by concrete, tangible marks on the whiteboard are routinely used as resources to investigate and talk about the general properties of the mathematical artifacts indexed by them. For example, the particular drawing of the hexagonal pattern in the excerpts studied above was illustrating one particular stage (i.e., n=3), yet it was treated in a generic way throughout the whole session as a resource to investigate the properties of the general pattern implied by the regularity/organization embodied in that shared artifact. Noticing of such organizational features motivated the joint development of counting practices, where relevant components of the pattern were first isolated and then systematically counted.

Another important aspect of the team's achievement of general formulas, which summarize the number of sticks and triangles included in the n^{th} case respectively, is the way they transformed a particular way of counting the relevant objects in one of the partitions (i.e., a geometric observation) into an algebraic mode of investigation. For instance, once the team discovered that a particular alignment of sticks that they referred to as "collinear sides" corresponded to triangular numbers, they were able to summarize the sequence of numbers they devised into the algebraic formula 9n(n+1)-

6n. The shift to this symbolic mode of engagement, which relied on the presence of shared drawings and prior narratives as resources, allowed the team to progress further in the task of generalizing the pattern of growth by invoking algebraic methods. In other words, the team co-constructed general symbolic formulas for their shared tasks by making *coordinated* use of multiple realizations (graphical and linguistic) of the mathematical artifact (the hexagonal array) distributed across the dual-interaction spaces.

Conclusion

Perhaps the most important contribution of online learning environments like VMT to research is that they make the collective mathematical meaning-making process visible to researchers through their logs. This allows us to explore the mechanisms through which participants co-construct mathematical artifacts in graphical, narrative and symbolic forms; and to study how they incrementally achieve a shared understanding of them. Careful analysis of team members' actions helps us identify important affordances (i.e., possibilities and limitations on actions) of digital environments for supporting collaborative discussion of mathematics online.

Our analysis reveals that group members display their reasoning by enacting *representational affordances* of online environments like VMT. The persistent nature of the contributions and the availability of their production/organization allow other participants to *witness the mathematical reasoning* embodied in those actions. Group members establish relevancies across graphical, narrative and symbolic realizations of mathematics artifacts by enacting the referential uses of the available system features. Verbal references, highlighting a drawing with different colors, and the explicit referencing feature of the system are used to establish such relationships between contributions. Through referential practices group members:

(a) isolate objects in the shared visual field,

(b) associate them with local terminology stated in chat, and

(c) establish sequential organization among actions performed in chat and whiteboard spaces, which can be expressed in algebraic symbolism.

Finally, this case study also showed us how mathematics terminology comes into being in response to specific communicational needs. Mathematical discourse has a deeply indexical nature; mathematics terminology often encodes certain ways of thinking about mathematical objects. As we have seen in the excerpts above, terminology such as "sides", "collinear set of sides", etc., emerge from the need to talk about and direct others' attention to specific aspects of the task at hand. Such glosses, names or indexicals become meaningful *mathematical narrative artifacts* through the ways participants enact them by organizing the shared space in particular ways and/or referring to some part of a drawing or a previous chat posting. Once a shared sense of a term is established in interaction, subsequent uses of the term encode certain ways of constructing/grouping/organizing some items and begin to serve as a convenient way to refer to an overall strategy of looking at a problem in a particular way. The term may then lead to a symbolic expression, drawing upon associated practices of computation and manipulation.

In short, *mathematical understanding at the group level is achieved through the organization of representational and referential practices*. Persistent whiteboard objects and prior chat messages form a shared indexical ground for the group. A new contribution is shaped by the indexical ground (i.e., interpreted in relation to relevant features of the shared visual field and in response to prior actions); it reflexively shapes the indexical ground (i.e., gives further specificity to prior contents) and sets up relevant courses of action to be pursued next. Shared mathematical understanding is an observable process, a temporal course of work in the actual indexical detail of its practical actions, rather than a process hidden in the minds of the group members. Deep mathematical understanding can be located in the *practices of collective multimodal reasoning* displayed by teams of students through the sequential and spatial organization of their actions. Mathematical results are reached through a sequence of discourse interactions that build successively (Stahl, 2011). The discourse moves within the media of graphical constructions, narrative terminology and manipulable symbolisms, allowing progress to be made through visual means, counting skills, encapsulation of knowledge in words, and generalization in symbols.

References

Barron, B. (2003). When smart groups fail. *The Journal of the Learning Sciences. 12*(3), 307-359.

CCSSI. (2011). High school -- geometry. In Common Core State Standards Initiative (Ed.), *Common core state standards for mathematics.* (pp. 74-78)

Çakır, M. P. (2009). The joint organization of visual, narrative, symbolic interactions. In G. Stahl (Ed.), *Studying Virtual Math Teams* (pp. 99-141). New York, NY: Springer.

Çakır, M. P., Zemel, A., & Stahl, G. (2009). The joint organization of interaction within a multimodal CSCL medium. *International Journal of Computer-Supported Collaborative Learning, 4*(2), 115-149.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. R. Romberg (Eds.), *Mathematics classrooms* *that promote understanding* (pp. 19-32). Mahwah, NJ: Lawrence Erlbaum Associates.

Clark, H. H., & Brennan, S. (1991). Grounding in communication. In L. B. Resnick, J. M. Levine & S. D. Teasley (Eds.), *Perspectives on Socially Shared Cognition* (pp. 127-149). Washington DC: American Psychological Association.

Dillenbourg, P., & Traum, D. (2006). Sharing Solutions: Persistence and Grounding in Multimodal Collaborative Problem Solving. *The Journal of the Learning Sciences, 15*(1), 121-151.

Dohn, N. B. (2009). Affordances revisited: Articulating a Merleau-Pontian view. *International Journal of Computer-Supported Collaborative Learning, 4*(2), 151-170.

Fuks, H., Pimentel, M., & de Lucena, C. (2006). R-U-Typing-2-Me? Evolving a chat tool to increase understanding in learning activities. *International Journal of Computer-Supported Collaborative Learning, 1*(1), 117-142.

Garcia, A., & Jacobs, J. B. (1998). The interactional organization of computer mediated communication in the college classroom. *Qualitative Sociology, 21*(3), 299-317.

Garfinkel, H. (1967). *Studies in Ethnomethodology*. Englewood Cliffs, NJ: Prentice Hall.

Garfinkel, H., & Sacks, H. (1970). On formal structures of practical actions. In J. Mckinney & E. Tirvakian (Eds.), *Theoretical sociology: Perspectives and developments* (pp. 337-366). New York, NY: Appleton-Century-Crofts.

Goodwin, C. (2000). Action and embodiment within situated human interaction. *Journal of Pragmatics, 32*, 1489-1522.

Goodwin, C., & Heritage, J. (1990). Conversation Analysis. *Annual Review of Anthropology, 19*, 283-307.

Hanks, W. F. (1992). The indexical ground of deictic reference. In A. Duranti & C. Goodwin (Eds.), *Rethinking context: Language as an interactive phenomenon* (pp. 43-76). Cambridge, UK: Cambridge University Press.

Hanks, W. F. (2000). Intertexts: Writings on language, utterance, and context. Lanham: Rowman & Littlefield.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers. *Mathematical Thinking and Learning, 1*(1), 59-84.

Heritage, J. (2002). Oh-prefaced responses to assessments: a method of modifying agreement/disagreement. In C. Ford, B. Fox & S. Thompson (Eds.), *The Language of Turn and Sequence* (pp. 196-224). Oxford: Oxford University Press.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., et al. (1996). *Making sense: Teachin and learning mathematics with understanding*. Portsmouth NH: Heinemann.

Hiebert, J., & Wearne, D. (1996). Instruction, Understanding, and Skill in Multidigit Addition and Subtraction. *Cognition & Instruction, 14*(3), 251-283.

Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *Journal of the Learning Sciences. 4*(1), 39-103. Web: http://lrs.ed.uiuc.edu/students/c-merkel/document4.HTM.

Kaput, J. (1998). Representations, Inscriptions, Descriptions and Learning: A Kaleidoscope of Windows. *Journal of Mathematical Behavior, 17*(2), 265-281.

Konold, C. (2007). Designing a data analysis tool for learners. In M. Lovett & P. Shah (Eds.), *Thinking with data: The 33rd annual Carnegie Symposium on cognition* (pp. 267-292). Hillside, NJ: Lawrence Erlbaum Associates.

Koschmann, T. (2001). Revisiting the paradigms of instructional technology. In G. Kennedy, M. Keppell, C. McNaught & T. Petrovic (Eds.), *Meeting at the crossroads. Proceedings of the 18th Annual Conference of the Australian Society for Computers in Learning in Tertiary Education* (pp. 15-22). Melbourne: University of Melbourne.

Koschmann, T., Stahl, G., & Zemel, A. (2007). The video analyst's manifesto (or the implications of Garfinkel's policies for the development of a program of video analytic research within the learning sciences). In R. Goldman, R. Pea, B. Barron & S. Derry (Eds.), *Video research in the learning sciences*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lagrange, J. B. (2005). Curriculum, classroom practices and tool design in the learning of functions through technology-aided experimental approaches. *International Journal for Computers for Mathematical Learning, 10*(2), 143-189.

Lenhart, A., Madden, M., Macgill, A. R., & Smith, A. (2007). Teens and social media. from http://www.pewinternet.org/~/media//Files/Reports/2007/PIP_Teens_Social_Media_Final.pdf.pdf

Livingston, E. (1987). *Making Sense of Ethnomethodology*. London: Routledge and Kegan Paul.

Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *International Journal of Computer-Supported Collaborative Learning, 1*(4), 441-465.

Mühlpfordt, M., & Stahl, G. (2007). *The integration of synchronous communication across dual interaction spaces*. Paper presented at the International Conference of CSCL:

Of mice, minds, and society (CSCL 2007). New Brunswick, NJ. Web: http://GerryStahl.net/vmtwiki/martin.pdf.

NCTM. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Olive, J. (1998). Opportunities to explore and integrate mathematics with the geometer's sketchpad. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space.* (pp. 395-418). Mahwah, NJ: Lawrence Erlbaum.

O'Neil, J., & Martin, D. (2003). *Text chat in action.* Paper presented at the The International ACM SIGGROUP conference on Supporting group work, Sanibel Island, Florida, USA.

Sacks, H. (1962/1995). *Lectures on Conversation*. Oxford, UK: Blackwell.

Sarmiento, J., & Stahl, G. (2008). *Extending the joint problem space: Time and sequence as essential features of knowledge building.* Paper presented at the International Conference of the Learning Sciences (ICLS 2008). Utrecht, Netherlands.

Schegloff, E. A., & Sacks, H. (1973). Opening up closings. *Semiotica, 7*, 289-327.

Sfard, A. (2002). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. In C. Kieran, E. Forman & A. Sfard (Eds.), *Learning discourse: Discursive approaches to research in mathematics* (pp. 13-57). Dordrecht, Netherlands: Kluwer.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* New York, NY: Cambridge University Press.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Stahl, G. (2006). *Group Cognition: Computer Support for Building Collaborative Knowledge.* Cambridge, MA: MIT Press.

Stahl, G. (Ed.). (2009). *Studying virtual math teams.* New York, NY: Springer.

Stahl, G. (2011). *How a virtual math team structured its problem solving.* Paper presented at the international conference on Computer-Supported Collaborative Learning (CSCL 2011). Hong Kong, China. Proceedings pp. 256-263.

Stahl, G., Koschmann, T., & Suthers, D. D. (2006). Computer-Supported Collaborative Learning. In R. K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp. 409-426). New York: Cambridge University Press.

Stahl, G., Zhou, N., Çakır, M. P., & Sarmiento-Klapper, J. W. (2011). *Seeing what we mean: Co-experiencing a shared virtual world.* Paper presented at the international conference on Computer Support for Collaborative Learning (CSCL 2011). Hong Kong, China. Proceedings pp. 534-541.

Suthers, D., Connelly, J., Lesgold, A., Paolucci, M., Toth, E., Toth, J., et al. (2001). Representational and Advisory Guidance for Students Learning Scientific Inquiry. In K. D. Forbus & P. J. Feltovich (Eds.), *Smart Machines in Education: The Coming Revolution in Educational Technology* (pp. 7-35). Menlo Park: AAAI Press.

Suthers, D. D. (2006). Technology affordances for intersubective meaning making: A research agenda for CSCL. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 1*(3), 315-337.

ten Have, P. (1999). *Doing conversation analysis, A practical guide*. Thousand Oaks, CA: Sage Publications.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Associates.

Zemel, A., & Çakır, M. P. (2009). Reading's Work in VMT. In Stahl, G. (Ed.), *Studying Virtual Math Teams*. (pp. 261-276). New York, NY: Springer.

4. "You can divide the thing into two parts": Analyzing Referential, Mathematical and Technological Practice in the VMT Environment

Timothy Koschmann, Gerry Stahl and Alan Zemel

Abstract. In keeping with the theme of this year's conference, we direct our attention to the analytic practices through which participants, when interacting via computers make sense of their own and others' actions. Participants' endogenous work of analysis has received little attention in prior research on collaborative learning. We would argue, however, that these are the very practices of greatest relevance for study in CSCL. The materials to be presented here come from the Virtual Math Teams (VMT) Project conducted under the auspices of the Math Forum at Drexel University. In this project, students situated at geographically diverse sites solve math problems together using text-based, synchronous chat communication and a shared graphical whiteboard. We examine the interaction of three students and a faculty moderator in their initial period of problem solving. We find evidence of manifold competencies related to discourse production, mathematics and technology use. We focus on the presentation of a prospective problem solution by one particular student and describe in detail how his practices provide for the analyzability of his actions.

The 'Practice Turn' in CSCL Research

The theme of this year's conference is research on practice. CSCL researchers have displayed an interest in practice from the very inception of the field. This interest might be seen as part of a more general "practice turn" (Schatzki, 2001) that has occurred in the human sciences over the last few decades.

Human practice, of course, is a very broad topic. If CSCL research is to take a practice turn, what kind of practice should we be studying? Lynch (2001), in a paper on the logic of practice, posited a form of "analytic work that is endogenous to the social production of coordinated talk" (p. 132). He wrote:

> For conversation analysts, 'analysis' is a pivotal term that identifies their own methodological activity with the objective domain they investigate. The concerted production of intelligible lines of talk is both the subject and the source of such analysis. (p. 132)

Conversational participants are already engaged in a form of analysis making sense of their own unfolding talk-in-interaction. Conversation Analysis (CA) seeks to study the practices whereby this form of analysis is done and document its underlying logic and methods (see, for example, Sacks, 1992). Our interests here are similar, but instead of studying methods of analysis endogenous to F2F conversation, we direct our attention to computer-mediated communication (CMC). The resources available to participants when interacting through computers are quite different from those in F2F exchanges in which intonation and other features of vocal delivery, gaze, gesture, etc. are so crucially important to sense making. Though these features are absent in CMC, as we will see in the case examined here, it is not without resources for sense making.

The Virtual Math Teams Project

The materials to be discussed come from a corpus assembled at the Math Forum at Drexel University. The Virtual Math Teams (VMT) Project, established in 2003, is one of a variety of programs conducted under the auspices of the Math Forum. In this project, teams of geographically dispersed students use an integrated suite of web-based software tools to explore proposed mathematics topics (Stahl, forthcoming). VMT sessions are run as an enrichment activity conducted outside of the regular school curriculum. Students are recruited through their math teachers at their home schools.

Here we study the interaction between three particular students self-identified as Aznx, Quicksilver and bwang8 (hereafter just "Bwang"), and a Math Forum facilitator ("Gerry"). The three students represented one team (Team B) in the 2006 VMT Spring Fest. Their collaboration continued for four online sessions, each of approximately one hour in length, and spaced out over a two-week period (see Medina, Suthers & Vatrapu [forthcoming] for an overview of Sessions I-III and Stahl [forthcoming] for a description of Session IV).

The VMT software environment supports collaboration at a distance using text-based, synchronous chat communication as well as a shared graphical whiteboard and an asynchronous community-wide wiki (Stahl, forthcoming). A screen image of the VMT user interface can be seen in Figure 1. Their moment-to-moment interaction was recorded by the system and can be replayed in real time using the VMT Replayer application. Unlike a video recording of a F2F encounter, in which we see what the camera operator chose to show us, here we see precisely what was made available to the participants themselves to see (i.e., a correspondence of the participants' and the observers' perspective). These recordings, therefore, provide a rich and comprehensive set of materials for examining practices of collaboration within computer-mediated interaction.[18]

We will examine Team B's initial period of joint activity in Session I. In a message posted early in the session, Gerry, the facilitator, provides instructions for where the worksheet for the first session might be found on the "View Topics" page, establishing the task for the day. Approximately 10 minutes later, Aznx asks, "So how do we submit this?" It would appear that in the intervening period something representing a solution to the posed task had been produced. Our analysis will focus on what that something might be and how it was developed interactionally.

"You can divide the thing into two parts"

Given the constraints of time, we offer here just a sketch of how an analysis might proceed. Let us first begin by examining the task description provided to the team in Session I (see Figure 1 in each of the previous two chapters). It contains three panels: a series of match-stick figures demonstrating a series graphically, a table representing the same series showing the number of match sticks and squares at each stage, and, finally, a list of instructions laying out the task itself. The instructions specify a sequence of actions designed to achieve a curricular goal. They are designed such that when the parties following the instructions reach the end of the directed steps, the instruction followers will have been led to a new understanding of some curricular matter. Such is the work of instruction (c.f., Lynch, 2000).

The curricular matter in this case is made visible in the two numeric series labeled in the table, "sticks" and "squares." The progression in both cases is based upon a

[18] It might be worth noting that the three co-authors conducted all but one of the research meetings to plan this report in the same environment and using the same tools as the participants.

simple summation function ($\Sigma_{i=1 \text{ to } N}$ (i) = 1 + 2 + 3 + … + N = (N+1) N/2), one employed ubiquitously in probability theory and statistics. The worksheet instructions are artfully designed to build not only toward an understanding of how this function arises in a variety of series, but also to familiarize the participants with the affordances of the VMT interface. The first task instruction asks the students to graphically represent, as match-stick figures, the next three elements in the series. This presumably provides a resource for then satisfying the second task step—filling in the next three rows in the table. The third step builds on the previous two and asks the students to articulate a "pattern of growth" for the series representing the number of sticks and squares.

A record of the team's interaction can be found in the Appendix. As in most chat interfaces, text, in the VMT environment, is composed in a "message entry box" (Garcia & Jacobs, 1999). When a carriage return is entered, the message is dispatched to the chat server and displayed in a serial list of postings visible to all in the "posting box" (Garcia & Jacobs, 1999). Prior to dispatching a chat post to the server, its content is only visible to the person typing it, but the fact that that person is preparing a message is made available to the others (e.g., "bwang8 is typing").[19] Though the participants are situated at different sites, therefore, their projected actions are available for the others to monitor.

Having located the worksheet, Bwang types, "are we supposed to solve it now?" (post 42). Posed as a procedural inquiry, his post addresses the interactional problem of how one might initiate concerted activity under circumstances in which one's collaborators are not co-present. His note not only displays his readiness to begin, but also characterizes the nature of the team's work as finding a solution. Bwang's query is nominally directed to the moderator who does not respond, but continues to provide instructions that will allow all team members to access the problem statement. Aznx (post 53) subsequently announces his own readiness to begin ("Let's start this thing.") But Bwang has already started.

He begins with an assertion that we have appropriated as the title for this talk ("you can divide the thing into two parts"). It is one that would appear to be riddled with referential puzzles. The referent of "the thing" is ambiguous. Perhaps it co-references the same matter as "it" in his earlier post (#42). But if that is so, and "the thing" references the problem that they are to solve, what exactly are they taking that to be? Is it one of determining how to perform the first assigned instruction? Is it related to computing the number of sticks and/or squares? Or does it have to do with the more general problem of seeing "a pattern of growth"? Given this uncertainty with regard to what "the thing" might be, we are even less

[19] If the message is not posted, the interval is marked in the Appendix as "Initiates a chat message but deletes without posting."

secure in our grasp of what "dividing it into two parts" might signify. Rather than seeking clarification, however, Bwang's correspondents "trust" (Garfinkel, 1963) that all these matters will be made clear in time.

Bwang wastes no time in making plain just what "this thing" might be. The VMT interface is a "dual-interaction space" (Çakir, Zemel, & Stahl, 2009), including not only a chat facility, but also a whiteboard panel. Actions preformed on the whiteboard (e.g., creation of a text or graphic object) are persistently available for all to see. Immediately after his post, Bwang turns to the whiteboard and scribes a series of lines. The resulting gestalt resembles a reconstruction of the third figure from the worksheet, opened like a book and isolating its vertical and horizontal elements (see Fig. 1). Though we now have a visual resource to help us resolve what "dividing the thing into two parts" might mean, we are still left unclear about how restructuring the third figure from the worksheet in this particular way is connected to the task at hand. It does not seem to be an action authorized by any of the worksheet instructions.

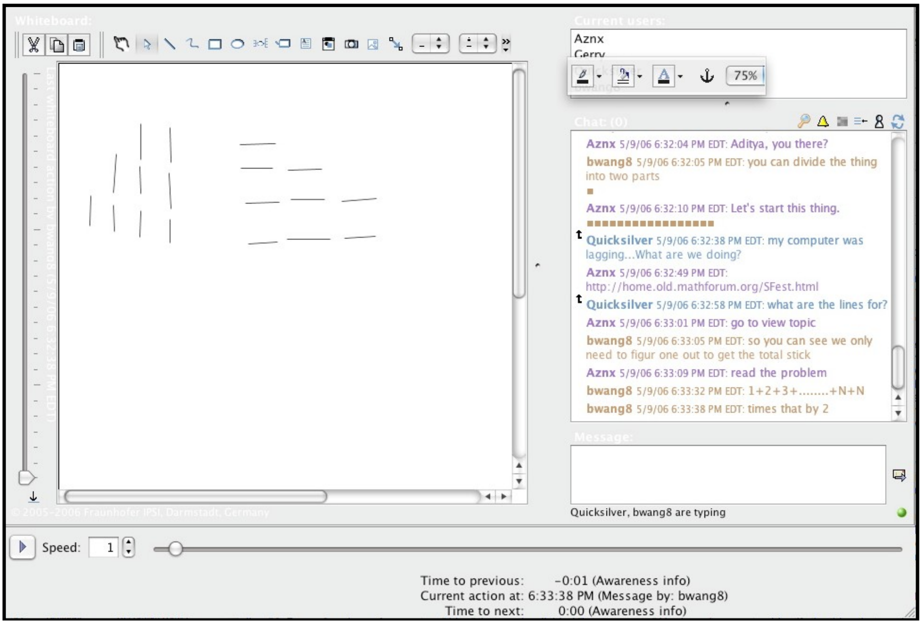


Figure 1. Screen image of the VMT interface.

On completing the last line on the whiteboard, Bwang returns to the chat panel and types, "so you can see we only need to figure one out to get the total stick" (post 58).

Chat posts are often not constructed grammatically as complete sentences, but consist instead of clausal units that must be re-composed by the reader to produce coherent utterances. This practice of building up utterances in installments allows readers to more closely monitor utterances in construction and increases interactivity (Garcia & Jacobs, 1999). Bwang's post, therefore, is read as part of an utterance in progress. The concatenated message, therefore, reads, "so you can see we only need to figure one out to get the total stick 1+2+3+……..+N+N times that by 2".

Bwang, through his actions, has cast the group's task as one of producing certain general formulas related to the generated patterns. Aznx's response, "Can we collaborate this answer even more? To make it even simpler?" (posts 63 & 64) is not clear. It is odd to find collaborate used as a transitive verb and there are ambiguities of meaning. Is he referring to the problem, which could potentially be further decomposed and clarified, or to Bwang's algebraic formulation? By labeling Bwang's contributions as "this answer", Aznx (post 63) implicitly endorses it as a candidate solution to the task at hand. It is, in fact, the first place in which Bwang's presentation is treated as such.

Bwang responds to Aznx's query by providing an algebraically restructured version of the right side of the formula ("(1+N)*N/2+N)*2", post 67). The rapidity with which this was produced would seem to allow little or no time for derivation, suggesting that the revised formula might have already been known before Aznx asked for it. In the posts that followed (69 to 85), the team continued to discuss the components of the developed formula. Bwang introduced a second formula for computing the number of squares (post 82).

It is, in fact, just the simple summation function. It is possible, though we have no way to know, that Bwang first recognized that the 'squares' series was based on a simple summation function and then extended this insight to produce the more elaborate formula for generating the 'sticks' series. Aznx's "so how do we submit this" (post 85) is closure implicative. His this casts a broad net over the whole approach developed by Bwang.

Referential, Mathematical and Technological Practices

We would now like to make certain general observations about the practices on display here. They evidence manifold competencies with regard to discourse production, mathematics and technology use. Bwang's elegant presentation of a prospective solution begins with a proleptic reference to "dividing this thing into two parts." It is not, however, until we get to the end of the presentation that we discover that "the thing" is not just the third figure in the worksheet, but a general formula for describing the patterns seen both in the set of figures and in the summarizing table. It is in the ways in which the functional description itself is revealed that we see the most profound evidence of mathematical practice.

Bwang's presentation of a prospective solution exhibits the properties of a derviation of sorts. It proceeds in logical steps that lead eventually to a known conclusion. The presentation had three parts: a graphical derivation, an informal formulaic presentation and, finally, a more conventional algebraic formulation.

The graphic presentation proceeded in four stages: [1] drawing 6 vertical lines, [2] drawing 6 horizontal lines, [3] drawing 3 vertical lines and, [4] producing 3 horizontal lines (see Fig. 2). The first six lines represent an application of the summation function, and the second six, a second application of the same function. But, the two subfigures are plainly incomplete—they are both missing an outer wall. The number of sticks needed to compete the subfigures is, in both cases, 3, which happens to be N. The final line count is: (6 + 3) + (6 + 3) = 18. Like a mathematician's board-work (c.f., Greiffenhagen, 2008), Bwang's presentation makes visible just how the 'sticks' series is generated. Had he chosen to present the case for N=4, as the first instruction step required, he and his audience would not have had a way to confirm the result. A feature of his demonstration was that the total number of lines produced could be checked against the value provided in the table.[20]

Bwang's "So you can see" (post 58) announces a derivation complete. Raymond (2004) described how "Speakers regularly use 'so' prefaced turn constructional units that articulate the upshot of prior talk to mark the completion of complex turns or activities" (p. 186). In this case, Bwang's opening bridges back, not to prior talk, but to what he had done on the whiteboard. The upshot is presented as already visible for all to see, but just what are we to see?

Like his previous "you can divide the thing into two parts," Bwang's "we only need to figur one out" is as rife with referential puzzles. One what? If the "thing" mentioned in his earlier post has now been divided in two, then each subfigure might be a candidate, but just what are we 'figuring out'? This is clarified when we

[20] There is probably more that could be said here. There is something about the selection of the N=3 case which gives enough scope for development of the more generic understanding that Bwang is seeking to achieve. In part, one could attribute the selection of the N=3 case to the peculiar affordances of the whiteboard demonstration. For N<3, the "four stage" presentation would not have been as effective or could have led to confusion on the part of others witnessing the construction. Because whiteboard actions cannot be narrated like traditional mathematical board-work (cf., Greiffenhagen, 2008), running through N=1 and N=2 before getting to N=3 would have been time consuming and pedantic

get to the end of this long post and are informed that the object is "to get the total stick."

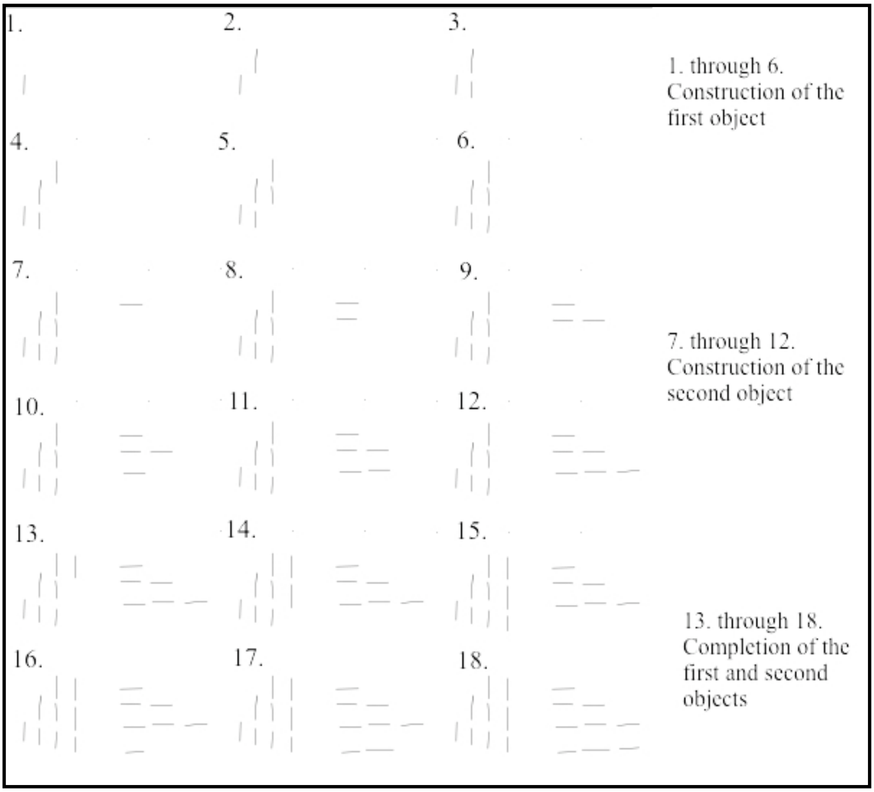


Figure 2. Bwang's re-construction of the third figure from the worksheet.

With this information in hand, the viewer can turn to the "stick" column in the table from the "View Topics" page, extract the entry for N=3 and check it against the count of sticks drawn on the whiteboard. This last part of the post, when concatenated with the two subsequent posts can be read as an informally presented equation (i.e., sticks (N) = (1 + 2 + 3 + … + N + N) 2). The first post informally presents the left-hand side of a functional equation; the second summarizes the two-stage production of each of the subfigures (i.e., $\Sigma_{i=1 \text{ to } N}$ (i) + N = 6 + 3); the last doubles the resultant obtained from the post before. It functions as if one had taken the previous post, wrapped it in parentheses and then multiplied it by two, in effect summing the two subfigures. Note that in presenting the formula in just this way, it recapitulates the demonstration from the whiteboard. One might envision how a mathematician might produce these two representations at a physical blackboard. Here Bwang's demonstration had to be adapted to fit the circumstances, but this was done seamlessly using the affordances of the VMT environment. Note, for example, the ways in which he was able to animate his graphical derivation on the whiteboard and was subsequently able to exploit the conventions of chat interaction to sequentially build his functional description.

We indicated earlier that Bwang's presentation exhibited the properties of an informal derivation in that it leads stepwise to a given conclusion. The known conclusion in this case is a stick (line) count of 18. The force of the demonstration, therefore, rests crucially on his audience recognizing that he is working with the third case from the worksheet for which the number of sticks and squares are known. He never indicates this in as many words, but he makes it clear in the way that he begins his drawing. He began his illustration with a figural quote of the third example from the worksheet (see Figure 3).

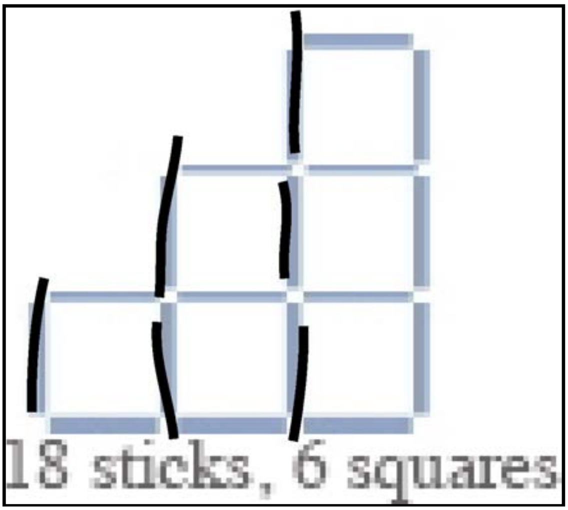


Figure 3. Bwang's figural quote of the N = 3 case from the worksheet.

Derivations only provide a gloss for the steps needed to reach the conclusion, each of the steps being incompletely specified. Decisions must be made at every turn as to how much specification is required. Bwang, in his three presentations of the formula, leaves certain aspects of the respective formulations to be worked out by his audience. This is a way in which the discourse is organized to display mathematical competence— Bwang treats his teammates as mathematically competent in his choices of what to make explicit and what to make implicit. It is also seen in his concluding tag line when he asks, "that's the formula, right?" (posts 68) in which he presents the formula as understood and his audience as competent to evaluate it.

We don't need a post-test to understand how the formula was to be understood—the understanding was made concrete in the participants' actions. In the chat log in the Appendix, one can study the ways in which the participants present matters for understanding to each other, build collaboratively on each others' actions and analyze each others' references. We observe them working to make sense of the scene before them populated with chat postings, whiteboard objects, and wiki entries. They can be viewed throughout to engage in a form of analysis. Our analysis here has focused upon the ways in which Bwang made his formula intelligible for the other participants. His practices for doing so provided for the analyzability of his actions. Participants' work of endogenous analysis has received little attention in prior studies of collaborative learning. We would argue, however, that these are the practices of greatest relevance for study in CSCL.

References

Çakir, M. P., Zemel, A., & Stahl, G. (2009). The joint organization of interaction within a multimodal CSCL medium. *International Journal of Computer-Supported Colalborative Learning, 4*, 115-149.

Garcia, A., & Jacobs, J. (1999). The eyes of the beholder: Understanding the turn-taking system in quasi-synchronous computer-mediated communication. *Research on Language and Social Interaction, 32*, 337-368.

Garfinkel, H. (1963). A conception of, and experiments with, 'Trust' as a condition of stable concerted actions. In O. J. Harvey (Ed.), *Motivation and social interaction* (pp. 187-238). New York: Ronald Press.

Greiffenhagen, C. (2008). Video analysis of mathematical practice? Different attempts to 'open up' mathematics for sociological investigation. *Forum Qualitative Sozialforschung, 9*(3).

Macbeth, D. (2000). On an actual apparatus for conceptual change. *Science Education, 84*, 228-264.

Sacks, H. (1992). *Lectures on conversation*. Oxford, U.K.: Blackwell.

Livingston, E. (2006). The context of proving. *Social Studies of Science, 36*, 39-68. Lynch, M. (2001). Ethnomethodology and the logic of practice. In T. R. Schatzki, K. K.

Cetina & E. von Savigny (Eds.), *The practice turn in contemporary theory* (pp. 131-148). London: Routledge.

Medina, R., Suthers, D., & Vatrapu, R. (2009). Representational practices in VMT. In G. Stahl (Ed.) (2009). *Studying virtual math teams*. New York: Springer.

Raymond, G. (2004). Prompting action: The stand-alone "So" in ordinary conversation. *Research on Language and Social Interaction*, 37, 185-218.

Schatzki, T. R. (2001). Practice theory. In T. R. Schatzki, K. K. Cetina & E. von Savigny (Eds.), *The practice turn in contemporary theory* (pp. 1-14). London: Routledge.

Stahl, G. (forthcoming). Meaning making in VMT. To appear in G. Stahl (Ed.), *Studying virtual math teams*. New York: Springer.

Zemel, A., & Çakir, M. P. (2009). Reading's work in VMT. To appear in G. Stahl (Ed.) (2009). *Studying virtual math teams*. New York: Springer.

Appendix.

#	chat	chat posting or	initiate	complete
40	Gerry	You can click on the button at the top that says "View Topic" to	18:28:45	18:29:12
41	bwang8	ok	18:29:32	18:29:32
42	bwang8	are we suppose to solve it now?	18:29:33	18:29:50
43	Gerry	Then you can click on the button in the little window that appears to open the topic	18:29:29	18:30:13
44	Gerry	browser*	18:30:32	18:30:36
45	Aznx	It didn't open.	18:30:33	18:30:40
46	Aznx	Now it did.	18:30:50	18:30:52
47	Aznx	So, are we supposed to work together?	18:31:25	18:31:32
	bwang8	((Initiates a chat message but deletes without posting))	18:31:25	18:31:45
48	bwang8	yeah	18:31:48	18:31:49
49	bwang8	ok	18:31:50	18:31:50
50	Gerry	Exactly!	18:31:51	18:31:54
	Quicksilver	((Initiates a chat message but deletes without posting))	18:31:51	18:31:57
51	Aznx	((Quicksilver's given name)), you there?	18:31:59	18:32:04
52	bwang8	you can divide the thing into two parts	18:31:52	18:32:05

	bwang8	((Created a line on whiteboard))		18:32:09
53	Aznx	Let's start this thing.	18:32:06	18:32:10
	bwang8	((Creates line objects on whiteboard))	18:32:11	18:32:38
54	Quicksilver	my computer was lagging...What are we doing?	18:32:30	18:32:38
55	Aznx	http://home.old.mathforum.org/SFest.html	18:32:48	18:32:49
56	Quicksilver	what are the lines for?	18:32:52	18:32:58
57	Aznx	go to view topic	18:32:57	18:33:01
58	bwang8	so you can see we only need to figur one out to get the total stick	18:32:42	18:33:05
59	Aznx	read the problem	18:33:04	18:33:09
60	bwang8	1+2+3+........+N+N	18:33:08	18:33:32
	Aznx	((Initiates a chat message but deletes without posting))	18:33:33	18:33:36
61	bwang8	times that by 2	18:33:34	18:33:38
62	Quicksilver	Never mind I figured it out..	18:33:35	18:33:40
	bwang8	((Initiates a chat message but deletes without posting))	18:33:41	18:33:43
63	Aznx	Can we collaborate this answer even more?	18:33:51	18:34:01
64	Aznx	To make it even simpler?	18:34:02	18:34:05
65	bwang8	ok	18:34:14	18:34:15
66	Aznx	Because I think we can.	18:34:07	18:34:16
67	bwang8	((1+N)*N/2+N)*2	18:34:25	18:34:50
68	bwang8	that's the formula, right?	18:34:52	18:34:58
	Aznx	((Initiates a chat message but deletes without posting))	18:34:43	18:35:05
69	Aznx	How did you come up with it?	18:35:09	18:35:15

70	bwang8	for total sticks	18:35:12	18:35:16
	Aznx	((Initiates a chat message but deletes without posting))	18:35:17	18:35:19
	bwang8	((Initiates a chat message but deletes without posting))	18:35:25	18:35:26
	Aznx	((Initiates a chat message but deletes without posting))	18:35:20	18:35:28
71	bwang8	is a common formual	18:35:27	18:35:34
	Aznx	((Initiates a chat message but deletes without posting))	18:35:29	18:35:35
72	bwang8	formula	18:35:38	18:35:40
73	Aznx	Yeah, I know.	18:35:43	18:35:46
74	bwang8	and just slightly modify it to get this	18:35:45	18:35:59
	bwang8	((Deletess some objects on whiteboard))	18:36:15	18:36:27
	bwang8	((Creates a line on whiteboard))		18:36:31
75	Aznx	Aditya, you get this right?	18:36:27	18:36:31
	bwang8	((Creates some lines on whiteboard))	18:36:32	18:36:37
	Quicksilver	((Initiates a chat message but deletes without posting))	18:36:35	18:36:39
	bwang8	((Creates some lines on whiteboard))	18:36:39	18:36:43
	Quicksilver	((Moves some objects on whiteboard))	18:36:44	18:37:05
	Gerry	START:TextEditing		18:37:44
76	Quicksilver	What does the n represent?	18:37:39	18:37:45
	bwang8	((Initiates a chat message but deletes without	18:37:35	18:37:52
77	bwang8	the given	18:37:54	18:37:57
78	bwang8	N	18:37:58	18:37:58

79	Aznx	Yeah.	18:38:00	18:38:02
	bwang8	((Initiates a chat message but deletes without posting))	18:38:02	18:38:04
80	Aznx	In the problem.	18:38:03	18:38:05
	Gerry	END:TextEditing		18:38:13
	Gerry	((Creates a textbox on whiteboard))		18:38:13
	Gerry	((Resizes some objects on whiteboard))		18:38:17
	Quicksilver	((Resizes some objects on whiteboard))		18:38:20
	Aznx	((Initiates a chat message but deletes without posting))	18:38:10	18:38:21
81	Aznx	Oh	18:38:36	18:38:37
82	bwang8	The number of squares is just (1+N)*N/2	18:38:06	18:38:38
	Aznx	((Creates some lines on whiteboard))	18:38:48	18:38:44
83	Quicksilver	We need that as well.	18:38:46	18:38:50
	Aznx	((Creates a line on whiteboard))		18:38:51
84	Gerry	I put BWang's formula on the whiteboard	18:38:36	18:38:52
	Aznx	((Creates a line on whiteboard))		18:38:55
	Aznx	START:TextEditing		18:39:04
	Aznx	END:TextEditing		18:39:19
	Aznx	((Creates a textbox on whiteboard))		18:39:19
	Aznx	((Moves some objects on whiteboard))		18:39:28
85	Aznx	So how do we submit this?	18:39:39	18:39:45

5. Repairing Indexicality in Virtual Math Teams

Gerry Stahl, Alan Zemel, Timothy Koschmann

Abstract: Meaning making in the online collaboration settings of CSCL takes special forms depending on the affordances of the software. Here we analyze how virtual math teams in a synchronous environment combining text chat and shared whiteboard repair problems of chat confusion. We observe the central role of indexicality in establishing common ground and facilitating group cognition.

Repairing Chat Confusion in Virtual Math Teams

The problem of "chat confusion" has been much discussed in analyses of computer-mediated communication (Herring, 1999). It is commonly attributed to the fact that the system of turn taking, which structures face-to-face conversation, does not operate in online text chat (Fuks, Pimentel, & Pereira de Lucena, 2006; Garcia & Jacobs, 1998, 1999; O'Neill & Martin, 2003). We have argued that the turn-taking structure of conversation is replaced by a threading structure of responses in chat (Çakir, Xhafa, Zhou, & Stahl, 2005; Zemel & Çakir, 2007). For this reason, we recommend that an analysis of text-chat interaction should typically start with a clarification of the threading structure of the responses of postings to each other (Stahl, 2009, Ch. 20, 26, 28). We took this approach to a particularly interesting but confusing chat excerpt in (Stahl, 2007a) and concluded that there was still ambiguity about what the participants were saying.

In this paper, we extend that analysis. We look at the source of confusion at a deeper level: as being a matter of issues of *indexicality*. For instance, when one student refers to "the second formula" another student misunderstands which formula is being indexed as the second one. The students are working in a virtual environment in which their text chat postings reference mathematical formulae and diagrams in a shared whiteboard. The team works hard to repair misunderstandings concerning indexicality. It is by working out a shared system of indexing that they are able to effectively use the deictic referencing that is taken to such an extreme in text chat, with its characteristically brief, elliptical use of pronouns, articles and numbers in place of noun phrases and clauses. This intersubjective indexical field (Hanks, 1992) can be seen as the basis for establishing common ground (Clark & Brennan, 1991) and a joint problem space (Teasley & Roschelle, 1993).

In mathematics, symbols like x or n are used to index things like the unknown value being sought or the current stage in an increasing pattern. In the interaction that we study in this paper, there is also a problem in understanding the indexicality of the symbol n in the formulae under discussion. This problem is of particular concern for the participants and—in contrast to the confusion about the indexicality of "second" in "the second formula"—this problem is never resolved. In fact, we will see that this confusion may be related to a subtle problem of the value of n in the formula, leading to an error in the student work, which is never brought to light or corrected. This may be a result of the novice status of the students as mathematicians and the fact that they have not adopted the full set of mathematical practices that might have avoided such a problem (Livingston, 1999; Sfard, 2008; Stahl, 2008), such as defining their terms explicitly and labeling indexed objects with persistently visible letters.

We will investigate problems of indexicality and their repair using data from the Virtual Math Teams project at Drexel University. This CSCL research project has previously been presented at ICCE (Stahl, 2005; Stahl, Wee, & Looi, 2007) and at CSCL (Stahl, 2007a, 2007b). The background for it is discussed in (Stahl, 2006) and many results are gathered in (Stahl, 2009). The specific excerpt is taken from the beginning of the last of four hour-long sessions. An initial analysis of the excerpt to determine its threading was undertaken in (Stahl, 2007a; revised version in Stahl, 2009, Ch. 26), This analysis was taken up in (Medina, Suthers, & Vatrapu, 2009), which traced back through the sessions to document the establishment of several group math problem-solving practices that were at work in the excerpt.

In the following section, we go back to the beginning of the fourth session (at 19:00:00) and review the interaction up to and including the previously studied excerpt (from 19:29:46 to 19:33:11). Figure 1 shows the VMT interface during much of this period.

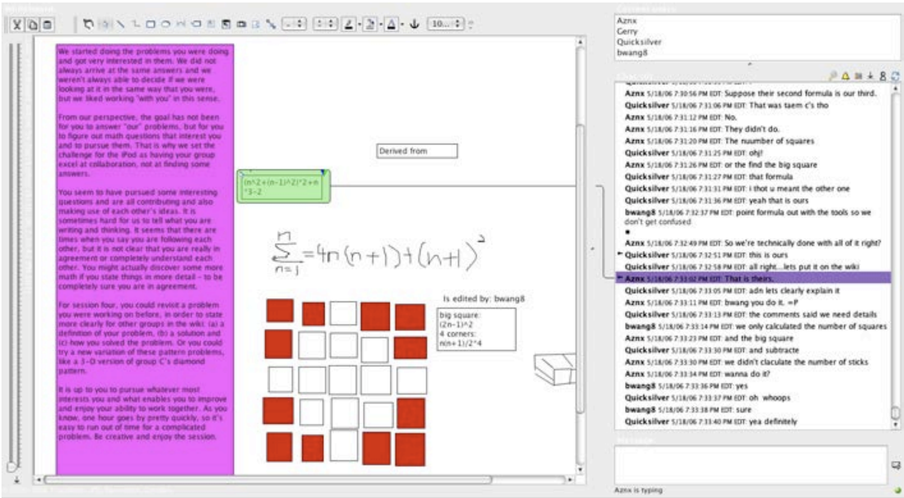


Figure 1. Aznx is pointing to Team C's formula

Analysis of the Work the Students Do in the Chat and Whiteboard

In this section, we proceed systematically through the log of student work to trace the chat references to various formulae in the shared whiteboard. In doing so, we can observe how their concern with the formulae arises, how confusion in indexing specific formulae unfolds and how the team repairs the confusion so that they can continue with their work.

[19:00:00 – 19:14:28] The students return to the chat room for their fourth and final session. They orient to a textbox of feedback on the whiteboard from the VMT mentors (the purple text box on the left in Figure 1). The feedback raises for them the issue of whether their discussion in the previous session was clearly expressed—both for them and for others: "it is not clear that you are really in agreement or completely understand each other." It suggests that they review the derivation of their math findings for posting to the wiki: "For session four, you could revisit a problem you were working on before, in order to state more clearly for other groups in the wiki: (a) a definition of your problem, (b) a solution and (c) how you solved the problem."

(Please see the Appendix at the end of this chapter for the chat transcript referred to in the following paragraphs.)

[19:14:38 – 19:17:15] The students discuss what topic to pursue during this session. They decide to continue to work on the diamond problem from their third session and to "solve it thoroughly, and then state the solution as they suggested in the feedback."

[19:17:15 – 19:20:23] They proceed to recap their previous findings. They want to post their findings to the wiki, but decide to conduct a thorough review in chat first to get their story straight. At 19:18:31 Bwang posts a textbox: "big square: (2n-1)^2" to start the review of their derivation. He indexes it in chat and with a graphical reference at 19:19:44, asking for agreement on the formula's correctness. All members associate Bwang's symbolic formula with the word "square."

[19:20:24 – 19:21:56] Bwang proposes that the number of blocks in the corners (the red squares in the whiteboard diagram of the red and white big square) grow like this: 0, 1, 3, 6, 10. The others identify this pattern with "triangular numbers," and Bwang affirms their responses in an instructor-like fashion. Bwang then provides a formula for the number of squares in the four corners, based on the (Gaussian sum) formula from previous sessions, which he had already posted: "4*n(n+1)/2= the four corners."

[19:21:39 – 19:22:06] While Bwang does that, Quicksilver drags a textbox from the top right margin of the whiteboard into a prominent position: "Derived from N(n+1)/2" and Aznx similarly drags another box, with two formulae: "(n^2+(n-1)^2)*2+n*3-2 n^2+(n-1)^2." Quicksilver asks if his box is correct and the others agree. No one—including Aznx—comments in the chat on Aznx' move in the whiteboard.

[19:22:28 – 19:22:51] Bwang posts the expression in chat: "(2n-1)^2-2n(n-1)" and says, "this is the equation for each level." This is visibly a combination of his two previous formulae, for the number of blocks in the big square minus the number of blocks in the four corners.

[19:22:52 – 19:23:19] Aznx responds to this expression with the question, "So how do we know what to multiply/change the formula by?" He then twice starts to type another posting, but erases it without posting. Bwang tries emphatically to ask Aznx what he meant by this. At 19:23:19, Bwang wrote, "wait what do you mean" and at 19:23:50 he asked, "can you explain this" and pointed back to Aznx' posting. Bwang's appeal that all discussion "wait" until Aznx explains his question and Bwang's use of the graphical reference to point back to the question a minute later indicate the high level of Bwang's concern about not understanding Aznx' strange question. As Bwang had said when he posted the expression, it is the "equation for each level"—where the variable "n" indicates the level and is the basis for change in the formula. Aznx' question raises the possibility that he does not understand the role of the variable "n" in equations like these. Aznx had previously expressed some uncertainty about the role of "n": at 19:18:08 he had responded to Quicksilver's statement, "our objective is to find the amount of squares and sticks in each level righrt?" with "Yeah, intending that it is n." When Quicksilver continued by saying, "that was step a," Aznx objected at 19:18:18, "no, step one." He later understood that Quicksilver was referring to step (a) of the feedback, but this could show that Aznx took the formula with "n" to be only for the first step, n=1, rather than for all values of n.

[19:23:19 – 19:25:40] Aznx next asks, "Suppose we didn't know the formula. . . . Not n(n+1)/2". The group discusses this formula and clarifies that it is the formula for the number of squares in each of the four corners. It is not clear where Aznx is going with this, but Quicksilver and Bwang try to clarify things for him.

[19:25:43 – 19:26:16] Aznx now says: "But that's not what it ends up to be. . . . If you double check with our already-given formula. . . . It's this. . . . The first one". He points to the textbox that he had dragged out at 19:22:01 with the content, "(n^2+(n-1)^2)*2+n*3-2 n^2+(n-1)^2." Bwang (19:26:39) and Quicksilver (19:27:27) clarify for Aznx that the first formula in his textbox is for the number of sticks, not the number of blocks. These formulae were not derived by Team B, but were copied from Team C's work on the wiki and remained on the side of the whiteboard from previous sessions until Aznx dragged the textbox into the center. Aznx concludes, "I got confused with all the formulas lol."

[19:28:22 – 19:30:25] The team then discusses posting the solution to the wiki and decides to review their derivation in the chat first. This brings us to the analysis in (Stahl, 2009, Chapter 26) and the confusion about "the second formula."

[19:30:32 – 19:30:56] Aznx says, "Well, I can explain the second formula." To this Quicksilver responds emphatically, "NO! . . . We don't know hte second formula". Aznx then responds, "Yes we do. . . . Suppose their second formula is our third." The group has repeatedly gone over their derivation of the formula for the number of blocks in the diamond pattern as the number of blocks in the big square minus the number of blocks in the four stair-step corners. So Aznx claims he can now explain this. However, he indexes the formula he is referring to in a way that is not clear to the other group members. He calls it "the second formula." Subsequently, he refers to "their second formula" and "our third". So now there is a system of indexicals distinguishing first, second and third formulas in sets of ours and theirs.

[19:31:06 – 19:31:36] Quicksilver says, "That was taem c's tho." Here, "that" is presumably referencing the subject of Aznx' previous statement, "their second formula." The "tho" indicates that the second formula is not a proper subject for Team B to report in the wiki because it is not theirs, but Team C's (at least originally, as indicated by "was"). Aznx explains that he can not be referring to a formula from Team C because, "No. . . . They didn't do. . . . The number of squares. . . . or the find the big square." Quicksilver then sees that Aznx must be referring to their own formula based on the number of squares in the big square.

[19:32:37 – 19:33:02] After a minute during which nothing was posted in the chat, Bwang suggested that they "point formula out with the tools so we don't get confused." Quicksilver then points with the graphical referencing tool to the textbox that he had dragged out, saying in the chat, "this is ours." Aznx points with the graphical referencing tool to the textbox that he had dragged out, saying in the chat, "That is theirs" (see reference line from chat to whiteboard in Figure 1). This clarifies the categorization of the three formulae: formula one and formula two in Aznx's textbox are Team C's formula for the number of sticks. Formula three in Quicksilver's textbox is Team B's own formula for the number of blocks.

[19:32:58 – 19:33:40] Having resolved the referential confusion, the group can now proceed with their work. The resolution made explicit that the group had only solved the problem for the number of blocks, not the number of sticks. So they decide to tackle the problem of the number of sticks.

Discussion of Indexicality

In the context of this VMT chat about math, the group of students has to coordinate the joint understanding of a complex system of tightly related graphical, symbolic and linguistic resources (e.g., the white diamond in a red square image in the whiteboard; the math formulae in the whiteboard and chat; the terms like "big square," "corner," "triangular numbers," "diamond"). The meaning-making context in which these resources are embedded stretches over multiple sessions (days), much of which is no longer visible in the currently displayed computer interface. To engage in their collaborative task, the students must be able to reference/index the resources in a mutually understood way. They need to recall, explain and reason with these resources in shared ways. For novices in mathematics and in online collaborative problem solving, the three students are confronted with an extremely complex set of resources, existing in multiple media, multiple times (previous sessions, prior actions, projected future activities) and multiple interaction spaces (chat, whiteboard areas, wiki pages, possibly private workspaces). The open-ended math problem may be more challenging than they are used to and they are being held to high standards of expressing their ideas clearly for each other (some of whom they have never met in person) and for various ill-defined audiences (other groups, VMT mentors).

Trained mathematicians take advantage of domain practices that were originally developed by the early Greek geometers (Latour, 2005; Netz, 1999). The rubric of a formal proof involves maintaining an ordered sequence of logical derivation steps that is persistently visible. Major representations, expressions and findings are often numbered, named or labeled to provide for unambiguous and easy referencing. Terms used in the proof are defined explicitly. The vocabulary used in a proof is limited and controlled. Students such as those in Team B have not been socialized into these practices and use the unmediated linguistic resources of ordinary language, causing referential ambiguities, interpersonal misunderstandings and indexical confusions.

In this episode, we see at least two indexical confusions: (a) what is indexed by "the second formula" in Aznx's post at 19:30:32 and (b) what is "n" in Team B's formula. (a) The first confusion is resolved with the use of VMT's explicit graphical referencing tool. It is attributed by Aznx to his confusion with "all the formulas" and by Quicksilver to a confusion between the group's equations and Team C's equations. Much ambiguity remains in this discussion, but the group is able to proceed productively to new work. (b) The second confusion results in a mathematical error that the group never recognizes, despite the fact that Bwang got it right at 19:22:28. Aznx seems to be confused about the role of "n" in the formula for number of blocks—see Bwang's concern regarding 19:22:52 at 19:23:19 and 19:23:50. This could be related at a deeper level to Aznx' confusion about variables in formulae generally. On the other hand, Aznx' confusion may have just had to do with referring to the wrong formulae—e.g., to Team C's when his group was discussing their own formulae.

For both the participants and the analysts, understanding what is taking place in a VMT session involves understanding the mathematical relationships that are being discussed—much of which is included in background knowledge that is not made explicit in the postings, but is implicit in the work done by the postings. A case in point involves the variable "n" in Team B's formula for the number of blocks in a diagonal pattern. If we take the pattern as starting with one block for n=1, then the big enclosing square contains (2n-1)^2 blocks, as the team noted. However, when n=1, there are no blocks in the corners. So the Gaussian sum is not for 1+2+…+n, but rather for 0+1+…+(n-1), as Bwang actually indicated at 19:20:43 when he said, "I think the 4 corner is growing like this. . . . 0, 1, 3, 6, 10." Accordingly, the sum is (n-1)n/2 rather than n(n+1)/2. Bwang seems to have used this correct formula at 19:22:28 when he wrote, "(2n-1)^2-2n(n-1)". However, when he added it to his textbox at 19:26:15 he wrote "big square: (2n-1)^2, 4 corners: n(n+1)/2*4". It was never explicitly noted that n started at 1 for the big square and at 0 for the corners. This difference in algebraic indexing was never shared and was lost in the discussion, resulting in a mathematically erroneous formula, unbeknownst to the team. Again, rigorous mathematical practices would have avoided this problem. Even checking the formula of simple cases would have raised questions that could have led to discovering the problem.

We have seen in this session how the group learns to conduct effective collaborative math work by indexing more clearly their references to resources. By reviewing the derivation of their prior findings, they make progress in tying together their complex system of resources in a mutually understood way.

Here we can see that the establishment of "common ground" in a situation like this is much different than Clark's (1991) concept of exchanging expressions of mental representations to assure their isomorphism or identity. Rather, what is needed is the co-construction of a joint indexical field (Hanks, 1992). Similarly, what could be construed as a conversational "repair"—namely clarifying what Aznx meant by "the second formula"—centrally involves determining which symbolic expression is being indexed.

The analysis also sheds light on Sfard's (2008) notion of multiple realizations of a math object. It is not just that the math object "diamond pattern" consists of a tree of realizations such as the drawings, symbolic formulae and narratives related to this pattern. Rather, these realizations only "make sense" within the context of a much larger indexical field, including other patterns, formulae and concepts. For instance, the formula that is the students' solution indexes the n^{th} stage of the pattern, the enclosing square, the excluded corners, the graphical illustrations, the phrase "diamond pattern," the original problem statement, and so on. In a phenomenological sense, the whole world is "given" (i.e., indexed implicitly) in the meaning of a single math object. Within the VMT context, it is clear that this whole world is an intersubjective one and the indexical field is necessarily a co-constructed and jointly reproduced one. The group production and maintenance of a shared indexical network is central to collaborative meaning making and group cognition.

References

Çakir, M. P., Xhafa, F., Zhou, N., & Stahl, G. (2005). Thread-based analysis of patterns of collaborative interaction in chat. In C. Looi, Mccalla, G., Bredeweg, B., and Breuker, J. (Ed.), *Artificial intelligence in education* (Vol. 125, pp. 120-127). Amsterdam, NL: IOS Press.

Clark, H., & Brennan, S. (1991). Grounding in communication. In L. Resnick, J. Levine & S. Teasley (Eds.), *Perspectives on socially-shared cognition* (pp. 127-149). Washington, DC: APA.

Fuks, H., Pimentel, M., & Pereira De Lucena, C. (2006). R-U-Typing-2-Me? Evolving a chat tool to increase understanding in learning activities. *International Journal of Computer-Supported Collaborative Learning, 1*(1), 117-142. Available at http://dx.doi.org/10.1007/s11412-006-6845-3.

Garcia, A., & Jacobs, J. B. (1998). The interactional organization of computer mediated communication in the college classroom. *Qualitative Sociology, 21*(3), 299-317.

Garcia, A., & Jacobs, J. B. (1999). The eyes of the beholder: Understanding the turn-taking system in quasi-synchronous computer-mediated communication. *Research on Language and Social Interaction, 34*(4), 337-367.

Hanks, W. (1992). The indexical ground of deictic reference. In C. Goodwin & A. Duranti (Eds.), *Rethinking context: Language as an interactive phenomenon*. Cambridge, UK: Cambridge University Press.

Herring, S. (1999). Interactional coherence in cmc. *Journal of Computer Mediated Communication, 4*(4). Available at http://jcmc.indiana.edu/vol4/issue4/herring.html.

Latour, B. (2005). The netz-works of Greek deductions.

Livingston, E. (1999). Cultures of proving. *Social Studies of Science, 29*(6), 867-888.

Medina, R., Suthers, D., & Vatrapu, R. (2009). Representational practices in VMT. In G. Stahl (Ed.), *Studying virtual math teams*. New York, NY: Springer.

Netz, R. (1999). *The shaping of deduction in Greek mathematics: A study in cognitive history*. Cambridge, UK: Cambridge University Press.

O'neill, J., & Martin, D. (2003). *Text chat in action*. Paper presented at the ACM Conference on Groupware (GROUP 2003), Sanibel Island, FL.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge, UK: Cambridge University Press.

Stahl, G. (2005). *Sustaining online collaborative problem solving with math proposals [winner of best paper award]*. Paper presented at the International Conference on Computers and Education (ICCE 2005), Singapore, Singapore. pp. 436-443. Available at http://GerryStahl.net/pub/icce2005.pdf & http://GerryStahl.net/pub/icce2005ppt.pdf.

Stahl, G. (2006). *Group cognition: Computer support for building collaborative knowledge*. Cambridge, MA: MIT Press. Available at http://GerryStahl.net/mit/.

Stahl, G. (2007a). *Meaning making in CSCL: Conditions and preconditions for cognitive processes by groups*. Paper presented at the international conference on Computer-Supported Collaborative Learning (CSCL '07), New Brunswick, NJ. Available at http://GerryStahl.net/pub/cscl07.pdf.

Stahl, G. (2007b). *Workshop: Chat analysis in virtual math teams*. Presented at the International Conference of Computer-Supported Collaborative Learning (CSCL 2007). New Brunswick, NJ. Available at http://vmt.mathforum.org/vmtwiki/index.php/Chat_Analysis_Workshop.

Stahl, G. (2008). Book review: Exploring thinking as communicating in CSCL. *International Journal of Computer-Supported Collaborative Learning, 3*(3), 361-368. Available at http://dx.doi.org/10.1007/s11412-008-9046-4.

Stahl, G. (Ed.). (2009). *Studying virtual math teams*. New York, NY: Springer. Available at http://GerryStahl.net/vmt/book.

Stahl, G., Wee, J. D., & Looi, C.-K. (2007). *Using chat, whiteboard and wiki to support knowledge building*. Paper presented at the International Conference on Computers in Education (ICCE 07), Hiroshima, Japan. Available at http://GerryStahl.net/pub/icce07.pdf.

Teasley, S. D., & Roschelle, J. (1993). Constructing a joint problem space: The computer as a tool for sharing knowledge. In S. P. Lajoie & S. J. Derry (Eds.), *Computers as cognitive tools* (pp. 229-258). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Zemel, A., & Çakir, M. P. (2007). *Reading's work: The mechanisms of online chat as social interaction.* Paper presented at the National Communication Association Convention, Chicago, IL. Available at http://GerryStahl.net/vmtwiki/alan2.pdf.

Appendix

1279	19:15:57	Quicksilver	so where were we?
1280	19:16:00	bwang8	so right now we know that we must calculate the number of squares on each level by making a big square and minus the 4 extra corners
1281	19:16:03	Aznx	I'd say, we work on the pyramid problem, solve it thoroughly, and then state the solution as they suggested in the feedback. Then, if we have enough time, which probably will ,we'll sytart on the pyramid problem.
1282	19:16:21	Quicksilver	u said two pyramid problems?
1283	19:16:27	Quicksilver	read ur thing again
1284	19:16:27	Aznx	OOps
1285	19:16:34	Aznx	I meant in the first part
1286	19:16:37	Aznx	the diamond problem
1287	19:16:41	Aznx	not the pyramid
1288	19:16:41	bwang8	lol
1289	19:16:45	Quicksilver	so do diamond?
1290	19:16:49	Aznx	so we first work on the diamond solutions
1291	19:16:51	Aznx	yeah
1292	19:16:57	Aznx	we pretty much solved it didnt we?
1293	19:17:09	bwang8	yeah
1294	19:17:11	Aznx	Well 50% of it I should say.
1295	19:17:15	Quicksilver	lets just recap the process
1296	19:17:27	Quicksilver	from the point of view who had never seen this problem
1297	19:17:32	bwang8	we know how to calculate the big square in a level
1298	19:17:44	Quicksilver	ok hold on
1299	19:17:50	bwang8	as in this
1300	19:17:56	bwang8	whole thing
1301	19:17:57	Quicksilver	our objective is to find the amount of squares and sticks in each level righrt?
1302	19:18:03	bwang8	yeo
1303	19:18:04	bwang8	yep
1304	19:18:08	Aznx	Yeah, intending that it is n.

1305	19:18:10	Quicksilver	that was stpe a
1306	19:18:15	Quicksilver	from the comments
1307	19:18:18	Aznx	no, step one
1308	19:18:21	Quicksilver	we defined the problem
1309	19:18:26	Aznx	oh
1310	19:18:27	Aznx	yes
1311	19:18:40	Quicksilver	lets put that in the wiki now\
1312	19:18:45	Aznx	So we dfined the problem.
1313	19:18:50	Aznx	Hold on.
1314	19:18:56	Aznx	Let's finish the ewntire thing up first.
1315	19:19:04	Aznx	We can always look back if we mess up.
1316	19:19:07	Quicksilver	ok
1317	19:19:24	bwang8	the formula is correct, right?
1318	19:19:24	Aznx	So now we should focus on integrating the solutioni and how we found it.
1319	19:19:42	Quicksilver	yup
1320	19:19:44	bwang8	this one
1321	19:19:47	bwang8	ok
1322	19:19:47	Aznx	Yeah.
1323	19:19:55	Aznx	We can always double check, and it's darn right.
1324	19:20:05	Aznx	So we solve it by really looking at a bigger picture.
1325	19:20:15	Quicksilver	or bigger square in this case
1326	19:20:20	Aznx	In this case, the "square" itself.
1327	19:20:23	Aznx	Yeah.
1328	19:20:34	bwang8	i think the 4 corner is growing like this
1329	19:20:43	bwang8	0,1,3,6,10
1330	19:20:48	bwang8	what is the pattern
1331	19:20:56	Aznx	Triagnular numbers.
1332	19:20:58	Quicksilver	triangular numbers!
1333	19:21:00	bwang8	yep
1334	19:21:03	Aznx	We had already figured that out.
1335	19:21:10	bwang8	we can use the equation from session 1
1336	19:21:11	Quicksilver	yes
1337	19:21:20	Aznx	Yup.
1338	19:21:36	bwang8	n(n+1)/2
1339	19:21:56	bwang8	4*n(n+1)/2= the four corners
1340	19:21:57	Quicksilver	this right?
1341	19:22:03	bwang8	yes
1342	19:22:06	Aznx	Yeah
1343	19:22:28	bwang8	(2n-1)^2-2n(n-1)

1344	19:22:48	bwang8	this is the equation for each level
1345	19:22:52	Aznx	So how do we know what to mulitply/change the formula by?
1346	19:23:04	Quicksilver	we can use the brute force method
1347	19:23:15	Quicksilver	burt im sure there's a better wayu
1348	19:23:19	bwang8	wait what do you mean
1349	19:23:19	Aznx	Suppose we didn't know the formula.
1350	19:23:36	Quicksilver	hmm..
1351	19:23:39	Aznx	Not n(n+1)/2
1352	19:23:47	Quicksilver	so we don't know that?
1353	19:23:50	bwang8	can you explain this
1354	19:23:57	Aznx	look
1355	19:24:02	Quicksilver	he means as the levels increase
1356	19:24:06	Aznx	first there's n(n+1)/2 right?
1357	19:24:09	Quicksilver	what is the pattern
1358	19:24:12	Aznx	So now we nkow
1359	19:24:19	Aznx	that the number of squares in the pattern
1360	19:24:24	Aznx	is related to this formula
1361	19:24:32	Aznx	becuase the numbers are triangular numbers
1362	19:24:43	Aznx	So from there, what do we know what to do?
1363	19:25:13	bwang8	n(n+1)/2*4
1364	19:25:28	Quicksilver	because of four corners
1365	19:25:30	Quicksilver	right?
1366	19:25:36	bwang8	that is the number of squares in four corners
1367	19:25:40	Quicksilver	ok
1368	19:25:43	Aznx	But that's not what it ends up to be.
1369	19:25:56	Aznx	If you double check with our already-given formula
1370	19:26:00	Quicksilver	why?
1371	19:26:07	Aznx	It's this
1372	19:26:12	Quicksilver	oh yeah
1373	19:26:14	Quicksilver	it doesn't work
1374	19:26:16	Aznx	The first one
1375	19:26:29	bwang8	no
1376	19:26:39	bwang8	it is the second one that calculate the square
1377	19:27:11	Quicksilver	are you talking about this?
1378	19:27:21	Aznx	Then what's the first one for?
1379	19:27:27	Quicksilver	the sticks\
1380	19:27:33	Aznx	Oh!
1381	19:27:40	Aznx	Then the formula makes sense.
1382	19:27:45	Quicksilver	but pretend we don't know those yet

1383	19:27:47	Aznx	Yeah, I got it.
1384	19:27:51	bwang8	lol
1385	19:28:01	Aznx	I got confused with all the formulas lol.
1386	19:28:16	Quicksilver	i suppose so
1387	19:28:22	Aznx	So is that all?
1388	19:28:37	Quicksilver	what is the actual solution then? those equations?
1389	19:28:43	Aznx	Yeah.
1390	19:28:59	Quicksilver	but when we put in the wiki how we did it....what will we write
1391	19:29:20	Aznx	Um.
1392	19:29:42	Aznx	I don't know how to exactly word it.
1393	19:29:46	Quicksilver	(a) was define the problem, (b) was the solution which we got...
1394	19:29:48	bwang8	we calculated the # of square if the diamond makes a perfect square
1395	19:29:48	Aznx	We can define the problem.
1396	19:29:55	Aznx	We got the solutions.
1397	19:30:12	Quicksilver	yes
1398	19:30:16	Quicksilver	the added corners
1399	19:30:18	Aznx	But I'm not sure how to explain how we got to the solutions, although it makes prefect sense to me.
1400	19:30:19	Quicksilver	to make a square
1401	19:30:24	Aznx	I'm just not sure how to explain it.
1402	19:30:25	Quicksilver	and we found those were triangular numbers
1403	19:30:32	Aznx	Well, I can explain the second formula.
1404	19:30:35	Quicksilver	lets go step by step
1405	19:30:37	Quicksilver	NO!
1406	19:30:42	Quicksilver	we don't know hte second formula
1407	19:30:45	Aznx	It was done through the method of finsing the pattern of triangular #s.
1408	19:30:50	Aznx	Yes we do.
1409	19:30:55	Quicksilver	?
1410	19:30:56	Aznx	Suppose their second formula is our third.
1411	19:31:06	Quicksilver	That was taem c's tho
1412	19:31:12	Aznx	No.
1413	19:31:16	Aznx	They didn't do.
1414	19:31:20	Aznx	The nuumber of squares
1415	19:31:25	Quicksilver	ohj!
1416	19:31:26	Aznx	or the find the big square
1417	19:31:27	Quicksilver	that formula
1418	19:31:31	Quicksilver	i thot u meant the other one
1419	19:31:36	Quicksilver	yeah that is ours
1420	19:32:37	bwang8	point formula out with the tools so we don't get confused

1421	19:32:49	Aznx	So we're technically done with all of it right?
1422	19:32:51	Quicksilver	this is ours
1423	19:32:58	Quicksilver	all right...lets put it on the wiki
1424	19:33:02	Aznx	That is theirs.
1425	19:33:05	Quicksilver	adn lets clearly explain it
1426	19:33:11	Aznx	bwang you do it. =P
1427	19:33:13	Quicksilver	the comments said we need details
1428	19:33:14	bwang8	we only calculated the number of squares
1429	19:33:23	Aznx	and the big square
1430	19:33:30	Quicksilver	and subtracte
1431	19:33:30	Aznx	we didn't claculate the number of sticks
1432	19:33:34	Aznx	wanna do it?
1433	19:33:36	bwang8	yes
1434	19:33:37	Quicksilver	oh whoops
1435	19:33:38	bwang8	sure
1436	19:33:40	Quicksilver	yea definitely

6. Group Creativity in Inter-Action: Collaborative Referencing, Remembering and Bridging

Johann Sarmiento and Gerry Stahl

Abstract

In this paper, we present a qualitative case study of group creativity online in the domain of mathematics. We define creative work broadly, ranging from the micro-level co-construction of novel resources for problem solving to the innovative reuse of ideas and solution strategies across virtual teams. We analyze the collaborative interactions of virtual math teams with an emphasis on describing the relationship between "*synchronic*" aspects of creative work (i.e. single episode interactions) and their "*diachronic*" evolution across time and across collectivities. Our analysis indicates that the synergy between these two types of interactions and the resulting creative engagement of the teams relies on three fundamental processes: (1) referencing and the "configuration of indexicals", (2) collective remembering, and (3) bridging across discontinuities. In addition we also reflect on the aspects of the online environment used by these virtual teams which promote, support or hinder diachronic and synchronic interactions and creativity as aspects of group cognition.

Introduction

We take a social and interactional view of creativity. We study creative accomplishments of small groups working together online. It may be that one can see the mechanisms and practices that are constitutive of creativity in the observable interactions of groups and then understand individual creativity as forms of "internalization" of these interactional processes.

Although the social dimension of individual creativity has been studied extensively in creativity research [e.g., 1, 2, and 3], collective creativity is a recent topic of study. In fact, understanding collective creativity as interactional phenomena of groups evolving over time can help us understand better the creative process itself. For instance, recent conceptual models of group creativity [5] propose that collective creative work can be better understood as the synergy between *synchronic* interactions (i.e., in parallel and simultaneously) and *diachronic* exchanges (i.e., interaction over long time spans, and mediated indirectly through creative products). In this paper we attempt to explore the interdependency between the synchronic and diachronic interactions and analyze its relationship with creative work, broadly defined. In our study of mathematics collaboration online we observe collective creative work as manifested in a wide range of interactions extending from the micro-level co-construction of novel resources for problem solving to the innovative reuse and expansion of ideas and solution strategies across multiple teams. This paper presents a case study of such collective creativity.

We start by describing the Virtual Math Teams project, the context from which our observations originate. Then we turn our attention to describing incrementally some central interactional aspects of online collectivities engaged in creative work. Our main goal is to better understand the synergy between single-episode collaboration and the creative work of multiple collectivities engaged together over time. In particular, we describe three interactional processes which appear to be fundamental for collective creativity: (1) referencing and the "configuration of indexicals", (2) collective remembering, and (3) bridging across discontinuities.

The emergence of computational environments that support collaborative work has opened up the opportunity for researchers to go beyond studies of "solo" action and investigate distributed systems of cognition and creativity that situate artifacts, tasks and knowing in the interactions of co-participants and activity systems over time. In addition to describing the interactions that the virtual teams observed engage in, we also reflect on the particular aspects of the online environment employed which promote, support or hinder *synchronic* and *diachronic* interactions.

The Virtual Math Teams Project

The Math Forum (http://mathforum.org) is an online community, active since 1992. It promotes technology-mediated interactions among teachers of mathematics, students, mathematicians, staff members and other interested parties committed to learning, teaching and doing mathematics. As the Math Forum community continues to evolve, the development of new interaction supports becomes increasingly essential for sustaining and enriching the mechanisms of community participation available. As an example of these endeavors, the Virtual Math Teams (VMT) project at the Math Forum investigates the innovative use of online collaborative environments to support effective secondary mathematics learning in small groups. The VMT project is an NSF-funded research program designed to investigate sustained collaborative problem-solving in computer-supported environments and to characterize how members of the Math Forum's community of learners constitute their interactions over time to foster their development as learners of mathematics.

Central to the VMT research program are the investigation of the nature and dynamics of group cognition [6] as well as the design of effective technological supports for quasi-synchronous small-group interactions. In addition, we investigate the linkages between synchronous interactions (e.g. collaborative chat episodes) and distributed, asynchronous interactions at the level of the online community. We are currently studying how upper middle school and high school students do mathematics collaboratively in online chat environments. We are particularly interested in the methods that they develop to conduct their interactions in such an environment. Taken together, these methods define a culture, a shared set of ways to make sense together. The methods are subtly responsive to the chat medium, the pedagogical setting, the social atmosphere and the intellectual resources that are available to the participants. These methods help define the nature of the collaborative experience for the small groups that develop and adopt them.

In our iterative design-based research approach, we started by conducting chats in a variety of commercially available environments. Based on these early investigations, we concluded that we needed to add a shared whiteboard for drawing geometric figures and for persistently displaying notes. We also found a need to minimize "chat confusion" by supporting explicit referencing of conversation threads. We decided to try ConcertChat [4], a research collaboration environment combining persistent chat with a shared whiteboard and a set of referencing tools. By collaborating with the software developers, our educational researchers have been able to successively try out versions of the environment with groups of students and to gradually modify the environment in response to our research. Some of ConcertChat's interactional supports include:

* Chat conversations are persistent during and after each session. Latecomers automatically receive the last ten messages when joining a session and can load all previous messages at will.
* A chat user can post a new message with an explicit graphical link pointing to one or more previous messages. The graphical link between the two messages is displayed until a new message gets posted in the chat, but can be shown again by a user clicking on the linking message.
* The shared whiteboard allows chat participants to create drawings and shared graphic information with each other. Every whiteboard action is recorded as part of the evolving

history of the whiteboard. Users can manipulate a slide bar to navigate through all changes made in the whiteboard since the creation of the chat room.

* When someone types a new chat message, they can also select and point to a rectangular area in the whiteboard. When that message appears in the chat as the last posting, a bold line appears connecting the text to the area of the drawing (see Figure 1).

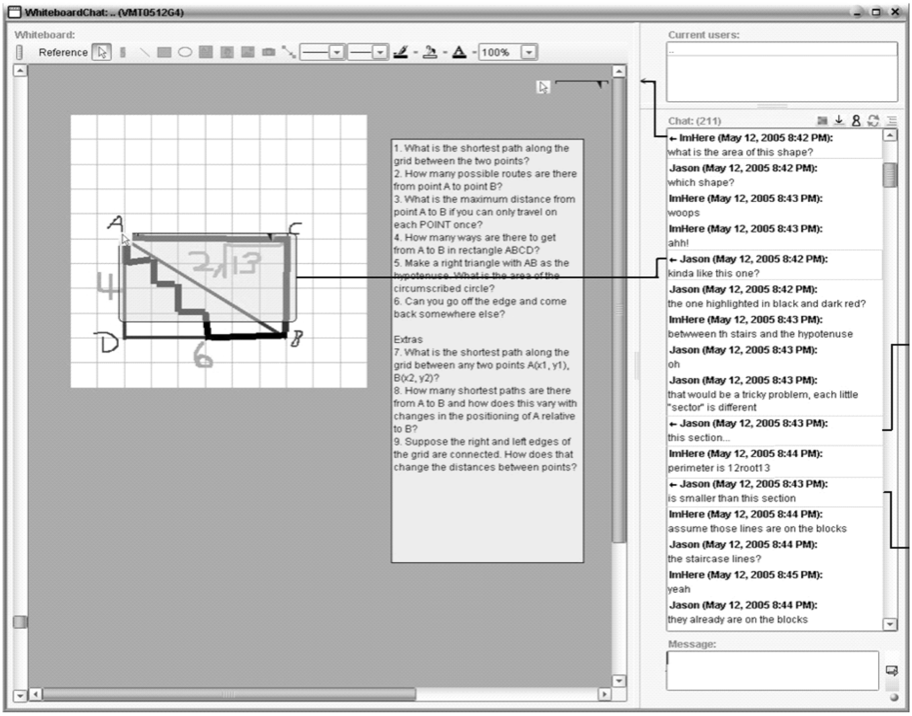
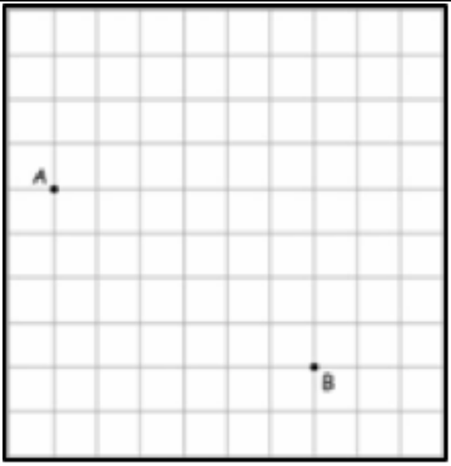


Figure 1. VMT/ConcertChat collaboration environment

In the Spring of 2005 and 2006, we conducted a series of pilot studies using ConcertChat. In each study we formed five virtual math teams, each containing about four middle-school students selected by volunteer teachers at different schools across the USA or abroad. The teams engaged in online math discussions for four hour-long sessions over a two-week period. They were given a brief description of an open-ended mathematical situation and were encouraged to explore this world, create their own questions about it, and work on those questions that they found interesting. For example, the teams participating in the 2005 study (and whose work we will use to illustrate our observations about collective creativity) worked in exploring a non-Euclidian world where the concept of distance between two points in space had to be redefined. The initial task as presented to the students is displayed in Figure 2.



Pretend you live in a world where you can only travel on the lines of the grid. You can't cut across a block on the diagonal, for instance Your group has gotten together to figure out the math of this place. For example, what is a math question you might ask that involves these two points?

Figure 2.Grid-world task

The observations that we will present in the following sections come from our qualitative analysis of resultant interaction logs. We will present these reflections starting at the micro-level of collaborative creative work and expanding towards more global interactional processes across collectivities and time spans.

Referencing and Indexicality

Indexicality, the referencing or symbolic pointing achieved through language and other means, is one of the unique aspects of group creativity which Sawyer [5] has described in his analysis of creative collaboration in music and theater groups. The role of indexicality is that of joining—in elaboration, contrast, reframing, etc.—the individual elements that the participants of a collectivity produce and reuse as part of their creative work. From this perspective, it is through a complex "configuration of indexicals" that the creative product is conceived to emerge through synchronous interactions. In our analysis of virtual math teams, we have been able to observe this primordial level of synchronous creative work promoted and supported by the online environment and its explicit referencing tools. Below, we describe an instance of such referencing work embedded in the collaborative mathematical work of one of the teams analyzed and offered as an initial and fundamental mechanism of creative work.

The chat log excerpt visible in Figure 1 is reproduced in Figure 3 (with line numbers added for referencing in this paper). In this interactional sequence, two team members discuss parts of a drawing that has already been constructed in the shared whiteboard. The students had created the drawing as part of discussions about shortest paths between points A and B in a grid-world were you can only travel along the lines of the grid (see Figure 2). In particular, a red triangle, ABD, was drawn with sides of length 4, 6 and $2\sqrt{13}$. A thick black staircase line was drawn as a path on the grid from A to B. In this excerpt, the students propose a math problem involving this drawing.

1 ImH: what is the area of this shape? [REF TO WB]

2 Jas: which shape?

3 ImH: woops

4 ImH: ahh!

5 Jas: kinda like this one? [REF TO WB]

6 Jas: the one highlighted in black and dark red?

7 ImH: between th stairs and the hypotenuse

8 Jas: oh

9 Jas: that would be a tricky problem, each little "sector" is different

10 Jas: this section [REF TO WB]]

11 ImH: perimeter is 12root3

12 Jas: is smaller than this section [REF TO WB]

13 ImH: assume those lines are on the blocks

14 Jas: the staircase lines?

15 ImH: yea

16 Jas: they already are on the blocks

Figure 3. Chat log. Line numbers added; names anonymized. Graphical references to the whiteboard indicated by [REF TO WB].

The message in line 1 of the chat excerpt (see Figure 3) proposes a mathematical question for the group to consider: "What is the area of this shape?" This is accompanied by a graphical reference to the whiteboard. The reference does not indicate a specific area—apparently ImH did not completely succeed in using this new referencing tool. Line 2 raises the question, "Which shape?" pointing out the incompleteness of the previous message's reference.

Lines 5 and 6 offer a repair of line 1's problem. First, line 5 roughs in the area that may have been intended by the incomplete reference. It includes a complete graphical reference that points to a rectangular area that includes most of the upper area of rectangle ACBD in the drawing. The graphical referencing tool only allows the selection of rectangular areas, so line 5 cannot precisely specify a more complicated shape. The text in line 5 ("kinda like this one?") not only acknowledges the approximate nature of its own referencing, but also acknowledges that it may not be a proper repair of line 1 and accordingly requests confirmation from the author of line 1. At the same time, the *like* reflects that this act of referencing is providing a model of what line 1 could have done.

Line 5 is accompanied by line 6, which provides a textual reference or specification for the same area that line 5 pointed to: the one highlighted in black (the staircase line) and dark red (lines AC and CB). The inexact nature of the graphical reference required that it be supplemented by this more precise textual reference. Note how the sequence of indexical attempts in lines 1, 2, 5 and 6 successively focuses shared attention on a more and more well-defined geometric object. This is an interactive achievement of the group (the interaction between ImH and Jas, observed by others and situated among the math objects co-constructed by all).

Lines 5 and 6 were presented as questions calling for confirmation by ImH. Clarification follows in line 7 from ImH: "between the stairs and the hypotenuse." Line 8's "Oh" signals a shift in the understanding of the evolving reference. Now that a complete reference has been co-constructed to a math object that is well enough specified for the practical purposes of carrying on the chat, Jas continues the problem-solving activity by raising an issue that must first be dealt with. Line 9 says that calculating the area now under consideration is tricky. The tricky part is that the area includes certain little "sectors" whose shapes and areas are non-standard. Line 9 textually references "each little 'sector'." *Little* refers to sub-parts of the target area. *Each* indicates that there are several such sub-parts and *sector*, put in scare quotes, is proposed as a name/description of these hard-to-refer-to sub-parts.

Lines 10 through 16 illustrate the kind of highly interactive work in which groups engage when creating and defining their problem space. Beyond simply clarifying an ambiguity in their vocabulary, this interaction represents the contingent and ongoing sense making that leads to the emergence of a fully meaningful math object that the group has created, started to specify and is about to start investigating.

In this example, the group has creatively produced a new mathematical object: a geometric area with interesting features that the group can explore and discuss. The ability of group members to discuss the new object relies on their establishment of a shared *configuration of indexicals* in terms of which features of the object ("this shape", "the stairs", "those lines") can be referenced. The intersubjective being-there-together in a chat is structured as a world of future possible activities with shared meaningful objects within this referential network. The possibilities for collaborative action are made available by the social, pedagogical and technical context of the VMT environment, but the group must creatively enact this by co-constructing a shared system of indexicality (its Heideggerian world of being-with, the situation, activity structure, network of relevant significance). The group creativity thereby consists in its establishment of the conditions and preconditions of its ability to engage in shared meaning making [7].

Next, we expand this synchronic characterization of the role of interactional referencing and indexicals to consider diachronic aspects of collective remembering and its role in creative work and learning.

Collective Remembering

The virtual teams involved in our studies demonstrated across their sessions a variety of methods for producing and managing relevant resources for their mathematical work. In each session they used the chat conversation and the shared whiteboard extensively to achieve this. In the previous section we analyzed how the chat conversation and the resources on the whiteboard where used to establish a new math object and begin its exploration. These textual and graphical resources and their co-constructed meaning allowed teams to create visualizations of strategies and ideas, to contrast several different representations of a problem situation, to coordinate different problem-solving paths among different team members, and to reconstruct collectively past work so that it can be continued in the present moment. We turn our attention now to this last aspect, which seems particularly interesting when considering the relationship between single-episode interactions (synchronic) and longer sequences of interaction (diachronic).

As an example, in the last session of one of the teams studied, the facilitator produces a summary of the teams' prior work and suggests that they continue from that point. However, the team has some new members and others who had missed the previous sessions, so this recommencement of the prior work was problematic. One of the participants who attended the last session attempts to remember what they were doing, and by doing so engages the group in the collective remembering of that prior work as can be seen in the following chat log (Figure 4).

121 MFmod: I think that the above section I wrote is where the group last was
122 MFmod: yes?

123 drago: well
124 gdog: i dont remember that
125 drago: actually, my internet connection broke on Tuesday
126 drago: so I wasn't here
127 MFmod: so maybe that is not the best place to pick up
128 estrickmcnizzle: i wasnt able to be here on tuesday either
129 gdog: how bout u meets
130 meets: uh...
131 meets: where'd we meet off....
132 meets: i remember
133 gdog: i was in ur group
134 meets: that we were trying to look for a pattern
135 gdog: but i didn't quite understand it
136 gdog: can u explain it to us again meets
137 meets: with the square, the 2by 2 square, and the 3by2 rectangle
138 meets: sure...
139 meets: so basically...
140 gdog: o yea
141 gdog: i sort of remember
142 meets: we want a formula for the distance between poitns A and B
143 drago: yes...
144 meets: ill amke the points
145 MFmod: since some folks don't remember and weren't here why don't you pick up with this idea and work on it a bit
146 meets: okay
147 meets: so there are those poitns A and B
148 meets: (that's a 3by2 rectangle
149 meets: we first had a unit square
150 meets: and we know that there are only 2 possible paths......

Figure 4. Chat excerpt from session 4 of Team 5.

One of the things that is remarkable about the way this interaction unfolds is the fact that although it might appear as if it is Meets who remembered what they were doing last time, the actual activity of remembering unfolds as a collective engagement in which different team members participate dynamically. In fact, later in this sequence there is a point where Meets remembers the fact that they had discovered that there are 6 different shortest paths between the corners of a 2-by-2 grid but he reports that he can only "see" four at the moment. Even though Drago did not participate in the original work leading to that finding, he was able to see the six paths when Meets presented the 2-by-2 grid on the whiteboard and proceeded to invent a method of labeling each point of the grid with a letter so that one can name each path and help others see it (e.g., "from B to D there is BAD, BCD …"). After this, Meets was able to see again why it is that there are six paths in that small grid and together with Drago, they proceeded to investigate, in parallel, the cases of a 3-by-3 and a 4-by-4 grid using the method just created. The result can be seen in Figure 5.

Despite the fact that Figure 5 is a restrictively static representation of the team's use of the whiteboard, it allows us to illustrate some unique aspects of this remarkable creative organization of their collective activity. First, we see again the crucial role of indexicals and referencing activity in the collective construction of the mathematical ideas of the team (e.g., through the use of labels, the witnessing of actions on the whiteboard, and the coordination of parallel activity).

The use of the whiteboard represents an interesting way of making visible the procedural reasoning behind a concept (e.g., shortest path). The fact that a newcomer can use the persistent history of the whiteboard to re-trace the team's reasoning seems to suggest a possible strategy towards preserving complex results of problem-solving activities. However, the actual meaning of these artifacts is highly situated in the doings of the co-participants, a fact that challenges the ease of their reuse despite the availability of detailed records such as those provided by the whiteboard history.

Despite these technical limitations, we could view the artifacts created by this team as "bridging" objects which, in addition to being a representation of the teams' moment-to-moment joint reasoning, could also serve for their own future work and for other members of the VMT online community. These particular objects are constructed *in situ* as a complex mix of resources that "bridge" different points in their own problem-solving and, potentially, those of others. As can be seen in Figure 5, the two team members combined the depiction of the cases being considered, the labeling and procedural reasoning involved in identifying each path, a summary of results for each case (i.e., the list of paths expressed with letter sequences) and a general summary table of the combined results of both cases. The structure of these artifacts represents the creative work of the team but also documents the procedural aspects of such interactions in a way that can be read retrospectively to document the past, or "projectively" to open up new possible next activities.

Despite the fact that the problem-solving artifacts and conversations are the result of the moment-by-moment interactions of a set of participants and, as such, require a significant effort for others to reconstruct their situated meaning, they can serve as one of the resources used to "bridge" problem-solving episodes, collectivities or even conceptual perspectives. Here, we use the term "bridging" to characterize interactional phenomena that cross over the boundaries of time, activities, collectivities, or perspectives as relevant to the participants themselves. Bridging thereby might tie events at the local small-group unit of analysis to interactions at larger units of analysis (e.g., the community). Bridging may reveal linkages among group meaning-making efforts, across collectivities or events in time, diachronically.

Next, we will present an instance of this type of interactional phenomena that is closely related to these diachronic aspects of group creativity.

Bridging the Past: Projecting to Others

So far, we have explored two aspects of the creative dimension of the work that virtual teams engaged in as part of our studies. We have seen that the use of referencing and the configuration of indexicals are necessary elements of the "synchronic" interactions of these teams but that they can also play a central role in processes such

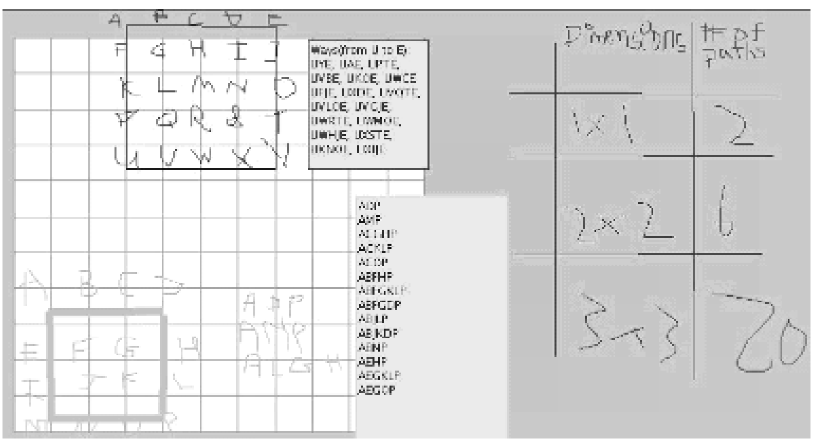


Figure 5. Shared whiteboard of Team 5, session 4.

as those that we have labeled "group remembering." As a matter of fact, we can see the central role of referencing as that of overcoming boundaries in joint activity.

Deictic expressions such as "the one highlighted in black and dark red" are sometimes used to overcome gaps in perception, while temporal deictic terms (e.g., last time, next time, etc.) can be used as part of the process of doing memory work and engaging with prior activities. In fact, in the contexts of extended sequences of collaborative knowledge work, where the membership of a team might change over time and where the trajectory of problem solving needs to be sustained over time, overcoming such boundaries might be especially challenging. We define this type of purposeful overcoming of boundaries through interaction as "bridging" work and turn our attention now to interactional strategies that virtual teams utilized to engage in these kinds of activity.

In order to investigate the dynamics of bridging we designed our studies so that a number of teams worked on the same task for a series of four sequential sessions. In our 2005 study, teams used a different ConcertChat room for each session and had no direct access to archives of their previous interactions. Despite this apparent limitation, they demonstrated several strategies to reconstruct their sense of history and to establish the continuity of their interactions.

The following excerpt represents an example of this, recorded during the second session of one of the participating teams, where two new team members, Gdo and Mathwiz, are joining the dyad that had collaborated in the first session, Drago and Estrickm.

302	gdo:	now lets work on our prob [Points to Whiteboard]
303	drago:	last time, me and estrickm came up
304	drago:	that
305	gdo:	……
306	drago:	you always have to move a certain amount to the left/right and a certain amount to the up/down
307	gdo:	what?
308	drago:	for the shortest path
309	drago:	see
310	drago:	since the problem last time
311	drago:	stated that you couldn't move diagonally or through squares
312	drago:	and that you had to stay on the grid
313	*gdo leaves the room*	
314	mathwiz:	would you want to keep as close to the hypotenuse as possible? or does it actually work against you in this case?

315	drago:	any way you go from point a to b
316	*gdo joins the room*	
317	drago:	is the same length as long as you take short routes
318	gdo:	opps
319	gdo:	internet problem
320	gdo:	internet problem
321	drago:	you always have to go the same ammount right, and the same ammount down

Figure 6. Excerpt from session two of Team 3.

This excerpt illustrates how the participants of this interaction chose to start a current collaborative task. Understandably, when teams sustain their collaborative work over multiple individual sessions, the task of recommencing knowledge-building activity becomes an issue that participants have to address. We can see that Drago's posting in line 303 ("last time, me and estrickm came up") stands as an uptake of the proposal for collective action put forward by Gdo in line 302 ("now lets work on our prob"). By contrasting "last time" with Gdo's "now", Drago attempts to establish a particular kind of episodic continuity or "relevant history" of the team (unavailable elsewhere in the collaboration environment), while at the same time categorizing Gdo and Mathwiz as newcomers and opening up the possibility of orienting to them as such.

Drago's posting in line 306 ("you always have to move a certain amount to the left/right and a certain amount to the up/down") completes the initiation of his bridging move, not only in a temporal sense but also as far as the problem-solving trajectory, since a prior discovery ("you always have to go…") is presented as relevant to re-start the problem-solving task of the team.

Naturally, it is not a simple task for the new members of the team to fully understand the meaning of Drago's summary but they engage in doing the situated work of making sense of it and using it. In fact, the reply posted in line 307 by Gdo ("what?") and the subsequent elaboration attempted by Drago suggest that the posting in 306 was taken as a problematic response to the proposal to initiate the problem-solving work. Perhaps additional work was necessary for line 306 to be fully sensible for the team—in other words, for Drago to successfully bridge prior work into the present. In the subsequent lines we can see the beginnings of an instance of the kind of interactional work that seems to be necessary for the team to engage with the reported past that Drago is presenting.

Even without a thorough understanding of the mathematical task at stake, one can see that Drago elaborates on his initial posting by providing additional problem information (308, "for the shortest path") and adding further references to elements of the past problem-solving activity (310-312, "since the problem last time stated that you couldn't…"). Furthermore, Mathwiz's posting in line 314 ("would you want to keep as close to the hypotenuse as possible? or does it actually work against you in this case?") engages with the bridging activity opened up by Drago in a particular way. Mathwiz seems to suggest a specific way of clarifying Drago's presentation of how the grid-world works while at the same time doing the interesting work of positioning Drago as the one to assess this suggestion (i.e., testing whether this case "works against you"). This short sequence signals only the beginnings of the type of interactional work necessary to fully bridge prior knowledge-work into present joint activity, and yet it is sufficient to provide significant evidence of the nuanced aspects of this type of activity.

This scenario of change in membership and continuity of task work is a clear example of the need for persistent supports in collaboration environments. However, simply providing Drago, Mathwiz, Gdo, and the rest of the team direct access to raw recordings of the team's prior sessions would probably be an inefficient solution. Even if the team was reusing the same persistent room for each session, the interactional ground that is so essential to the meaning of the chat and whiteboard records is not easily recovered and certainly not easily transferred or summarized. On the other hand, the successful bridging achieved by this team can be partially linked to the sophistication of their problem-solving work in the last session, especially when compared with other teams, which did not establish such a strong sense of continuity of their problem-solving trajectory.

At the moment, our analysis suggests that these attempts to establish continuity in collaborative problem solving involve: (a) the recognition and use of discontinuities or boundaries as resources for interaction, (b) changes in the participants' relative alignment toward each other as members of a collectivity, and (c) the use of particular orientations towards specific knowledge resources (e.g., the problem statement, prior findings, what someone professes to know or remember, etc). Bridging activity defines the interactional phenomena that cross over the boundaries of time, activities, collectivities, or perspectives. It defines a set of methods through which participants deal with the discontinuities relevant to their joint activity.

As a result of our initial findings, we designed in our 2006 study a setting in which "bridging" could be investigated more conspicuously. We arranged for the teams to reuse the same persistent chat rooms so that they had direct access to the entire history of their conversations and their manipulations on the whiteboard across the four sessions. In addition, mentors provided explicit feedback by leaving a note on the whiteboard of each team's room in between sessions. Finally, we also provided a wiki space to help the teams share their explorations (e.g., formulae found, new problems suggested by their work, etc.). We have just begun to analyze the results of this study in which we hope to better analyze the interrelationship between synchronic and diachronic interactions. Below, we provide some of our initial observations.

The reuse of the same room by teams that were much more stable in their membership over time proved effective in stimulating the constructive establishment of continuity in the creative and problem-solving activity of the teams. The feedback provided by the external mentors, however, was in several cases problematic since it re-framed past experiences in ways that seemed unfamiliar or curious to the participants themselves. In addition, the use of the wiki space provided us with a set of interesting examples of new “bridging” activity being conducted by the teams.

Through the wiki postings, teams working on the same or similar task were made aware of the parallel work being conducted by their counterparts. In several cases, the wiki acted as an effective third workspace from which materials generated by one team could be used, validated, and advanced by other teams. The authors of the postings also used them to sustain their own problem-solving across the four sessions. Postings and trajectories of use in the wiki also showed a structure that was very different from the conversational and interactional style of the chat room artifacts. Some postings were purposively vague and others resembled highly elaborate summaries of the teams’ findings. In a few cases, postings included a narrative structure abstracted from the chat sessions (e.g., “So in session 3, our team tried to understand Team C's formula …”).

In one instance, the wiki presented evidence of cross-team asynchronous interactions: Team B found a new problem generated by another team in addition to a possible solution. Team B proceeded to work on the problem, found a mistake in the solution formula originally reported, and proceeded to re-work the original solution and post the corrected result back to the wiki.

These preliminary findings seem to suggest both the potential of explicit bridging spaces to promote continuity and sustain creativity in problem-solving work, especially in the context of an online community formed of multiple virtual teams with overlapping interests and activities. Naturally, the availability of bridging resources like the wiki does not by itself shape the ways participants interact over time. In addition, the fact that certain social practices were promoted (e.g. reporting to others, imitating, reflecting, etc.), also influenced the way these resources were used.

Conclusions

Several models have been proposed to characterize components of individual creativity, such as the ability to concentrate efforts for long periods of time, to use "productive forgetting" when warranted, and to break “cognitive set” [1]. We can predict that these individual skills also play a role that is distinctively critical in the context of long-term collective knowledge building. In fact, we have seen in our analysis of virtual math teams that some of these individual accomplishments are also crucial as social interactions and rely on basic interactional mechanisms such as referencing, group remembering and the bridging of discontinuities.

When one looks seriously at the interactional activity that goes into the formulation and communication of creative ideas, one sees the limitations of traditional, ahistorical views of creativity as decontextualized and instantaneous "inspiration" that mysteriously comes to the lone genius. Creativity involves extended efforts to articulate, criticially consider and communicate notions that are not already part of the taken-for-granted life-world. Even when accomplished largely by an individual person, this generally involves trials with physical and/or textual artifacts [8]. Such internal monologue generally incorporates skills learned from dialogues in dyads or small groups [9]. The study of creative accomplishments in groups, where their interactions can be made visible for analysis, may provide insights about individual and group creativity.

Recent models of group creativity [5] argue that collective creative work has to be understood as the synergy between synchronic interactions (i.e. parallel and simultaneous) and diachronic exchanges (i.e. interaction over long time spans, and mediated by ostensible products). Our analysis validates this model in the context of the creative and problem-solving work of virtual math teams and starts to provide an interactional description of some of the processes underlying these two types of interaction.

Because continuity in itself is important to the success of virtual teams, we have observed how participants develop a series of bridging methods to co-construct mathematical knowledge within single collaborative episodes as well as over time, evolve a sense of collectivity, and interlink their collaborative interactions with those of others.

Just as we have argued that cognition should not be conceptualized solely or even predominantly as a fundamentally individual phenomenon [6], so we claim that creativity is often rooted in social interaction and that innovative creations should often be attributed to collectivities as a feature of their group cognition.

Acknowledgments

The Virtual Math Teams Project is a collaborative effort at Drexel University. Gerry Stahl directs it, with co-PIs Stephen Weimar and Wesley Shumar. Johann Sarmiento manages the project, with fellow graduate RAs Murat Cakir, Ramon Toledo and Nan Zhou. Alan Zemel leads the data sessions. A number of Math Forum staff work on the project. The following visiting researchers have spent 3 to 6 months on the project: Jan-Willem Strijbos (Netherlands), Fatos Xhafa (Spain), Stefan Trausan-Matu (Romania), Elizabeth Charles (Canada), Weiqin Chen (Norway). The ConcertChat software was developed at the Fraunhofer Institute IPSI in Darmstadt, Germany, by Martin Wessner, Martin Mühlpfordt and colleagues. The VMT project is supported by grants from the NSDL, IERI and SLC programs of the US National Science Foundation.

References

1. Amabile, T. M. The *Social Psychology of Creativity*. New York: Springer-Verlag, 1983.
2. Csikszentmihalyi, M. *Society, culture, person: A systems view of creativity*. In R.J. Sternberg (Eds.) The nature of creativity (pp. 325-339). New York: Cambridge University Press, 1988.
3. Csikszentmihalyi, M. *The domain of creativity*. In M.A. Runco & R.S. Albert (Ed.) Theories of creativity (pp. 190-212). Newbury Park, CA: Sage, 1990.
4. Mühlpfordt, M., & Wessner, M. *Explicit referencing in chat supports collaborative learning.* Paper presented at the international conference on Computer Support for Collaborative Learning (CSCL 2005), Taipei, Taiwan, 2005.
5. Sawyer, R. K. *Group Creativity: Music, Theater, Collaboration.* Mahwah, NJ: Lawrence Erlbaum, 2003.
6. Stahl, G. Group Cognition: Computer Support for Building Collaborative Knowledge. Cambridge, MA: MIT Press, 2006.
7. Stahl, G. *Meaning making in CSCL: Conditions and preconditions for cognitive processes by groups.* Paper presented at the international conference on Computer Support for Collaborative Learning (CSCL 2007), New Brunswick, NJ, 2007.
8. Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. New York, NY: Basic Books.
9. Vygotsky, L. (1930/1978). *Mind in society*. Cambridge, MA: Harvard University Press.

7. Polyphonic Support for Collaborative Learning

Stefan Trausan-Matu, Gerry Stahl and Johann Sarmiento

Abstract. This paper argues that one reason for the success of collaborative problem solving where individual attempts failed is the polyphonic character of work in small groups. Polyphony, a concept taken from music, may occur in chats for problem solving, transforming dialog into a "thinking device": Different voices jointly construct a melody (story, or solution) and other voices adopt differential positions, identifying dissonances (unsound, rickety stories or solutions). This polyphonic interplay may eventually make clear the correct ("sound") construction. The paper illustrates the polyphonic character of collaborative problem solving using chats. It also proposes prototyped software tools for facilitating polyphony in chats.

1 Introduction

This paper considers the role of polyphonic inter-animation of multiple voices in collaborative learning. Inspired by the work of Mikhail Bakhtin, this idea sheds new light on the dialogic nature of discourse in human language. It could also have consequences for the design of collaborative learning environments.

In polyphony, several voices jointly construct a melody (or a story, or a potential solution in the textual-chat case) while other voices situate themselves on a differential position, identifying dissonances (unsound, rickety stories or solutions). This polyphonic game may eventually make clear the correct, sound solution.

The ideas are exemplified with chat excerpts for collaborative learning of mathematics problem solving, investigated in the Virtual Math Teams (VMT) project at Math Forum @ Drexel University. Inter-animation patterns in two dimensions were discovered: longitudinal (chronologically sequential) and vertical, towards two opposite trends: unity vs. difference. We consider that even individual thinking is also an implicit collaborative (dialogic) process that involves multiple voices. However, actual collaborations, in small groups of different personalities empower the dialogic process.

An environment for collaborative learning (that may be seen also as a groupware) based on the polyphonic inter-animation principles is introduced. Several modules are already implemented while others are in a final stage.

The paper continues by introducing discourse, the dialogic theory of Mikhail Bakhtin and polyphony. The next section of the paper introduces Computer-Supported Collaborative Learning (CSCL) and analyses the polyphonic welding of longitudinal-vertical unity-difference dimensions. Software tools that support the polyphonic inter-animation are presented in the fourth section. The paper ends with conclusions and references.

2 Discourse, Dialogic and Polyphony

Learning may be seen as directly related to discourse building, as Sfard remarked: "rather than speaking about 'acquisition of knowledge,' many people prefer to view learning as *becoming a participant in a certain discourse*" [11]. Koschmann [5] emphasized the social dimension of learning and discourse, quoting Deborah Hicks [4]: "Learning occurs as the co-construction (or reconstruction) of social meanings from within the parameters of emergent, socially negotiated, and discursive activity" (p. 136).

The above ideas follow the socio-cultural learning paradigm initiated by Vygotsky. He has a permanently increasing influence on learning theories, stating that learning is a social process, mediated by specific tools, in which symbols and especially human language plays a central role [15]. However, he did not investigate in more detail how the language and discourse are actually used in collaborative activities. It is the merit of Mikhail Bakhtin to propose a sound theory of how meaning is socially constructed.

Bakhtin extended Vygotsky's ideas in the direction of considering the role of language and discourse, with emphasis on speech and dialog. Bakhtin raises the idea of dialogism to a fundamental philosophical category, *dialogistics*. For example, Voloshinov (a member of Bakhtin's circle who, according to many opinions, signed a book written by his more famous friend because the former has an interdiction to publish during Stalin regime) said: "… *Any true understanding is dialogic in nature*. Understanding is to utterance as one line of dialogue is to the next" [14]. This is in consonance with Lotman's conception of text as a "thinking device" [17], determining that: "The semantic structure of an internally persuasive discourse is not *finite*, it is *open*; in each of the new contexts that dialogize it, this discourse is able to reveal ever new *ways to mean*" [1].

Any discourse may be seen as an intertwining of at least two threads belonging to dialoguing voices. Even if we consider an essay, a novel or even a scientific paper, discourse should be considered implying not only the voice of the author. The potential listener has an at least as important role. The author makes a thread of ideas, a narrative. Meanwhile, in parallel to it, he must take into account the potential flaws of his discourse; he must see it as an utterance that can be argued by the listener. In this idea, discourse is similar to dialog and to music polyphony (in fact, it should not be a surprise that different art genres like music, literature and conversation have similar features), where different voices interanimate.

Discursive voices weave sometimes in a polyphonic texture, a feature which Bakhtin admired so much in Dostoyevsky's novels. They are characterized by Bakhtin as "a plurality of independent and unmerged voices and consciousnesses" [2]. However, polyphony is not only a random overlay of voices. It has also musicality; it is in fact one of the most complex types of musical compositions, exemplified by the complex contrapuntal fugues of Johann Sebastian Bach. "When there is *more than one independent melodic line happening at the same time* in a piece of music, we say that the music is contrapuntal. The independent melodic lines are called counterpoint. The music that is made up of counterpoint can also be called polyphony, or one can say that the music is polyphonic or speak of the polyphonic texture of the music." [7].

In polyphonic music, the melodic, linear dimension is not disturbing the differential, vertical harmony. Moreover, for example, in Bach's fugues, the voices inter-animate each other. The main theme is introduced by a voice, reformulated by the others, even contradicted sometimes (e.g. inverted) but all the voices keep a vertical harmony in their diversity.

Starting from Bakhtin's ideas, we extend these ideas to collaborative learning. Therefore, we will further describe how polyphony may arise in collaborative learning and we will propose ways of supporting it in learning environments.

3 The Polyphony of Problem Solving Chats

3.1 Collaborative Learning in Virtual Math Teams

Computer and communication technologies offer now new possibilities for collaboration, by virtualizing classroom group interaction. New types of artifacts like hypertext, the World Wide Web, chats or forums of discussions, are changing the classical learning scenarios. In addition to classical sheets of paper or blackboards for drawing diagrams and writing formulas and sequences of problem solving steps, computer animations, simulations or even virtual participants in the dialog (artificial agents) may be used now for collaboration. It is extremely important to analyze the particularities of discourse in this new context. A good example is the fact that in chats we can much more easily use a multiple threaded discourse, similar to contrapuntus in classical music than in face-to-face conversations.

The (VMT) research program investigates the innovative use of online collaborative environments to support effective K-12 mathematics learning as part of the research and development activities of the Math Forum (mathforum.org) at Drexel University. VMT extends the Math Forum's "Problem of the Week (PoW)" service by bringing together groups of 3 to 5 students in grades 6^{th} to 11^{th} to collaborate online in discussing and solving non-routine mathematical problems. Currently, participants interact using a computer-supported collaborative learning environment, which combines quasi-synchronous text-based communication (e.g., chat) and a shared whiteboard among other interaction tools.

At the core of VMT research is the premise that primarily, group knowledge arises in discourse and is preserved in linguistic artifacts whose meaning is co-constructed within group processes [10]. Key issues addressed by the VMT include the design challenge of structuring the online collaborative experience in a meaningful and engaging way, and the methodological challenge of finding appropriate methodological approaches to study the forms of collaboration and reasoning that take place.

3.2 Polyphonic Inter-animation in Chats

Let us consider the following problem:

> Three years ago, men made up two out of every three internet users in America. Today the ratio of male to female users is about 1 to 1. In that time the number of American females using the internet has grown by 30,000,000, while the number of males who use the internet has grown by 100%. By how much has the total internet-user population increased in America in the past three years? (A) 50,000,000 (B) 60,000,000 (C) 80,000,000 (D) 100,000,000 (E) 200,000,000

This problem was one of an eleven-problem set that were used for an experiment in which a group of students had to solve first individually and after that collaboratively, using chat. It was one of the two that were not solved individually by any students but it was solved collaboratively.

Let us now consider a chat excerpt that includes the main utterances that contributed to the finding of the solution (see figure 1).

350 Mic how do we do this..
351 Mic Without knowing the total number
352 Mic of internet users?
....
357 Dan it all comes from the 30000000
358 Mic did u get something for 10?
359 Dan we already know
360 Mic 30000000 is the number of increase in american females
361 Mic and since the ratio of male to female
362 Mic is 1 to 1
363 Mic thats all i got to give. Someone finish it
364 Mic Haha
365 Cosi haha you jackass
366 Mic Haha
367 Dan Hahaha
368 Mic u all thought i was gonna figure it out didn't
369 Mic U
370 Mic huh?
371 Hal it would be 60,000,000
372 Mic Hal
373 Mic its all u
374 Mic See
375 Mic i helped
376 Cosi ok, so what's 11 – just guess on 10
....
386 Mic lets get back to 5
387 Cosi i think it's more than 60,00000
388 Mic way to complicate things
389 Cosi haha sorry
390 Mic life was good until you said that
391 Mic :(
392 Cosi they cant get higher equally and even out to a 1 to 1 ratio
393 Cosi oh, no wait, less than that
394 Cosi 50000000
395 Cosi yeah, it's that
396 Cosi im pretty sure
397 Mic Haha
398 Mic how?
399 Cosi because the women pop had to grow more than the men in order to even out
400 Cosi so the men cant be equal (30)
401 Mic oh wow...
402 Mic i totally skipped the first sentencwe
403 Cosi therefore, the 50,000,000 is the only workable answer
404 Dan very smart
405 Cosi Damn im good

Fig. 1. An excerpt illustrating the collaborative solution construction

Discourse begins with Dan's idea of starting from the 30000000 number specified in the problem statement (line 357). It continues with Mic's problem solving buffoonery (lines 360-364, 366 and 368-370), remarked by Cosi (line 365) and Dan (line 367): Mic seems to start writing a reasoning but he only fakes, writing fragments of the problem statement linked by a typical phrase "… and since … ". However, this fake discourse fragment seems to belong to a mathematics speech genre and, even being a pastiche, is continued by Hal which extrapolates the 1:1 ratio from the present (as stated in problem) to the whole 3 years and advances 60000000 as a solution (line 371).

Mic continues the buffoonery (lines 372-375). After about one minute, Cosi's (incorrect) utterance "i think it's more than 60,00000" appears as a critique or as an intuition of something wrong, of some kind of an "unsuccessful story". Nevertheless, after less than another minute, she realizes that her own supposition is wrong because the ratio cannot be 1:1 or bigger.

The collaborative discourse enabled Cosi to solve the problem. She didn't solve it in the first phase, when they had to solve it individually. However, when she listened to the discourse proposing a solution (correct in the case of Dan's beginning proposal, fake at Mic and wrong at Hal), she felt the need to put herself on a different position. Therefore, the discourse acted as a tool, as an artifact that enabled Cosi to find the correct answer.

Discourse in chat collaborative problem solving has an obvious sequential, longitudinal, time-driven structure in which the listeners are permanently situated and in which they emit their utterances in a threaded manner. In parallel with this linear threading dimension, the participants situate themselves meanwhile also on a critical, transversal (or differential) position. For example, in the excerpt considered in this section, Dan's theme was continued by Mic's buffoonery, continued itself by Hal and then contradicted by a first theme of Cosi that was eventually totally changed, in its opposite. We could say that the critique of Cosi appeared as a need to bring the harmony of a correct solution.

In this longitudinal-transversal space, voices behave in an unity-difference manner. This phenomenon is not specific solely to chats. It appears also to polyphonic music: "The deconstructivist attack (…) – according to which only the difference between difference and unity *as an emphatic difference* (and not as a return to unity) can act as the basis of a differential theory (which dialectic merely claims to be) – is the methodical point of departure for the distinction between polyphony and non-polyphony." [6].

The unity and difference trends take different shapes in chat problem solving. We can include in the unity category cumulative talk [8] or collaborative utterances [9], repetitions [12], socialization or jokes. For example, many times participants in chats feel the need to joke, probably in the need to establish a closer relation with other participants, in order to establish a group flow state [3]. In fact, in all the chats we examined there is a preliminary socialization phase, inter-animation appearing not immediately after the beginning of chats.

4 Groupware for Polyphonic Inter-animation

Difference making has a crucial role in chats for collaborative learning, role which may be best understood from a polyphonic, musical perspective. The possibility of contemplating (listening), from a critical position, the ideas (melodies) of other peoples and entering into an argumentation (polyphony of voices), enhance problem solving and enables learning through a trial-error process. Such processes appear also in individual problem solving (we can say that thinking is also including multiple inner voices) but the presence of multiple participants enhance both the possibility of developing multiple threads and, meanwhile, of differences identification. The inter-animation of the multiple perspectives of the participants, the opposition as result of contemplation and the presence of a third opinion in case of conflict, and sometimes the synthesis it brings are a better asset to success than a multi-voiced discourse performed by an individual (as inner thinking), that is inherently much less critique.

Evidence that participants permanently keep a differential position is also provided by the statistics of personal pronouns usage in chat sessions. For example, in a corpus of chats recorded in May 2005, "I" was used 727 times, much more than the usage of "we", with 472 occurrences. First person "me" was used 84 times comparing to "us", used only 34 times. However, the second person addressing is very well represented by 947 uses of "you."

A natural consequence of the theoretical considerations discussed above is the need for a software support for small groups that facilitates polyphonic development. Such a groupware, named "POLYPHONY", is now under development. The system is built around a chat system, which has some additional modules, not present in usual instant messaging. These modules offer abstractions of the ongoing chat, in the idea of making clear the flow of ideas and the other "voices" (the melody) and, the most important, to induce polyphonic, differential ideas .

In figure 2, a snapshot of one of the first implemented modules, the summarizer, is illustrated. This module builds a summary using natural language processing and heuristics, It automatically assigns an importance score to each utterance, and selects the most important utterances. Summarization is important in chats because knowing what came before, starting from clear summaries would help people to respond, to carry on the "melody" and to contribute to the polyphony with a personal, differential voice.

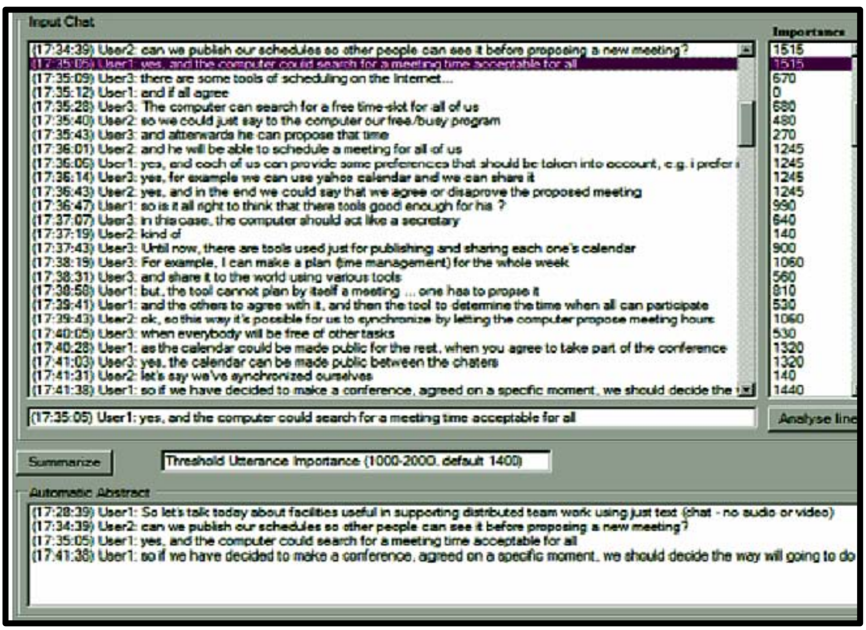


Fig. 2. A summarization module that offers an abstraction of the flow of main ideas

In addition to the summarization module, other facilities for chats, based on natural language processing are developed in POLYPHONY. They abstract and display facts about each participant, for example, the emotional state, the degree of relevance of the utterances of each participant. A module for speech acts identification has been already implemented [13]. The goals aimed by these modules are to induce self-reflection and images about the others, to facilitate inter-animation, and finally to encourage multiple voices to enter into a polyphonic framework.

5 Conclusions

Discourse in chats implies an inter-animation of multiple voices along two dimensions, the sequential, utterance threading and the transversal, differential one. These two dimensions correspond to a unity-difference (or centrifugal-centripetal, [1]) basic feature of polyphony. The unity directed dimension is achieved at diverse discourse levels by repetitions, collaborative utterances, socializing and negotiation discourse segments.

The second, differential dimension could be better understood if we consider discourse as an artifact that, taking into account that every participant in collaborative activities has a distinct personality, is a source of a critical, differential attitude. Even if individual, inner discourse may be multi-voiced, difference and critique are empowered in collaborative contexts, in a community of different personalities.

A consequence of the sequential-differential perspective for the design of CSCL environments is that they must facilitate inter-animation not only on the longitudinal dimension, through threading but also the transversal, differential, critical dimension. Tools that may enter in this category should be able to provide abstractions or summarizations of previous discourse, in order to facilitate differential position taking. They should also allow the participants to emphasize the different proposed themes and to relate them in threads, polyphonically.

Wegerif also advocates the use of a dialogic framework for teaching thinking skills by inter-animation: "meaning-making requires the inter-animation of more than one perspective" [16]. He proposes also that questions like "'what do you think?' and 'why do you think that ?' in the right place can have a profound effect on learning" [16]. However, he did not remark the polyphonic feature of inter-animation.

Acknowledgements

The authors wish to express their appreciation to the members of the Virtual Math Teams research project at Drexel University, whose voices are present in different ways in the paper. They also want to thank to the anonymous reviewers for their very useful remarks. The research presented here has been partially performed under a Fulbright Scholar post-doc grant (awarded to Stefan Trausan-Matu) and was also supported by the NSF grants REC 0325447 and DUE 0333493. Any opinions, findings, or recommendations expressed are those of the authors and do not necessarily reflect the views of the sponsors

References

1. Bakhtin, M.M. (1981). *The Dialogic Imagination: Four Essays*, University of Texas Press.
2. Bakhtin, M.M. (1984). *Problems of Dostoevsky's Poetics*, Theory and History of Literature Series, vol. 8, Minneapolis.
3. Csikszentmihalyi, M. (1990). Flow: The Psychology of Optimal Experience, Harper Collins.
4. Hicks, D. (1996). Contextual inquiries: A discourse-oriented study of classroom learning. In D. Hicks (Ed.), Discourse, Learning, and Schooling (pp. 104-141), Cambridge University Press.
5. Koschmann, T. (1999). Toward a Dialogic Theory of Learning: Bakhtin's Contribution to Understanding Learning in Settings of Collaboration, in C.Hoadley and J. Roschelle (eds.), Proceedings of the Computer Support for Collaborative Learning 1999 Conference, Stanford, Laurence Erlbaum Associates.
6. Mahnkopf, C.S. (2002). Theory of Polyphony, in Mahnkopf CS, Cox F & Schurig W (eds). *Polyphony and Complexity*, Hofheim, Germany: Wolke Verlags Gmbh.
7. Polyphony (2005). http://cnx.rice.edu/content/m11634/latest/, retrieved on 4th May, 2005
8. Mercer, N. (2000). Words and Minds. How we use language to think together, Routledge.
9. Sacks, H. (1992). Lectures on conversation. Oxford, UK: Blackwell.
10. Schegloff, E. A. (1997). "Narrative Analysis" Thirty Years Later. Journal of Narrative and Life History, 7(1-4), 97-106.
11. Sfard, A. (2000). On reform movement and the limits of mathematical discourse, Mathematical Thinking and Learning, 2(3), 157-189.
12. Tannen, D. (1989). Talking Voices: Repetition, Dialogue, and Imagery in Conversational Discourse, Cambridge University Press.
13. Trausan-Matu, S., C. Chiru, R. Bogdan, (2004). Identificarea actelor de vorbire în dialogurile purtate pe chat, in Stefan Trausan-Matu, Costin Pribeanu (Eds.), Interactiune Om-Calculator 2004, Editura Printech, Bucuresti, pp. 206-214.

14. Voloshinov (1973). Marxism and the Philosophy of Language, New York Seminar Press.
15. Vygotsky, L. (1978). Mind in society. Cambridge, MA: Harvard University Press.
16. Wegerif, R. (2005). A dialogical understanding of the relationship between CSCL and teaching thinking skills. In T. Koschman, D. Suthers, & T.W. Chan (Eds.). *Computer Supported Collaborative Learning 2005: The Next 10 Years!* Mahwah, NJ,.pp. 707-716.
17. Wertsch, J.V. (1991), *Voices of the Mind*, Harvard University Press.

8. Book review: Exploring thinking as communicating in CSCL

Anna Sfard, *Thinking as communicating: Human development, the growth of discourses and mathematizing*, Cambridge University Press, 2008.

Reviewed by Gerry Stahl

Anna Sfard raised the methodological discourse in the CSCL community to a higher niveau of self-understanding a decade ago with her analysis of our two prevalent metaphors for learning: the acquisition metaphor (AM) and the participation metaphor (PM). Despite her persuasive argument in favor of PM and a claim that AM and PM are as incommensurable as day and night, she asked us to retain the use of both metaphors and to take them as complementary in the sense of the quantum particle/wave theory, concluding that

> Our work is bound to produce a patchwork of metaphors rather than a unified, homogenous theory of learning. (Sfard, 1998, p. 12)

A first impression of her new book is that she has herself now come closer than one could have then imagined to a unified, homogenous theory of learning. It is a truly impressive accomplishment, all the more surprising in its systematic unity and comprehensive claims given her earlier discussion. Of course, Sfard does not claim to give the last word on learning, since she explicitly describes how both learning and theorizing are in principle open-ended. One could never acquire exhaustive knowledge of a domain like math education or participate in a community culture in an ultimate way, since knowledge and culture are autopoietic processes that keep building on themselves endlessly.

Sfard does not explicitly address the tension between her earlier essay and her new book. To reconcile her two discourses and to assess their implications for the field of CSCL, one has to first review her innovative and complex analysis of mathematical thinking.

Understanding Math Objects

Sfard introduces her presentation by describing five quandaries of mathematical thinking. I will focus on just one of these, which seems particularly foundational for a theory of math cognition, though all are important for math education: What does it mean to *understand* something in mathematics? Sometimes we ask, What is *deep understanding* in math (as opposed to just being able to go through the procedures)? I am particularly interested in this question because in my research group we are observing the chat of an algebra student who repeatedly says things like, "the formula makes sense to me… but I do not see why it should either" (see chat screenshot in Figure 1). For us as analysts, it is hard to know how Aznx cannot see why the equation is right if it makes sense to him; the nature of his understanding seems to be problematic for him as well as for us. One assumes that either he "possesses" knowledge about the applicability of the formula or he does not.

According to Sfard's theory, a math object—like the equation that Bwang is proposing in the chat for the number of blocks in stage N of a specific kind of pyramid—is an *objectification* or reification of a discursive process, such as counting the blocks at each stage (see also Wittgenstein, 1944/1956, p.3f, §3). In fact, we observe the team of students in the chat environment visibly constructing the pyramid in their shared whiteboard. Looking through Sfard's eyes, we can watch the students counting in a variety of ways, sometimes by numbering the graphical representations of blocks, other times by referencing shared drawings of the blocks from the chat postings, or by coordinating the sequential drawing of arranged blocks with the chat discussion in ways that make visible to the other students the enumeration of the pattern.

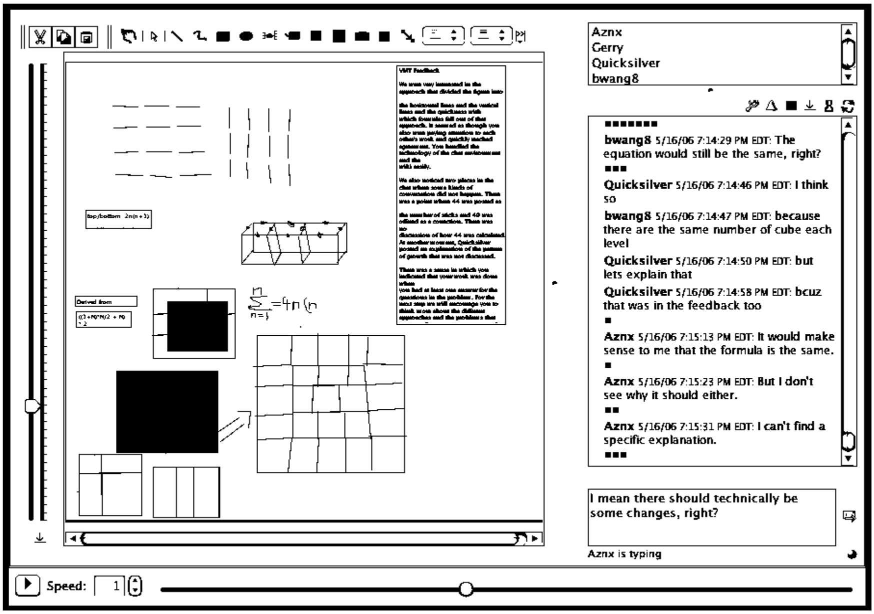


Figure 1. Three students chat about the mathematics of various formations of stacked blocks. Aznx expresses uncertainty about his understanding of Bwang's proposal about a formula and his ability to explain the formula in response to Quicksilver.

Sfard's central chapters spell out the ways in which math objects are subsequently co-constructed from these counting communication processes, using general procedures she names *saming*, *reification,* and *encapsulation*. Note, for instance, that Bwang is explicitly engaged in a process of saming: claiming that a set of already reified math objects (previous and current equations the students are discussing) are "the same." He states, " The equation would still be the same, right? … Because there are the same number of cube[s on] each level." He has reified the counting of the blocks into the form of a symbolic algebraic expression, which looks like an object with investigable attributes, rather than a discursive counting process. If he were a more expert speaker of math discourse, Bwang might even encapsulate the whole set of same equations as a new object, perhaps calling them *pyramid equations*. And so it goes.

In our case study, Aznx, Bwang, and Quicksilver engage in four hours of online collaborative math discourse. They consider patterns of several configurations of blocks that grow step by step according to a rule (see also Moss & Beatty, 2006). They develop recursive and quadratic expressions for the count of blocks and number of unduplicated sides in the patterns. They decide what to explore and how to go about it, and they check and question each other's math proposals, collaboratively building shared knowledge. Their group knowledge[21] is fragile, and the team repeatedly struggles to articulate what they have found out and how they arrived at it, encouraged to explain their work by the facilitator, who places the textbox of feedback in their whiteboard. During their prolonged interaction, the group creates a substantial set of shared drawings and chat postings, intricately woven together in a complex web of meaning.

Sfard describes the discursive construction of math objects, which—as Husserl (1936/1989) said—is *sedimented* in the semiotic objects themselves. To paraphrase and reify Sfard's favorite Wittgenstein quote[22], the use (the construction process) is embodied in the sign as its meaning. She lays out the generative process by which a tree of *realizations* is built up through history and then reified by a new symbolic realization that names the tree. The algebraic equation that Bwang proposes is one such symbolic expression. The students have built it to encapsulate and embody various counting processes and graphical constructions that they have produced together. The equation also incorporates earlier math objects that the group has either co-constructed or brought into their discourse from previous experience (e.g., Gauss' formula for the sum of N consecutive integers, previously learned in their math classrooms).

A centerpiece of Sfard's theory is the definition of a math object as the recursive tree of its manifold visual realizations. I will not attempt to summarize her argument because I want to encourage you to read it first hand. It is presented with all the grace, simplicity, insight and rigor of an elegant mathematical proof. It is itself built up from quasi-axiomatic principles, through intermediate theorems, illustrated with persuasive minimalist examples.

It is this definition of math object that, I believe, provides the germ of an answer to the conundrum of deep math understanding. That is, to understand a math object is to understand the realizations of that object. One must be able to unpack or de-construct the processes that are reified as the object. To be able to write an equation—e.g., during a test in school, where the particular equation is indicated—is not enough. One must to some extent be able to re-create or derive the equation from a concrete situation and to display alternative visual realizations, such as graphs, formulas, special

[21] The use of the term *group cognition* for referring to the discursive methods that small groups collaboratively use to accomplish cognitive tasks like solving problems often raises misunderstandings because readers apply AM when they see the noun *cognition*. They wonder where the acquired cognitive objects are possessed and stored, since there is no individual physical persisting agent involved. If one applies PM instead, in line with Sfard's theory, then it makes much more sense that discursive objects are being built up within a publicly available group discourse.

[22] "For a large class of cases—though not for all—in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language." (1953, p.20, §43)

cases and tables of the equation. There is not a single definition of the equation's meaning, but a network of inter-related realizations. To deeply understand the object, one must be conversant with multiple such realizations, be competent at working with them, be cognizant of their interrelationships and be able to recognize when they are applicable.

Routines of Math Discourse

Sfard then moves from ontology to pedagogy—from theory of math objects to theory of discourses about such objects, including how children come to participate in these discourses and individualize the social language into their personal math thinking. Based on her intensive work with data of young children learning math, she describes with sensitivity and insight how children come to understand words like *number*, *same*, *larger* and other foundational concepts of mathematical cognition. It is not primarily through a rationalist process of individual, logical, mental steps. It is a discursive social process; not acquisition of knowledge, but participation in co-construction of realizations. Sfard describes this as participation in social *routines*—much like Wittgensteinian language games. She describes in some detail three types of routines: *deeds*, *explorations,* and *rituals*. *Routines* are meta-level rules that describe recurrent patterns of math discourse. Like Sfard's discussion itself, they describe math discourses rather than math objects. *Deeds* are methods for making changes to objects, such as drawing and enumerating squares on the whiteboard. *Explorations* are routines that contribute to a theory, like Bwang's proposal.

Rituals, by contrast are socially oriented. The more we try to understand Aznx's chat postings, the more we see how engaged he is in social activity rituals. He provides group leadership in keeping the group interaction and discourse moving; reflecting, explaining, responding to the facilitator, positioning his teammates, assigning tasks to others. His mathematical utterances are always subtly phrased to maintain desirable social relations within the group and with the facilitator—saving face, supporting before criticizing, leaving ignorances ambiguous, checking in with others on their opinions and understandings, positioning his teammates in the group interaction, and assigning tasks to others. Each utterance is simultaneously mathematical and social, so that one could not code it (except for very specific purposes) as simply *content*, *social,* or *off-topic* once one begins to understand the over-determined mix of work it is doing in the discourse. Similarly, Bwang's explicitly mathematical proposals (explorations) are always intricately situated in the social interactions. Quicksilver often reflects on the group process, articulating the group routines to guide the process. Sfard's analysis helps us see the various emergent roles the students' participations play in their discourse—without requiring us to reduce the complexity of the social and semantic interrelationships.

Just as Vygotsky (1930/1978, 1934/1986) noticed that children start to use new adult words before they fully understand the meaning of the words (in fact, they learn the meaning by using the word), so Sfard argues that children advance from passive use of math concepts to routine-driven, phrase-driven, and finally object-driven use. They often begin to individualize group knowledge and terminology through *imitation*. Again, the part of the book on routines requires and deserves careful study and cannot be adequately presented in a brief review. I would encourage trying to apply Sfard's analysis to actual data of children learning math.

In our case, we see Aznx *imitating* his partners' routines and thereby gradually individualizing them as his own abilities. He often makes a knowledgeable-sounding proposal and then questions his own understanding. He does not *possess* the knowledge, but he is learning to *participate* in the discourse. In a collaborative setting, his partners can correct or accept his trials, steering and reinforcing his mimetic learning. During our four-hour recording, we can watch the group move through different stages of interaction with the symbols and realizations of math objects. The students we observe are not fully competent speakers of the language of math; as they struggle to make visible to each other (and eventually through that to themselves) their growing understanding, we as analysts can see both individual understanding and group cognition flowering. We can make sense of the discourse routines and interactional methods with the help of Sfard's concepts.

Participation in the discourse forms of math routines—such as exploration, ritual and imitation—can expose students to first-hand experiences of mathematical meaning making and problem solving. As they individualize these social experiences into their personal discourse repertoire, they thereby construct the kind of deep understanding that is often missing from acquisitionist/transmission math pedagogies (see Lockhart, 2008, for a critique of the consequences of AM schooling).

Situating Math Discourse

Sfard's theory resolves many quandaries that have bothered people about participationist and group cognitive theories. How can ideas exist in discourses and social groupings rather than in individual minds? It provides detailed analyses of how people participate in the discourses of communities—at least within the domain of math discourses, both local and historical. It provides an account of some basic ways in which individual learning arises from collaborative activities. It indicates how meaning (as situated linguistic use) can be encapsulated in symbols. It explains how children learn, and that creativity is possible, while suggesting ways to foster and to study learning. It describes some of the mediations by which public discourses—as the foundational form of knowledge and group cognition—evolve and are individuated into private thinking.

Sfard has done us the great service of bringing the "linguistic turn" of twentieth century philosophy (notably Wittgenstein) into twenty-first century learning science, elaborating its perspective on the challenging example of math education. She shows how to see math concepts and student learning as discourse phenomena rather than mental objects.

The kind of theoretical undertaking reported in this book must restrict its scope in order to tell its story. However, if we want to incorporate its important accomplishments into CSCL research, then we must also recognize its limitations and evaluate its contributions *vis a vis* competing theories. In addition to noting its incomplete treatment of socio-cognitive theory, knowledge building, activity theory, ethnomethodology, or distributed cognition, for instance, we should relate it more explicitly to the characteristics of CSCL.

First CSCL. By definition of its name, CSCL differs from broader fields of learning in two ways: its focus on *collaborative learning* (e.g., small group peer learning) and its concern with *computer support* (e.g., asynchronous online discussion, synchronous text chat, wikis, blogs, scripted environments, simulations, mobile computing, video games). Sfard does not present examples of small group interaction; her brief excerpts are from dyadic face-to-face discussions or adult-child interviews. Her empirical analyses zero in on individual math skills and development, rather than on the group mechanisms by which contributions from different personal perspectives are woven together in shared discourse. We now need to extend her general approach to computer-mediated interaction within small groups of students working together on the construction and deconstruction of math objects.

Fine-grained analysis of collaboration requires high-fidelity recordings, which—as Sfard notes—must be available for detailed and repeated study. She makes the tantalizing hypothesis that Piaget's famous distinction between successive developmental stages in children's thinking during his conservation experiments may be a misunderstanding caused by his inability to re-view children's interactions in adequate detail. Tape recordings and video now provide the technological infrastructure that made, for instance, conversation analysis possible and today allows multi-modal observation of micro-genetic mechanisms of interaction and learning. Computer logs offer the further possibility of automatically recording unlimited amounts of high quality data for the analysis of group cognition.

For instance, in our study of the case shown in Figure 1, we used a replay application that lets us step through exactly what was shared by everyone in the chat room. Our replayer shows the window as the participants saw it and adds across the bottom controls to slow, halt, and browse the sequential unfolding of the interaction. This not only allows us to review interesting segments in arbitrarily fine detail in our group data sessions, but also allows us to make our raw data available to other researchers to evaluate our analyses. Everyone has access to the complete data that was shared in the students' original experience. There are no selective interpretations and transformations introduced by camera angles, lighting, mike locations, transcription, or log format.

Of course, the analysis of group interaction necessarily involves interpretation to understand the meaning-making processes that take place. The analyst must have not only general human understanding, but also competence in the specific discourse that is taking place. To understand Aznx's utterances, an analyst must be familiar with both the "form of life" of students and the math objects they are discussing. As Wittgenstein (1953, p. 223, §IIxi) suggests, even if a lion could speak, people would not understand it. Sfard's talk about analyzing discourse from the perspective of an analyst from Mars is potentially misleading; one needs *thick descriptions* (Geertz, 1973; Ryle, 1949) that are meaning-laden, not "objective" ones (in what discourse would these be expressed?).

Sfard's discussion of the researcher's perspective (p. 278f) is right that analysis requires understanding the data from perspectives other than those of the engaged participants—for instance, to analyze interactional dynamics and individual trajectories. However, it is important to differentiate this removed, analytic perspective (that still understands the meaning making) from a behaviorist or cognitivist assumption of objectivity (that recognizes only physical observables or hypothetical mental representations). The analyst must first of all understand the discourses in order to "explore" it from an outsider's meta-discourse, and neither a lion nor an analyst from Mars is competent to do so.

Sfard defines the unit of analysis as the discourse (p. 276). The use of CSCL media for math discourses problematizes this, because the discourse is now explicitly complex and mediated. Although Sfard has engaged in classroom analyses elsewhere, in this book her examples are confined to brief dyadic interchanges or even utterances by one student. In fact, some examples are made-up sentences like linguists offer, rather than carefully transcribed empirical occurrences. Moreover, the empirical examples are generally translated from Hebrew, causing a variety of interpretive problems and lessening the ability of most readers to judge independently the meaning of what took place. Computer logs allow us to record and review complex interactions involving multiple people over extended interactions. The unit of analysis can be scaled up to include: groups larger than dyads (Fuks, Pimentel, & de Lucena, 2006), the technological infrastructure (Jones, Dirckinck-Holmfeld, & Lindström, 2006), the classroom culture (Krange & Ludvigsen, 2008), or time stretches longer than a single session (Sarmiento & Stahl, 2008). One can observe complex group cognitive processes, such as problem-solving activities, from group formation and problem framing, to negotiation of approach and sketching of graphical realizations, to objectification and exploration of visual signifiers, to reflection and individualization.

The encompassing discourse can bring in resources from the physical environment, history, culture, social institutions, power relationships, motivational influences, collective rememberings—in short, what activity theory calls the activity structure or actor-network theory identifies as the web of agency.

While Sfard uses the language of sweeping discourses—like the discourse of mathematics from the ancient Greeks to contemporary professional mathematicians—her specific analyses tend to minimize the larger social dimension in favor of the immediate moment. This is particularly striking when she uses terms like *alienation* and *reification* to describe details of concept formation. These terms are borrowed from social theory—as constructed in the discourses of Hegel, Marx and their followers, the social thought of Lukacs, Adorno, Vygotsky, Leontiev, Engeström, Lave, Giddens, and Bourdieu. Sfard describes the reification of discursive counting processes into sentences about math objects named by nouns as eliminating the human subject and presenting the resultant products as if they were pre-existing and threatening. She does this in terms that all but recite Marx's (1867/1976, pp. 163-177) description of the fetishism of commodities. However, whereas Marx grounded this process historically in the epochal development of the relations of social practice, the forces of material production and the processes of institutional reproduction, Sfard often treats mathematics as a hermetic discourse, analyzable independently of the other discourses and practices that define our world, though in her concluding chapter she emphasizes the need to go beyond this in future work.

Mathematics develops—both globally and for a child—not only through the inter-animation of mini-discourses from different personal perspectives, but also through the interpenetration of macro-discourses. Math is inseparable from the world-historical rise of literacy, rationalism, capitalism, monotheism, globalization, logic, individualism, science, and technology. CSCL theory must account for phenomena across the broad spectrum, from interactional details contained in subtle word choices to the clashes of epochal discourses. While Sfard has indicated a powerful way of talking about much of this spectrum, she has not yet adequately located her theory within the larger undertaking. One way to approach this would be to set her theory in dialog with competing participationist theories in CSCL and the learning sciences.

Continuing the Discourse

Issues of situating math discourse in social practice return us to the quandary of the metaphors of acquisition and participation. Sfard's book works out an impressive edifice of participation theory. Math can be conceptualized as a discourse in which people participate in the social construction of math objects; because of such participation they can understand and individualize elements of the discourse. In doing so, Sfard follows a path of dialogical and discursive theory starting at least with Bakhtin, Vygotsky, and Wittgenstein, and propounded by numerous contemporaries. Within the domain of math discourse, Sfard has pushed the analysis significantly further.

Her argument 10 years ago was that there is something to the metaphor of *objects* of math but that the ontological status of such objects was unclear and was perhaps best described by AM. In addition, she felt that multiple conflicting metaphors breed healthy dialog. But now she has shown that math objects are products of math discourse (so they now exist and make sense within PM). As for healthy dialog, there is plenty of opportunity for controversies among multiple discourses within PM itself. Thus, we can conclude that Sfard is justified in moving to a fully PM metaphor because this stream of thought is capable of resolving former quandaries and it contains within itself an adequate set of potentially complementary, possibly incommensurable discourses to ensure a lively and productive on-going debate. Sfard has provided us with one of the most impressive unified, homogenous theories of learning; it remains for us to situate that theory within the specific field of CSCL and within the broader scope of competing theoretical perspectives. This includes extending and applying her analysis to group cognition and to computer-mediated interaction. It also involves integration with a deeper theoretical understanding of social and cultural dimensions.

At the other end of the spectrum, one must also resolve the relationship of "thinking as communicating" with the psychological approach to individual cognition as the manipulation of private mental representations. Is it possible to formulate a cognitivist view without engaging in problematic acquisitionist metaphors of a "ghost in the machine" (Ryle, 1949)? Assuming that one already understands the mechanisms of math discourse as Sfard has laid them out, how should hypothetical-deductive experimental approaches then be used to refine models of individual conceptualization and to determine statistical distributions of learning across populations? Questions like these raised by the challenge of Sfard's book are likely to provoke continuing discourse and meta-discourse in CSCL—and in *ijCSCL*—for some time to come, resolving intransigent quandaries and building more comprehensive (deeper) scientific understandings.

References

Fuks, H., Pimentel, M., & de Lucena, C. J. P. (2006). R-u-typing-2-me? Evolving a chat tool to increase understanding in learning activities. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 1* (1), 117-142.

Geertz, C. (1973). *The interpretation of cultures*. New York, NY: Basic Books.

Husserl, E. (1936/1989). The origin of geometry (D. Carr, Trans.). In J. Derrida (Ed.), *Edmund Husserl's origin of geometry: An introduction* (pp. 157-180). Lincoln, NE: University of Nebraska Press.

Jones, C., Dirckinck-Holmfeld, L., & Lindström, B. (2006). A relational, indirect and meso level approach to design in CSCL in the next decade. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 1* (1), 35-56.

Krange, I., & Ludvigsen, S. (2008). What does it mean? Students' procedural and conceptual problem solving in a CSCL environment designed within the field of science education. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 3* (1), 25-52.

Lockhart, P. (2008). Lockhart's lament. *MAA Online, 2008* (March). Retrieved from http://www.maa.org/devlin/devlin_03_08.html.

Marx, K. (1867/1976). *Capital* (B. Fowkes, Trans. Vol. I). New York, NY: Vintage.

Moss, J., & Beatty, R. A. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *International Journal of Computer-Supported Collaborative Learning (ijCSCL), 1* (4), 441-466.

Ryle, G. (1949). *The concept of mind*. Chicago, IL: University of Chicago Press.

Sarmiento, J., & Stahl, G. (2008). *Extending the joint problem space: Time and sequence as essential features of knowledge building.* Paper presented at the International Conference of the Learning Sciences (ICLS 2008), Utrecht, Netherlands.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27* (2), 4-13.

Vygotsky, L. (1930/1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Vygotsky, L. (1934/1986). *Thought and language*. Cambridge, MA: MIT Press.

Wittgenstein, L. (1944/1956). *Remarks on the foundations of mathematics*. Cambridge, MA: MIT Press.

Wittgenstein, L. (1953). *Philosophical investigations*. New York, NY: Macmillan.

Notes