Shared Referencing of Mathematical Objects in Online Chat

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Abstract. We conceptualize referencing as a primary means of establishing intersubjective meaning and, therefore, as an important consideration in supporting collaborative learning. What we call group cognition is a discourse-centered analysis of the interactional basis of collaborative knowledge building, establishing common ground and co-constructing shared meaning. This epistemological perspective has methodological, technological and pedagogical implications. It directs empirical analysis toward the manifold forms of referencing that constitute small-group communication. Studies of how students actually make sense of, adapt and adopt referential affordances in computational media are used to inform the design of educational software environments. In particular, we are developing an online service for groups of people to discuss mathematical themes. This paper looks at how a group of students used methods of referencing to co-construct geometric objects in a chat room with graphical referencing tools.

Epistemology of Referencing

Referencing is a primary means for humans to establish joint attention and to make shared meaning. Vygotsky, in a particularly rich passage, described the interactional origin of pointing as an example of how gestures become meaningful artifacts for individual minds through social interaction:

> A good example of this process may be found in the development of pointing. Initially [e.g., for an infant], this gesture is nothing more than an unsuccessful attempt to grasp something, a movement aimed at a certain object which designates forthcoming activity…. When the mother comes to the child's aid and realizes this movement indicates something, the situation changes fundamentally. Pointing becomes a gesture for others. The child's unsuccessful attempt engenders a reaction not from the object he seeks but from another person. Consequently, *the primary meaning* of that unsuccessful grasping movement *is established by others*…. The grasping movement changes to the act of pointing. As a result of this change, the movement itself is then physically simplified, and what results is the form of pointing that we may call a true gesture. (Vygotsky, 1930/1978, p. 56, italics added)

The pointing gesture is perhaps the most basic form of deictic referencing. In its origin where the infant begins to be socialized into a shared world, the meaning of the gesture emerges interactionally as the participants orient to the same object and recognize that they are doing so jointly. In grasping, the infant's being-in-the-world is intentionally directed at the object; it is a being-at-the-object (Husserl, 1929/1960). When the mother joins the infant by transforming this grasp into a joint engagement with the object, the intentionality of the infant's grasp becomes intersubjective intentionality, constituting the infant and child as being-there-together-at-the-object (Heidegger, 1927/1996, §26). This fundamental act of collaborative existence simultaneously comes to be symbolized for them by the pointing gesture, which is practiced, repeated and abstracted by them together over time. The mother and infant become a small group, caring for shared objects by being-in-the-world-together and understanding as collaborative practice the symbolic meaning of the physical gesture as a referencing artifact.

As researchers, we can see new referencing gestures being created within interactions among collaborating people, particularly when their interaction is taking place via a new medium that they must learn how to use. In the analysis below, a chat posting—"What is the area of this shape?"—constitutes the participants in the chat as a group by designating them as the intended collective recipient and as the expected respondent to the question (Lerner, 1993). The group is the intended agent who will work out the mathematics of the proposal to compute the area. Simultaneously, by referencing a mathematical object ("this shape"), the posting constitutes the group as a being-there-together-at-the-object. We shall see that both these aspects of being a group may necessitate considerable interactional work by the participants. Before the elicited answer about area can be given in response to the question, the group has to negotiate what it as a group takes the object to be. Also, it may require a number of actions for group participants to co-construct the shared object and their being-there-together-at-the-object. In attempting to do this, they constitute themselves as a group and they may also establish referential gestures or terms that take on a shared meaning of intending the new math object. The interactional work of the group involves making use of the resources of the environment that mediates their interaction. This is particularly noticeable in online interaction. Vygotsky's infant and mother could use fingers, gaze, touch, voice. Online participants are restricted to exchanging textual postings and to using features of the mediating software (Garcia & Jacobs, 1999; Stahl, 2005b).

As designers of online education, we are interested in understanding how students collaboratively create new communicative gestures or interactional methods, including ways of referencing objects for joint consideration. More generally, an interactional understanding of referencing and meaning making leads to a theory of group cognition—rather than individual cognition based on mental representations—as a basis for studying collaborative learning (Stahl, 2006).

Technology for Referencing in a Chat Environment

In our design-based research at the Virtual Math Teams project (Stahl, 2005a), we started by conducting chats in a variety of commercially available environments: AOL Instant Messenger, Babylon, Blackboard, WebCT. Based on these early investigations, we concluded that we needed to add a shared whiteboard for drawing geometric figures and for persistently displaying notes. We also found a need to minimize "chat confusion" by supporting explicit referencing of response threads (Cakir *et al.*, 2005; Fuks, Pimentel, & de Lucena, 2006). We decided to try ConcertChat, a research chat environment with special referencing tools (Mühlpfordt & Wessner, 2005). By collaborating with the software developers, our educational researchers have been able to successively try out versions of the environment with groups of students and to gradually modify the environment in response to what we find by analyzing the chat logs.

ConcertChat provides a variety of referencing supports for math chats:

- A shared whiteboard allows chat participants to create drawings. As new objects appear in the drawing, an implicit form of referencing occurs. Participants typically refer with a deictic term in their textual chat to a new addition to the drawing, whose recent appearance for the group makes it salient.
- When someone types a new chat message, they can select and point to a rectangular area in the whiteboard. When that message appears in the chat as the last posting or as a selected posting, a bold line appears connecting the text to the area of the drawing (see figure 1).
- Similarly, a chat message can point to one or more earlier textual postings. ConcertChat includes a threaded view of the chat postings that, based on the explicit references between postings, displays them like a typical threaded discussion with responses indented under the posting that they reference.
- Of course, one can also make all the usual verbal references: using deictic terms (*that*, *it*, *his*, *then*); quoting part of an earlier posting; or citing the author of a previous posting.

In May 2005, we conducted a series of chats using ConcertChat. We formed five virtual math teams, each containing about four middle-school students selected by volunteer teachers at different schools across the USA. The teams engaged in online math discussions for four hour-long sessions over a two-week period. They were given a brief description of a non-traditional geometry environment: a grid-world where one could only move along the lines of a grid (Krause, 1986). The students were encouraged to come up with their own questions about the grid-world, such as questions about shortest paths between points A and B in this world.

The chats were facilitated by a member of our research project team. The facilitator welcomed students to the chat, pointed them toward the task, briefly demonstrated the graphical referencing tool and then kept generally quiet until it was time to end the session. We are now analyzing the resultant chat logs in order to draw design implications for a full-scale online math discussion service.

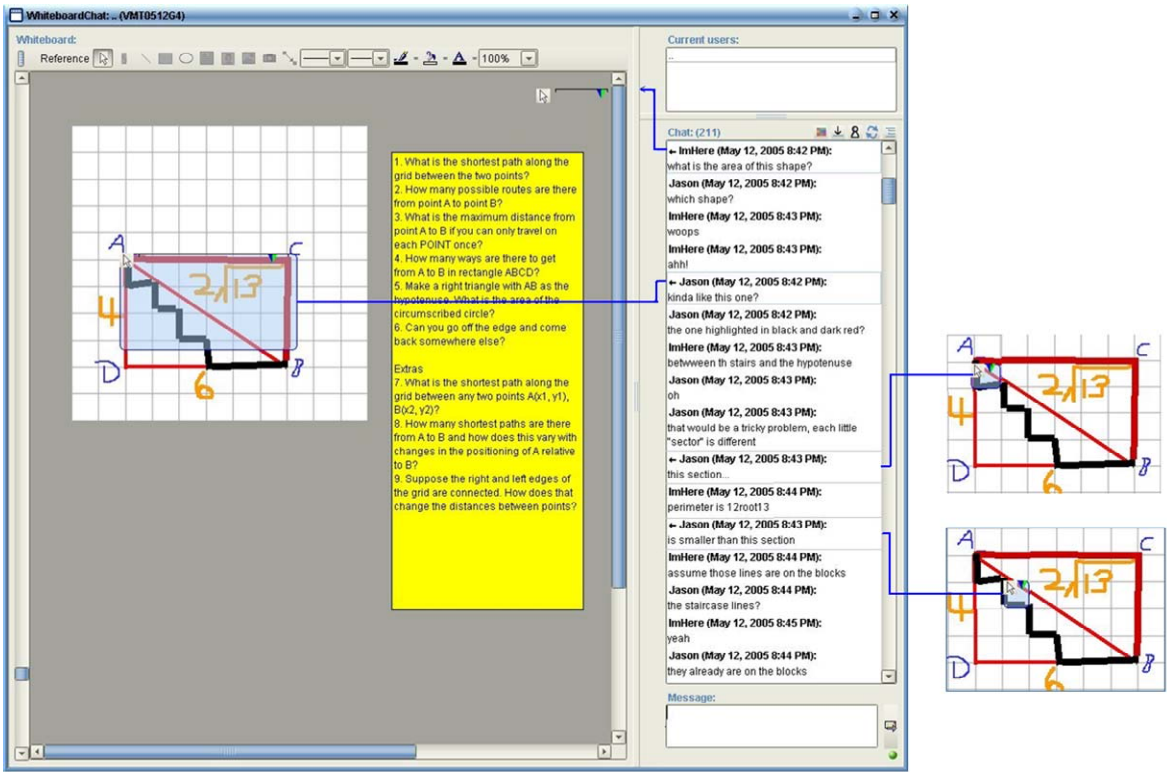

Figure 1. Screen view of ConcertChat with referencing. The image has been modified to show graphical references from chat lines 1, 5, 10 and 12 to the whiteboard. Only the reference from a single selected chat line would actually appear at any given time.

An Analysis of a Case of Referencing

The chat log excerpt visible in figure 1 is reproduced in figure 2 (with line numbers added for referencing in this paper). In this interactional sequence, two students discuss parts of a drawing that has already been constructed in the shared whiteboard. The students had created the drawing as part of discussions about shortest paths between points A and B in a grid-world. In particular, a red triangle, ABD, was drawn with sides of length 4, 6 and 2√13. A thick black staircase line was drawn as a path on the grid from A to B. In this excerpt, the students propose a math problem involving this drawing.

The message in line 1 of the chat (see figure 2) proposes a mathematical question for the group to consider: “What is the area of this shape?” This is accompanied by a graphical reference to the whiteboard. The reference does not indicate a specific area—apparently ImH did not completely succeed in using this new referencing tool. Line 2 raises the question, “Which shape?” pointing out the incompleteness of the previous message’s reference. The proposal in line 1 calls for a response, such as an attempt to answer the question. However, the question was incompletely formed because its reference was unclear so it received a call for clarification as its immediate response. Lines 3 and 4 display a recognition and agreement of the incomplete and problematic character of the referencing.

Lines 5 and 6 offer a repair of line 1’s problem. First, line 5 roughs in the area that may have been intended by the incomplete reference. It includes a complete graphical reference that points to a rectangular area that includes most of the upper area of rectangle ACBD in the drawing. The graphical referencing tool only allows the selection of rectangular areas, so line 5 cannot precisely specify a more complicated shape. The text in line 5 (“kinda like this one?”) not only acknowledges the approximate nature of its own referencing, but also acknowledges that it may not be a proper repair of line 1 and accordingly requests confirmation from the author of line 1. At the same time, the *like* reflects that this act of referencing is providing a model of what line 1 could have done. Peer instruction in the use of the software is taking place among the students as they share their growing understanding of the new chat environment. Line 5 is accompanied by line 6, which provides a textual reference or specification for the same area that line 5 pointed to: the one highlighted in black (the staircase line) and dark red (lines AC and CB). The inexact nature of the graphical reference required that it be supplemented by this more precise textual reference. Note how the sequence of indexical attempts in lines 1, 2, 5 and 6 successively focuses shared attention on a more and more well-defined geometric object. This is an interactive achievement of the group (the interaction between ImH and Jas, observed by others and situated among the math objects co-constructed by all).

Lines 5 and 6 were presented as questions calling for confirmation by ImH. Clarification follows in line 7 from ImH: "between the stairs and the hypotenuse." Line 8's "Oh" signals mutual understanding of the evolving reference and the establishment of an agreed upon boundary object (Star, 1989) for carrying on the mathematical investigation incompletely proposed in line 1 (Clark & Brennan, 1991). Now that a complete reference has been co-constructed to a math object that is well enough specified for the practical purposes of carrying on the chat, Jas launches into the problem solving by raising an issue that must first be dealt with. Line 9 says that calculating the area now under consideration is tricky. The tricky part is that the area includes certain little "sectors" whose shapes and areas are non-standard. Line 9 textually references "each little 'sector'." *Little* refers to sub-parts of the target area. *Each* indicates that there are several such sub-parts and *sector*, put in scare quotes, is proposed as a name/description of these hard-to-refer-to sub-parts.

Clarification of the reference to sectors is continued by lines 10 and 12. These lines compare two sectors, demonstrating that they are different by showing that one is smaller than the other. Lines 10 and 12 reference two different sectors, both with the same textual, deictic description: *this section*. It is possible to use the identical description twice here because the text is accompanied by graphical references that distinguish the two sectors. Line 10 points to the small grid square inside of rectangle ACBD in the upper left-hand corner adjacent to point A. Line 12 points to the next grid down the hypotenuse (see figure 1). Because of the roughness of the graphical reference tool, lines 10 and 12 can only indicate the squares of the grid, not the precise odd-shaped sectors that are of concern to the group discourse. On the other hand, the textual clause, *this section* has been given the meaning of the odd-shaped sub-areas of the area "between the stairs and the hypotenuse," although it cannot differentiate easily among the different sections. The carefully constructed combination of graphical and textual referencing accomplished in lines 10 and 12 was needed to reference the precise geometric objects. The combination of the two textual lines, with their two contrasting graphical references, joined into one split sentence was necessary to contrast the two sectors and to make visible the tricky circumstance. In this way, the discourse succeeded in constituting the complicated geometric sectors despite the limitations of the tool and of textual description by themselves.

Line 13 responds to the tricky issue by treating it as a non-essential consequence of inaccurate drawing. By proposing that the group "assume those lines are on the blocks," this posting treats the difference among the sectors as due to the inaccuracy of the drawing of the thick black staircase line in not precisely following the grid lines.

1	ImH:	what is the area of this shape? [REF TO WB]
2	Jas:	which shape?
3	ImH:	woops
4	ImH:	ahh!
5	Jas:	kinda like this one? [REF TO WB]
6	Jas:	the one highlighted in black and dark red?
7	ImH:	between th stairs and the hypotenuse
8	Jas:	oh
9	Jas:	that would be a tricky problem, each little "sector" is different
10	Jas:	this section [REF TO WB]]
11	ImH:	perimeter is 12root3
12	Jas:	is smaller than this section [REF TO WB]
13	ImH:	assume those lines are on the blocks
14	Jas:	the staircase lines?
15	ImH:	yea
16	Jas:	they already are on the blocks

Figure 2. Log from chat. Line numbers added; names anonymized. Graphical references to the whiteboard indicated by [REF TO WB].

Physical drawings are necessarily rough approximations to idealized mathematical objects in geometry. Lines drawn with a mouse on a computer screen tend to be particularly rough representations. The implication of line 13 is that the tricky issue is due to the inaccurate appearance of the lines, but that the faults of the physical drawing do not carry mathematical weight and can be stipulated away. But line 14 questions this move. It first makes sure that line 13's reference to *those lines* was a reference to *the staircase lines* that form part of the perimeter of the target area and of its different-sized sectors. When line 15 confirms that line 13 indeed referenced the staircase lines, line 16 responds that "they already are on the blocks"—in other words, the tricky situation was not due to inaccuracies in the drawing but the staircase lines were indeed already *taken as* following the grid for all practical purposes. The problem was still seen to be a tricky one once the mathematical object was clearly referenced and specified.

We see here that referencing can be a complex process in online mathematical discourse. In a face-to-face setting, the participants could have pointed to details of the drawing, could have gesturally described shapes, could have traced outlines or shaded in areas either graphically or through gestures with ease. Conversationally, they could have interrupted each other to reach faster mutual orientation and understanding. Online, the interaction is more tightly constrained by the affordances and more burdensome due to the restricted nature of the software environment. On the other hand, we have seen that middle-school students who are new to the graphical tools of ConcertChat, as well as to online collaborative mathematics, can call upon familiar resources of textual language, drawing, pointing and school mathematics to construct interaction methods that are seen to be amazingly sophisticated, efficient, creative and effective when analyzed in some detail.

Methods of Making Referential Sense

We have here only been able to look at some of what took place in a single effort to reference a mathematical object (see Mühlpfordt & Wessner, 2005, for other usages of the referencing functionality). In the series of chats that this effort was taken from, we observed groups of students engaging in a variety of other referencing methods within ConcertChat. Common methods included:

- Graphical references to previous messages were sometimes used to make salient a message from relatively far back in the chat. Without the graphical referencing functionality, this would require a lengthy textual explanation justifying change of topic and quoting or describing the previous message.

- Some students used graphical references to previous messages to specify a recipient for their new posting. If a student wanted to address a question to a particular student rather than to the group as a whole, he or she would accompany the question with a graphical reference to a recent posting by that student. (This was a use of graphical referencing not at all foreseen by the software designers.)

- It is common in chat for someone to spread a single contribution over two or more postings (e.g., lines 10 and 12). In conversation, people often retain their turn at talk by indicating that they are not finished in many ways, such as saying "ummm." In generic chat systems, people often end the first part of their contribution with an ellipsis (…) to indicate that they will continue in a next posting. In ConcertChat, students sometimes graphically referenced their first posting while typing their second. Then the two parts would still be tied together even if someone else's posting (like line 11) appeared in the meantime.

- Similarly, students graphically referenced their own previous posting when repairing a mistake made in it. The reference indicates that the new posting is to replace the flawed one.

- In chat, where the flow of topics is not as constrained as in conversation, it is possible for multiple threads of discussion to be interwoven. For instance, line 11 starts to discuss perimeter while area is still being discussed. Graphical references are used to tie together contributions to the same thread. For instance, line 12 might have referenced line 10 graphically.

- The graphical referencing tool is treated as one of many available referencing resources. Deictic terms are frequently used—sometimes in conjunction with graphical referencing (e.g., line 5).

- In chat, as in conversation, sequential proximity is a primary connection. By default, a posting is a response to the preceding posting. Chat confusion arises because sequentiality is unpredictable in chat; people generally respond to the most recent posting, but by the time their response is posted other postings may intervene.

Interestingly, the recentness of drawings may function as a similar default reference. Students frequently refer to a line that was just added to the whiteboard as *that line* without creating a graphical reference to it.

- Of course, purely textual references are also widely used to point to postings, people, groups, drawings, abstractions and math objects.

The many forms of referencing in chat tie together the verbal and graphical contributions of individual participants into a tightly woven network of shared meaning. Each posting is connected in multiple ways—explicit and implicit—to the flow of the shared chat (Stahl, 2005a). The connections are highly directional, granting a strong temporality to the chat experience (hard to fully appreciate from the log). The being-there-together in a chat is temporally structured as a world of future possible activities with shared meaningful objects. The possibilities for collaborative action are made available by the social, pedagogical, technical and discourse context (world, situation, activity structure, network of relevant significance) (Heidegger, 1927/1996, §18). While the shared context is opened up, enacted and made salient by the group in its chat, aspects of the context appear as designed, established or institutionalized in advance. They confront the participants as a world filled with meanings, priorities, resources and possibilities for action. It is a situation whose features, meanings and co-inhabitants are initially largely unknown. Students who enter this world must choose to either become engaged or be silent and log out.

Pedagogy of Referencing Math Objects

In the investigation reported here, we tried to encourage relatively open-ended explorations of mathematical inquiry by online teams of math students. We presented them with a non-traditional form of geometry in which notions like distance, area or shortest-path have to be renegotiated. While trains of inquiry can go in many directions, in a collaborative effort each step of the path may be clarified and shared. New math objects emerge and develop out of the discourse, including geometric figures and terminology. In this study, the analysis of a snippet of this process in a concrete empirical case has suggested the centrality of joint referencing to collaboration. This may serve as an additional clarification of what is meant by defining collaboration as "a continued attempt to construct and maintain a shared conception of a problem …. an emergent, socially-negotiated set of knowledge elements that constitute a Joint Problem Space" (Roschelle & Teasley, 1995, p. 70). The persistent whiteboard serves as a "group external memory" that plays a useful role in grounding shared understanding at the meso level of CSCL problem solving (Dillenbourg & Traum, 2006, p. 122f), in contrast to Clark & Brennan's (1991) linguistic micro level. The intertwining uses of the dual workspaces of whiteboard and chat mirror the intertwining of problem solving and communication that is characteristic of collaborative learning (Barron, 2003, p. 310).

Widespread conceptions of math learning as the memorization of "math facts" or the mastery of formulaic algorithmic solutions are oriented to the routine application of arithmetic rather than to the creative process that inspires mathematicians. The history of mathematics as a branch of scientific inquiry and knowledge building is a systematic unfolding of new domains through the shared construction of new math objects, like complex numbers, fractals, curved spaces. To share these boundary objects (Star, 1989), mathematicians have had to define new vocabularies, procedures and representations for referencing objects that do not exist as such in the physical world.

People who do not understand mathematical references can scarcely be expected to share the wonder and excitement that mathematicians feel who can "see" (Goodwin & Goodwin, 1994) what is being referenced. It is likely that much of the general population simply does not share the understanding of what is referenced in most mathematical proofs and discussions. Since our goal is to increase mathematical appreciation and participation through opportunities for online math discourse, we are keen to understand shared referencing and to support it in our environments.

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