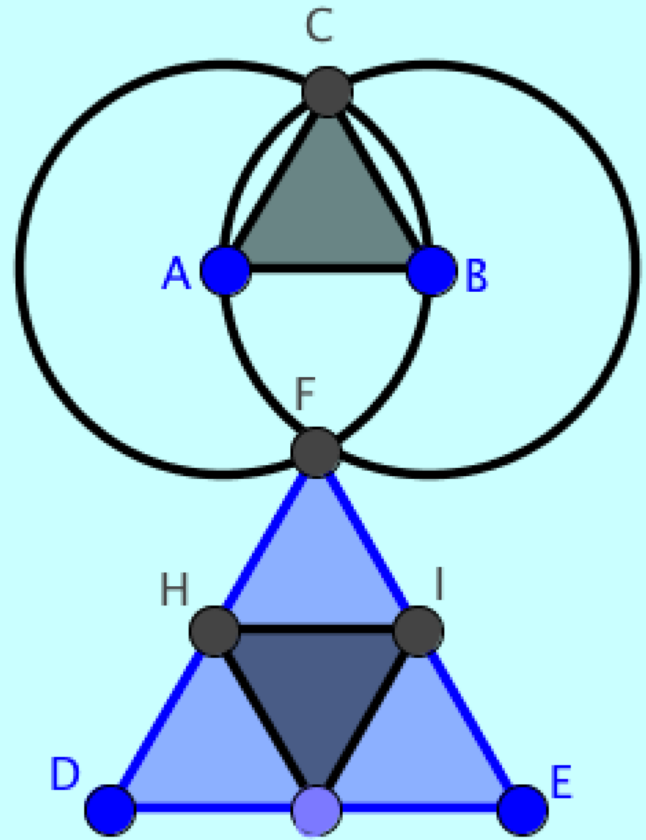

DIDACTICS OF HUMAN-CENTERED DYNAMIC GEOMETRY

LinCS Seminar, Göteborg

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Didactics of human-centered dynamic geometry

1. Introduction
2. Constructing geometric objects in GeoGebra
3. Abstracting dependencies in custom tools
4. Constructing propositions using dependencies
5. Proving propositions using dependencies
6. Creative-discovery as a human-centered approach
7. Teaching this approach to teachers collaboratively
8. Teaching this approach to students collaboratively
9. Analyzing how teachers and students learn this approach in their discourse and interaction

1. Introduction

Euclidean geometry has trained students for over 2,000 years in rational, deductive thinking and mathematical practices.

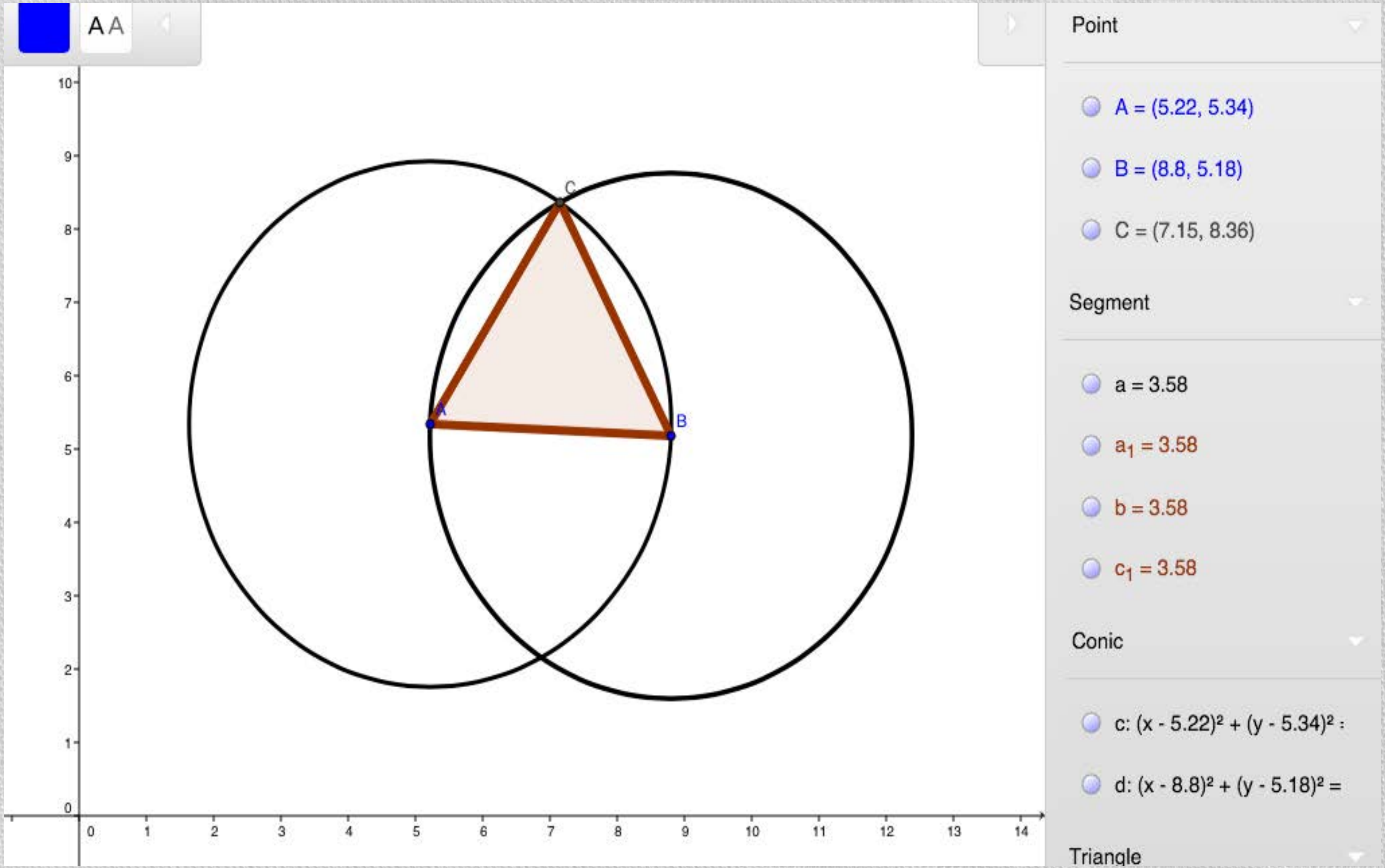
Computer-based dynamic geometry (Geometer's Sketchpad, Cabri, GeoGebra, etc.) adds three dimensions:

- **Dynamic dragging,**
- **Dynamic construction (including programming custom tools),**
- **Dynamic dependencies.**

With these, students can:

- **Drag figures to *discover* dependencies**
- **As well as design the *creation* of dependencies**
- **For the *human-centered* construction of figures**

GeoGebra on an iPad

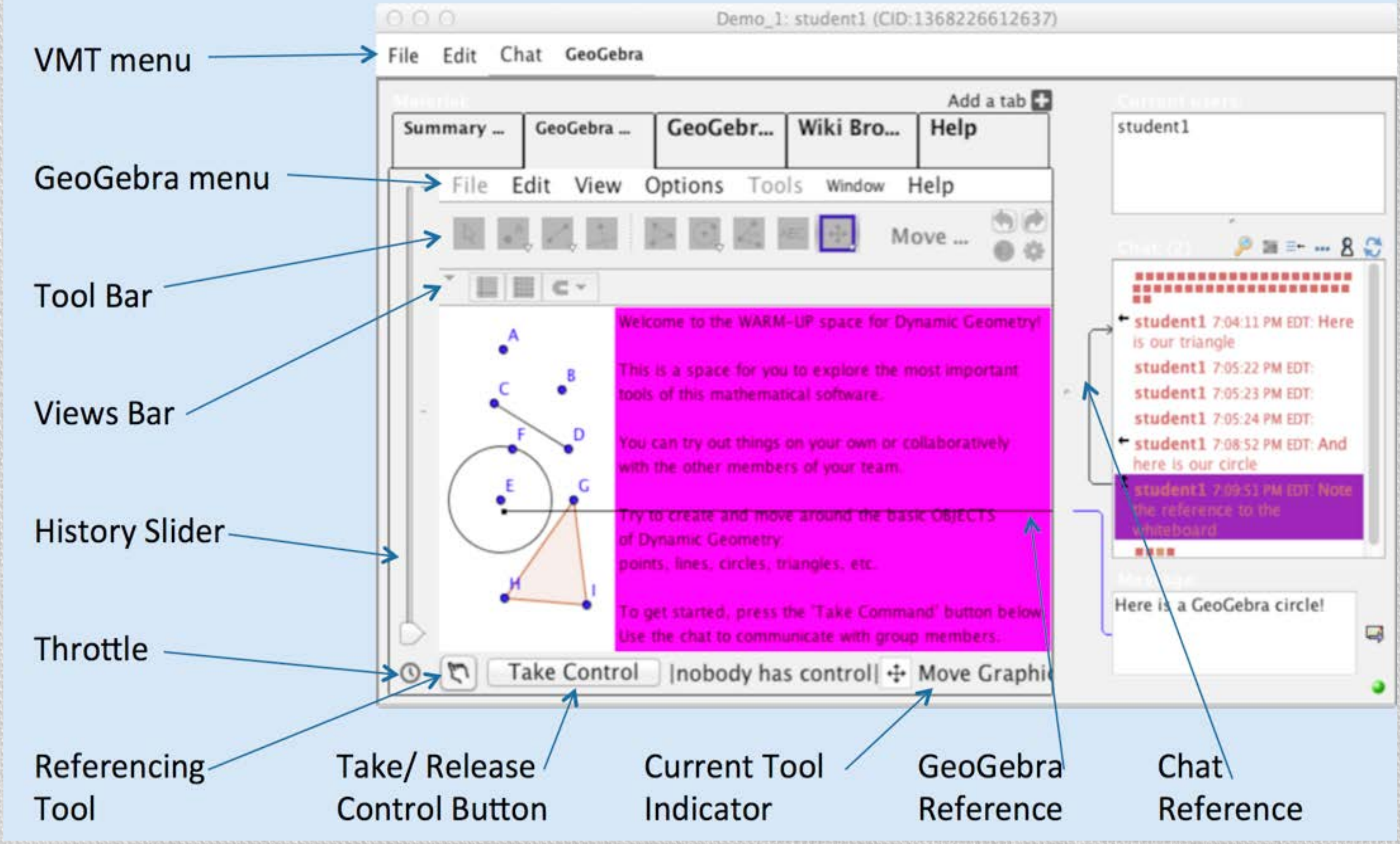


Rather than accepting geometric propositions as otherworldly truths, students can now conceive them as results of their own "creative discovery" within a local knowledge community.

How can geometry education be structured to incorporate this orientation to the social construction of geometric knowledge?

The VMT technology and curriculum adds an emphasis on significant mathematical discourse through support of collaborative learning by embedding multi-user GeoGebra in a collaborative-learning environment.

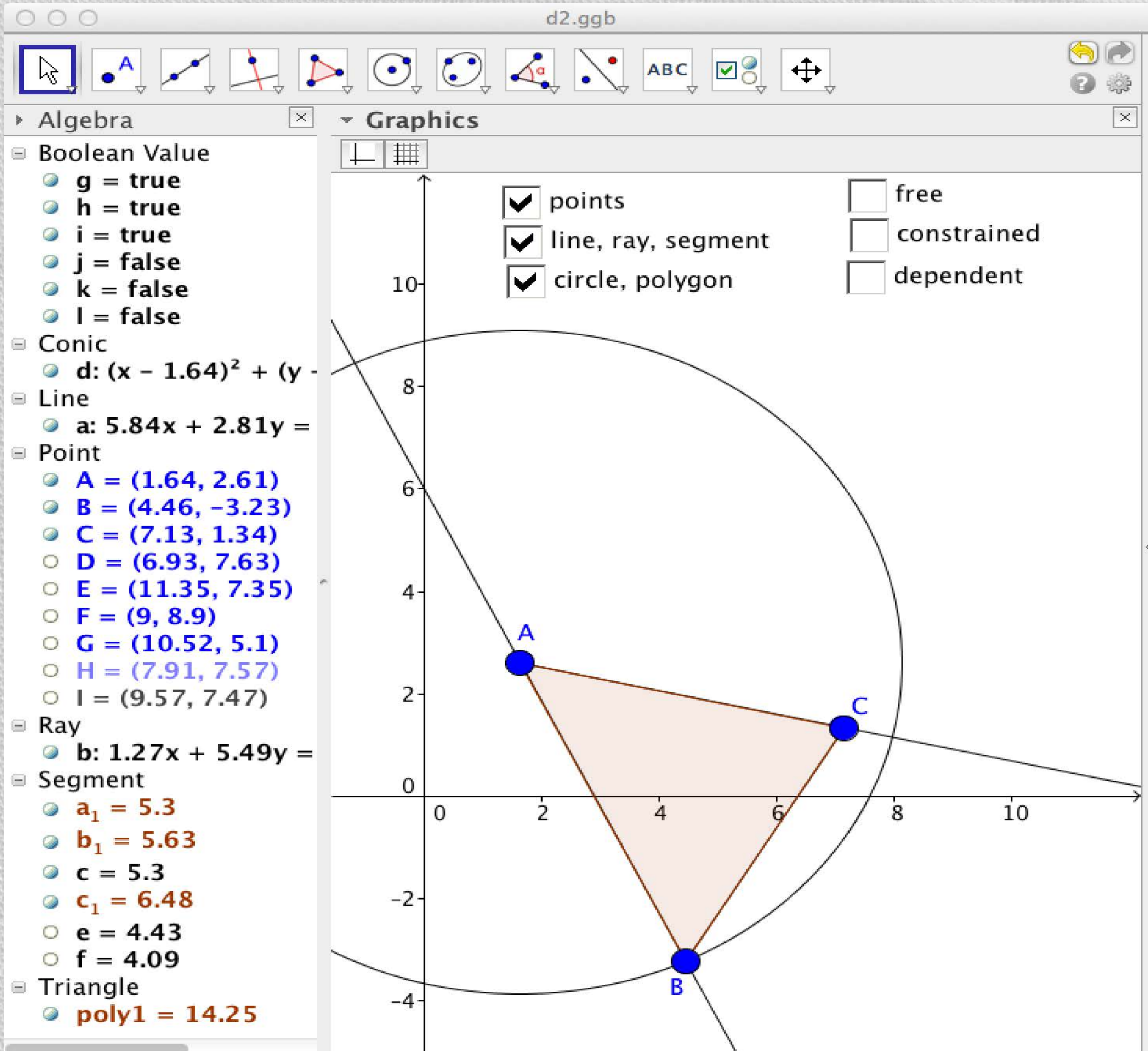
VMT with multiple GeoGebra tabs



2. Constructing geometric objects in GeoGebra

- Point constructed as a location on a plane surface. Digital approximation to infinite, continuous surface.
- Line constructed as linear equation passing through 2 points. Segment, ray and vector constructed similarly.
- Circle constructed as equation of points whose distance from a center point is equal to the distance between the center point and a second point. Conics constructed similarly.
- N-sided polygon constructed as defined by N points connected by segments joined at vertices.

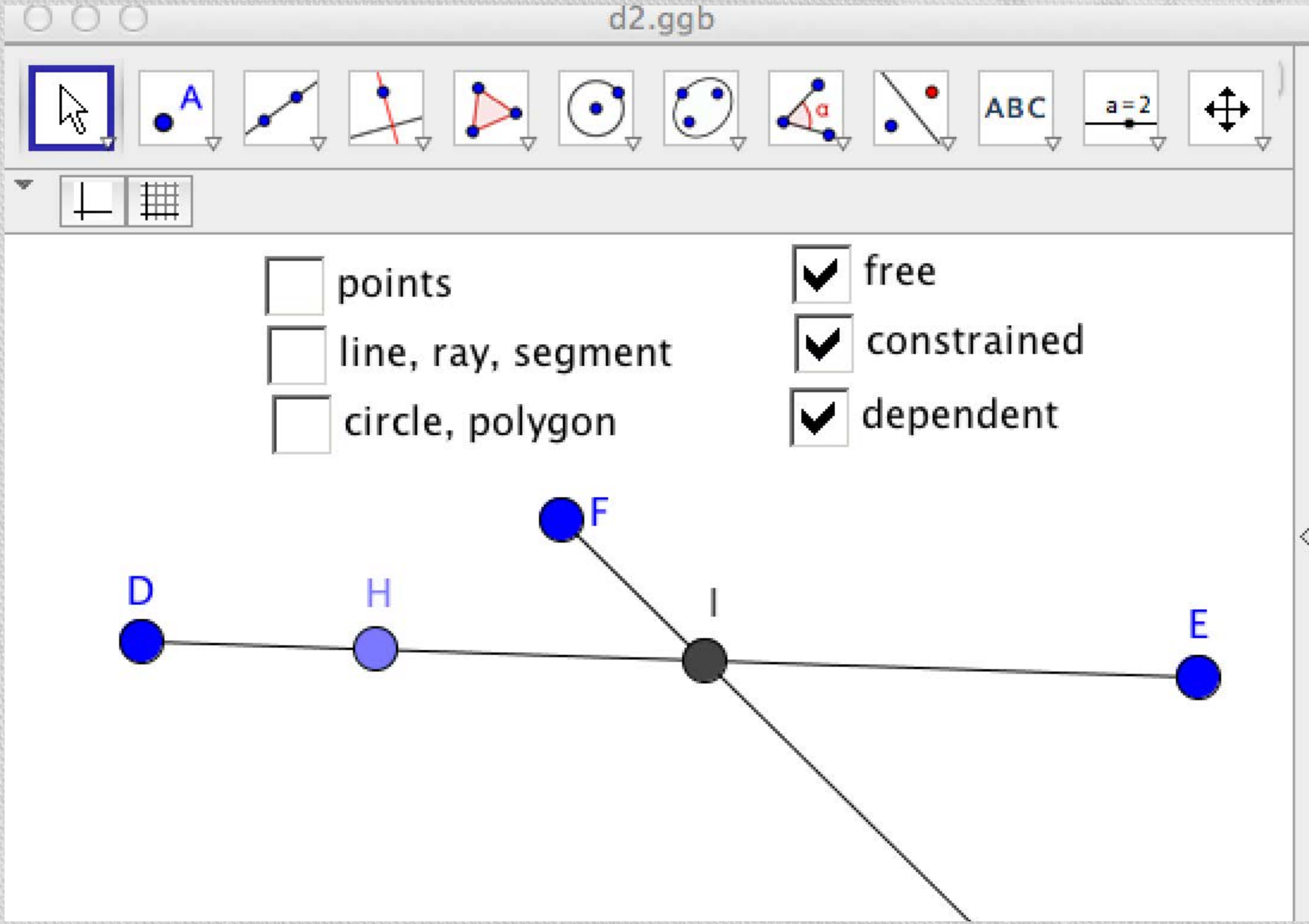
Primitive objects of GeoGebra



- **Point can be constructed to be:**
  - **Free – able to be dragged to any location.**
  - **Constrained – confined to be dragged only along a line or conic.**
  - **Dependent – cannot be dragged; location determined by intersection of two lines and/or conics.**

- **All constructed relationships are maintained under dragging:**
  - **If a point is dragged, any lines, conics or polygons constructed with that point are re-calculated.**
  - **If a line, conic or polygon is dragged, points defining it are dragged.**
  - **Dependency relations are maintained recursively.**

Free, constrained, dependent points



3. Constructing propositions using dependencies

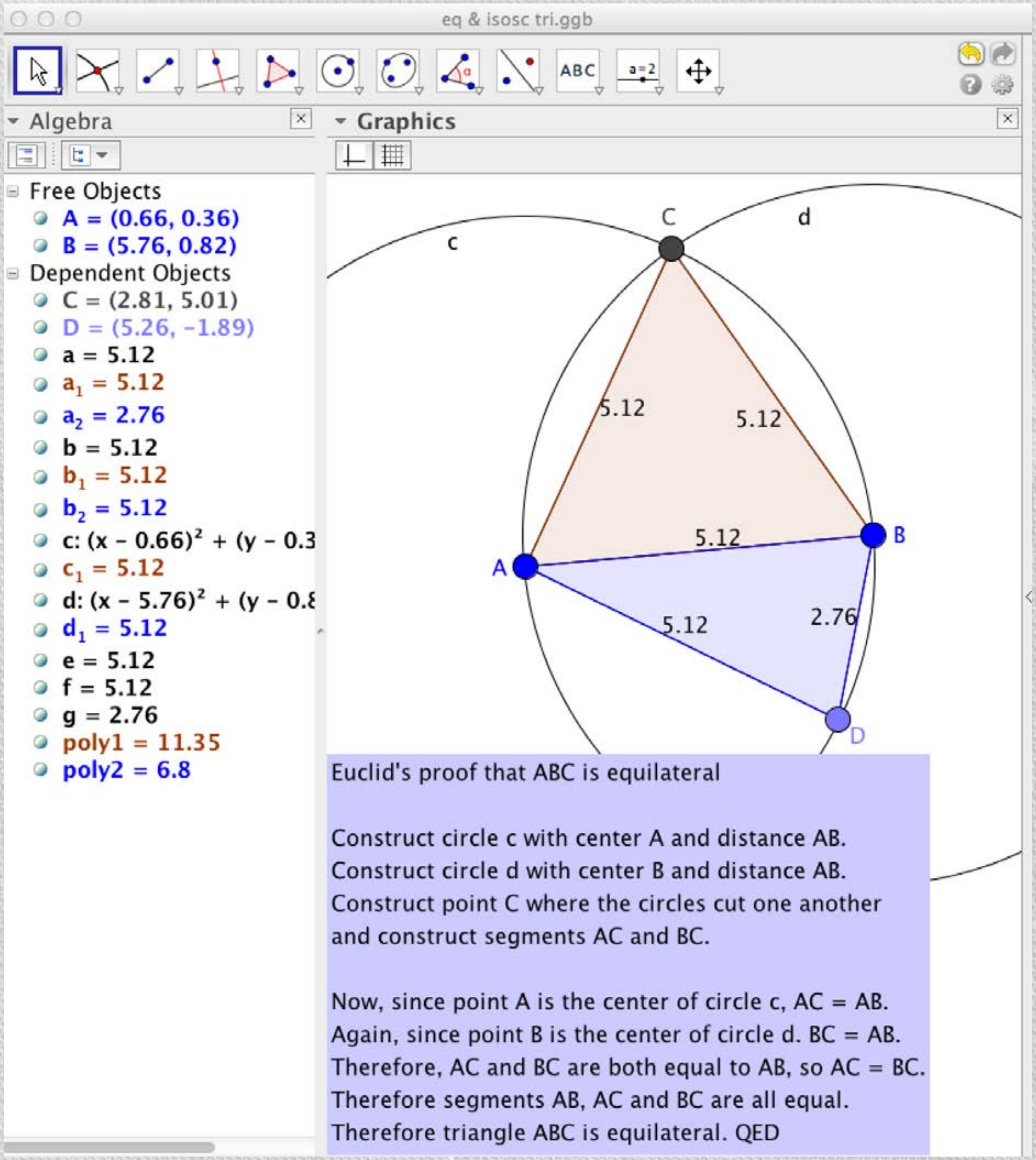
Euclid's Proposition #1 as a dynamic construction procedure:

- **In Euclid's construction of an equilateral triangle, he made the lengths of the three sides of the triangle dependent on each other by constructing each of them as radii of congruent circles.**

Euclid's proof of the equilateral triangle:

- **Then to prove that the triangle was equilateral, all he had to do was to point out that the lengths of the three sides of the triangle were all radii of congruent circles and therefore they were all equal. Of course, he had created this by his construction!**

Constructing an equilateral triangle

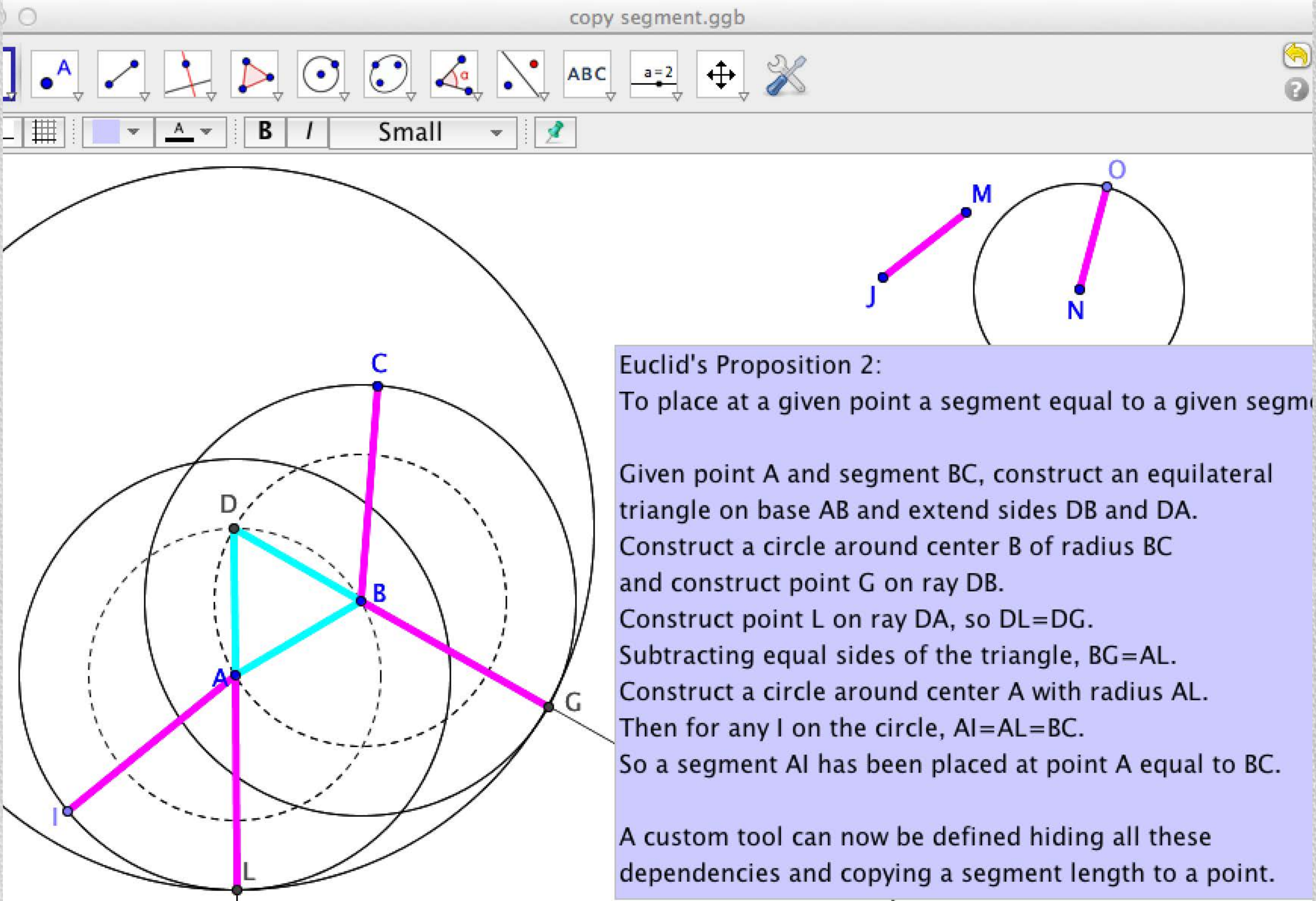


Euclid's Proposition #2:

- Using his construction of an equilateral triangle, Euclid showed how to copy the length of a line segment to an arbitrary new endpoint. He used the equality of radii extensively as well as the transitivity of equality of length: if lengths x=y and y=z, then x=z.

- I add another circle to allow the copied segment to point in any direction.
- Then I define a GeoGebra custom tool equivalent to compass tool.
- Given points A, B, C as input, the custom tool produces a circle around center C with radius =AB. This provides useful functionality while hiding a complicated sequence of 10 construction steps and several logical steps.

Copying a segment length



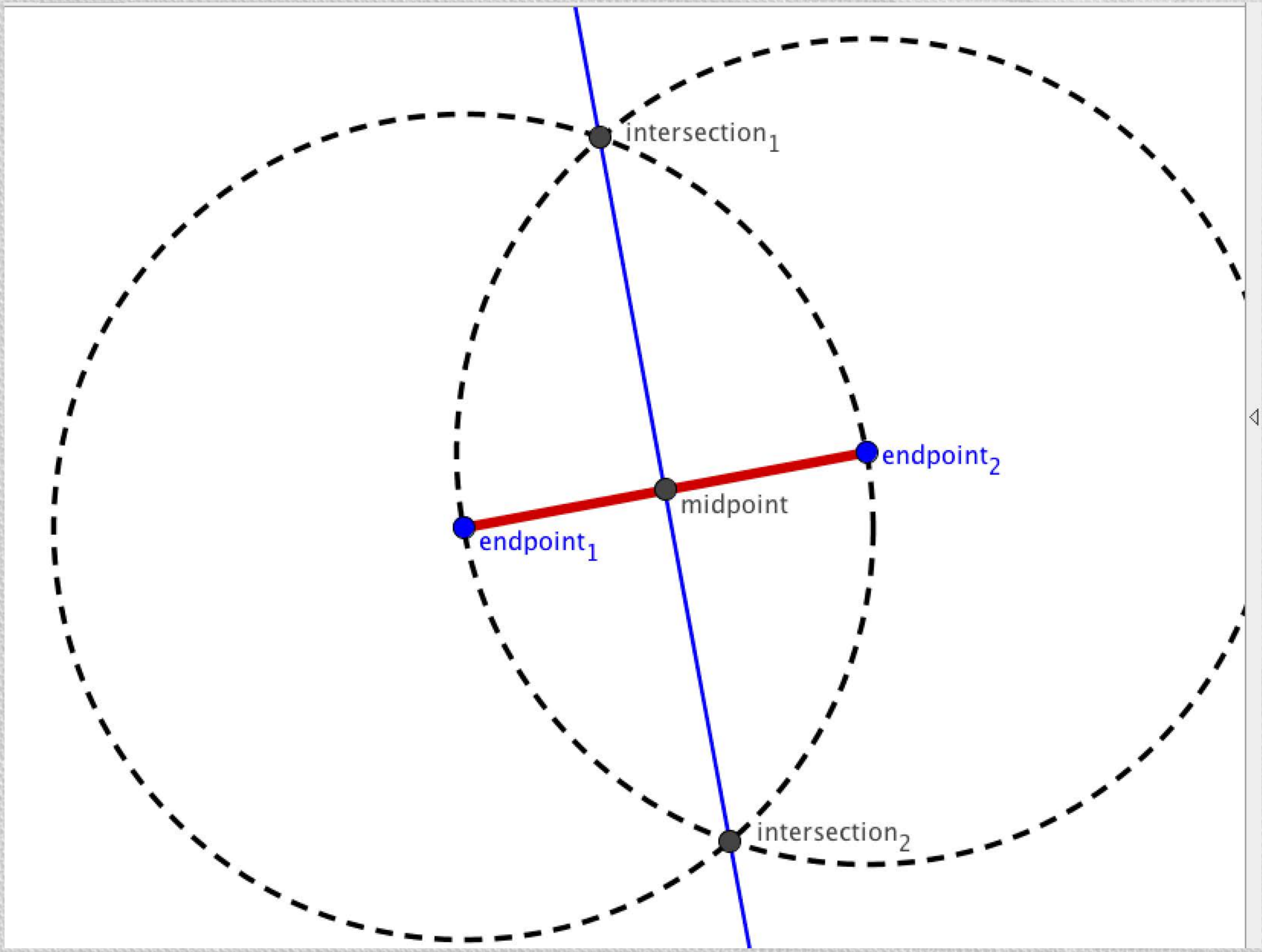
4. Abstracting dependencies in custom tools

The tools in GeoGebra consist of the following sets:

1. **Tools to construct individual points, lines, conics, polygons.**
2. **Tools to construct figures consisting of dependent objects, such as perpendicular lines, equilateral triangles, centers of triangles.**
3. **Tools to modify the appearance of the interface.**

- **Set (1) provides the primitive objects of GeoGebra (Euclid's definitions and postulates).**
- **Set (2) abstracts sequences of constructions with their dependencies. E.g., there are tools for an equilateral triangle, for copying a segment length to a given location, to bisect an angle, to locate the midpoint of a segment, to construct a perpendicular line from a point on or off the first line (Euclid's propositions #1, 2, 9, 10, 11, 12).**
- **Set (3) is for convenience and is not of mathematical relevance.**

Construct the midpoint & perpendicular bisector of a segment



Set (2) can be constructed by users from set (1) by defining custom tools – in theory.

(In fact, there are some limitations to the GeoGebra implementation of custom tools, in part due to a theoretical continuity problem.)

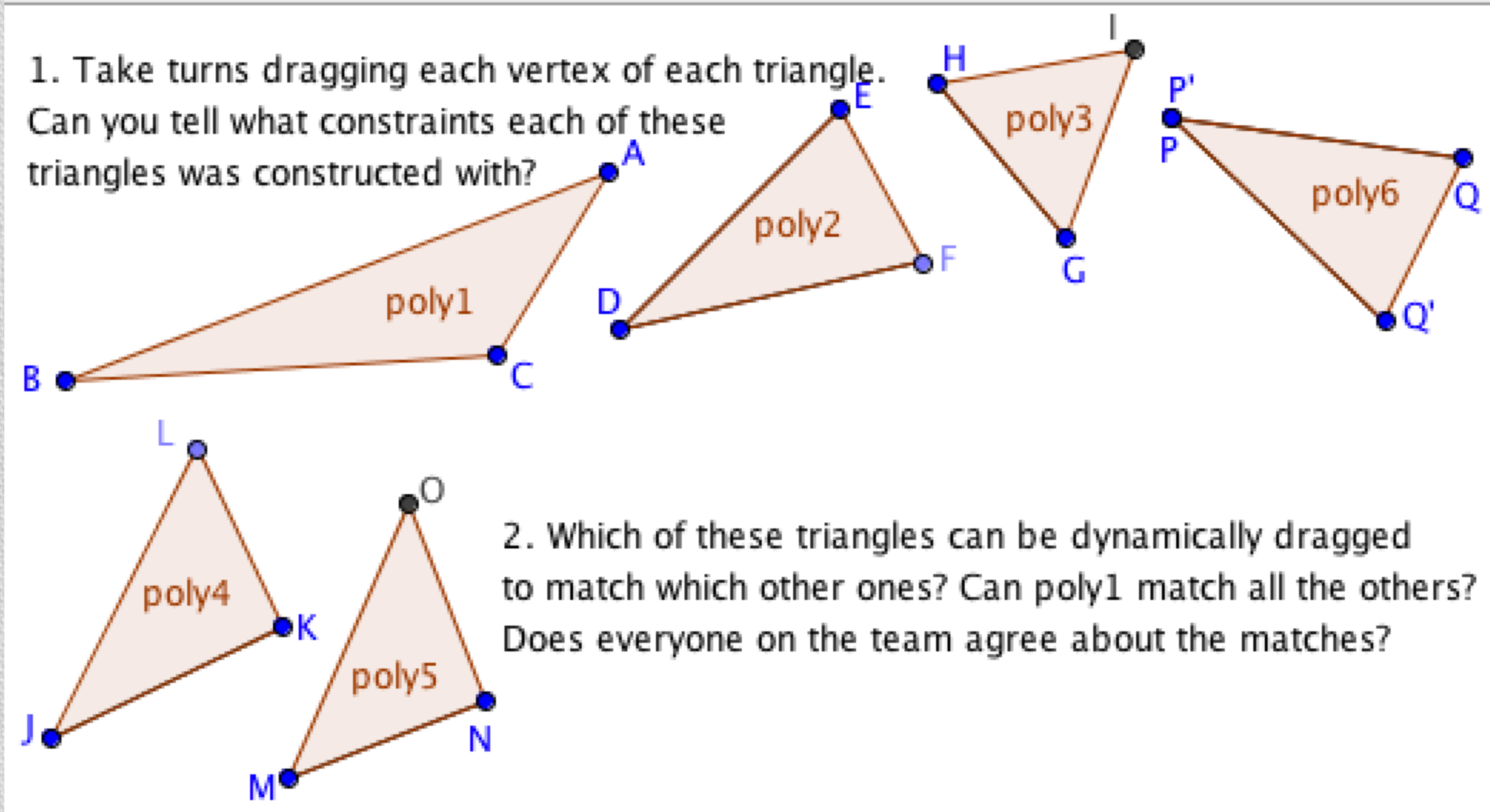
- **For instance, students can build their own tools for Euclid's propositions #1, 2, 9, 10, 11, 12.**
- **They can build tools to construct the different centers of triangles for exploring the Euler segment.**
- **They could build their own geometries.**

If students build their own tools, they may better understand the dependencies which are abstracted and hidden in the tools. The power of geometry consists in the dependencies, which are designed into various constructions. When these dependencies are hidden by hiding circles that constrain locations of points, using tools from set (2), providing students with pre-constructed figures to drag or just to view, the relationships among geometric objects are mystified. When dependencies are hidden behind ancient terminology and memorization based on accepted authority—rather than having students create and take ownership of the dependencies—the objects of geometry are alienated from the students.

Compare:

- → **Drilling students on the definitions of "special" triangles (equilateral, right, isosceles) or "special" quadrilaterals (square, rectangle, rhombus, kite), with**
- → **Having students explore the dependencies that distinguish all the possible kinds of triangles or quadrilaterals (number of equal sides, equal angles, parallel sides, etc.).**

A curricular unit on classifying kinds of triangles based on construction dependencies



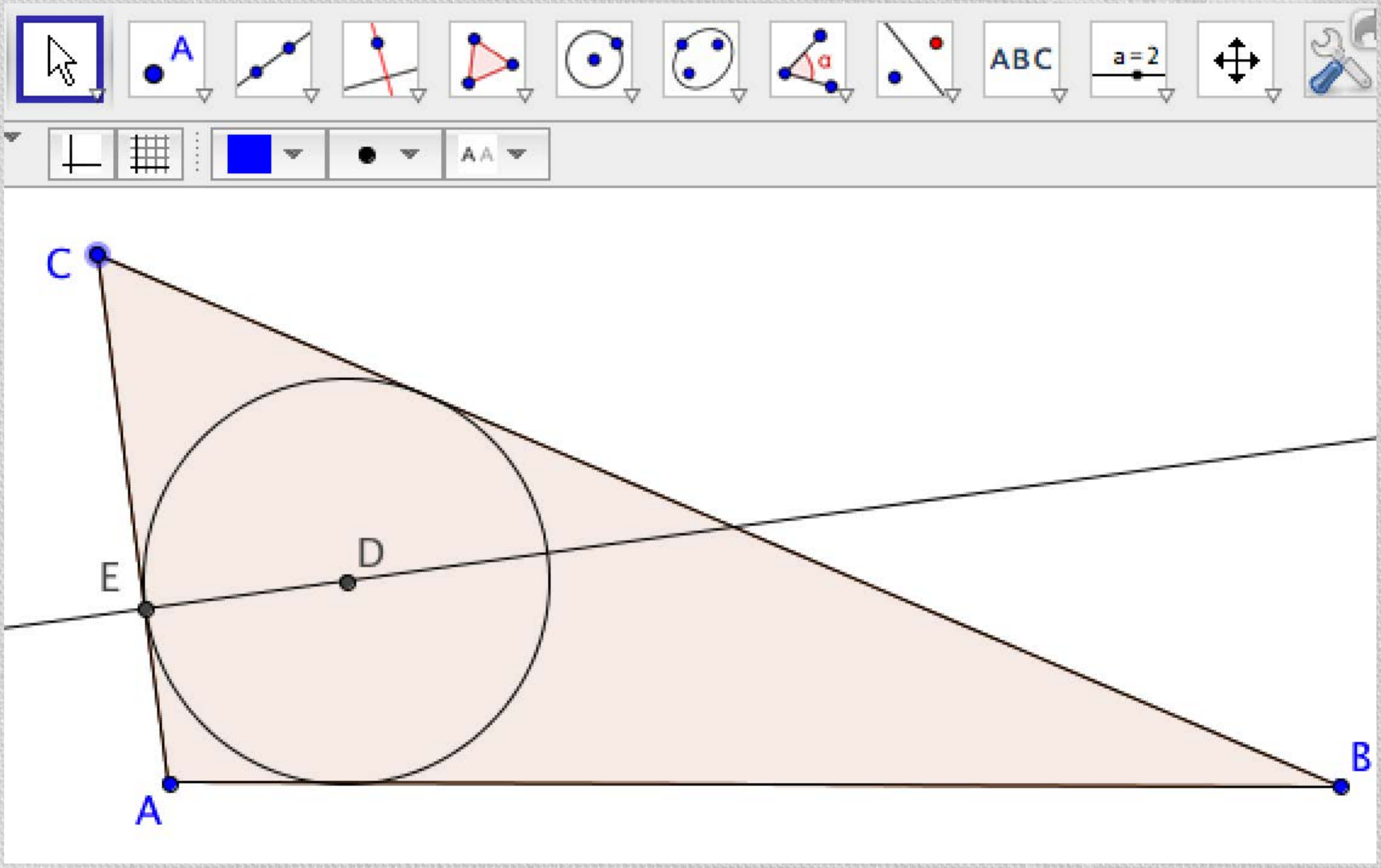
5. Proving propositions using dependencies

Students of basic geometry are supposed to be surprised that a simple generic triangle has special properties of an "incenter" that can be discovered and proven.

Students of modern (post-1900) geometry are supposed to be surprised at the complex innate properties of a simple triangle associated with "Euler's segment".

But we can see that these are nothing but dependencies built in by our construction of the incenter and of Euler's segment

The incenter of triangle ABC with a radius of the inscribed circle



A conjecture about the incenter

1. **The three bisectors of the vertex angles all meet at a single point.** **(It is unusual for three lines to meet at one point. For instance, do the angle bisectors of a quadrilateral always intersect at one point?)**

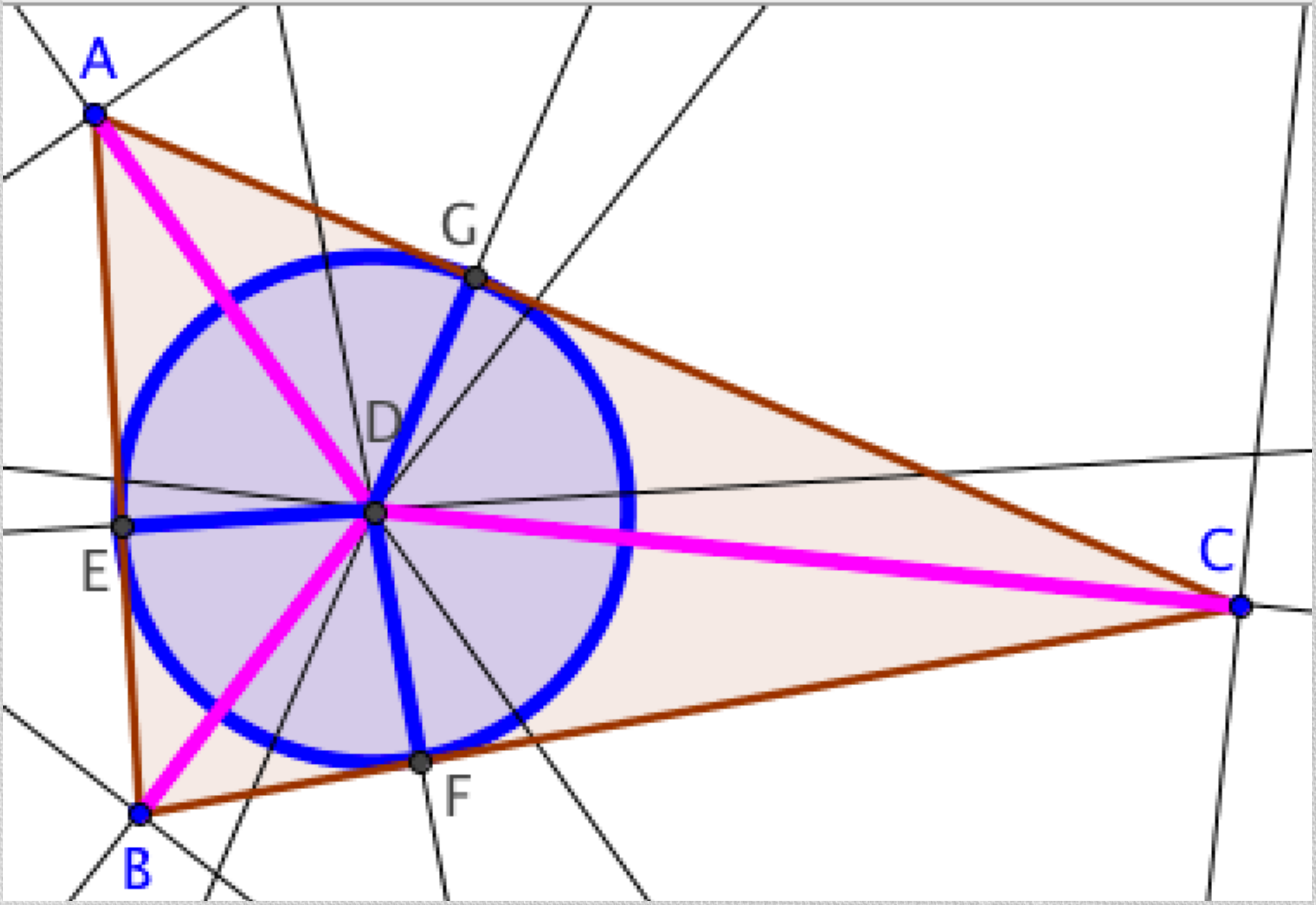
2. **The incenter of any triangle is located inside of the triangle.** **(Other kinds of centers of triangles are sometimes located outside of the triangle. For instance, can the circumcenter of a triangle be outside the triangle?)**

3. **Line segments that are perpendiculars to the three sides passing through the incenter are all of equal length.**

4. **A circle centered on the incenter is inscribed in the triangle if it passes through a point where a perpendicular from the incenter to a side intersects that side.**

5. **The inscribed circle is tangent to all three sides of the triangle.**

Illustration of the conjecture

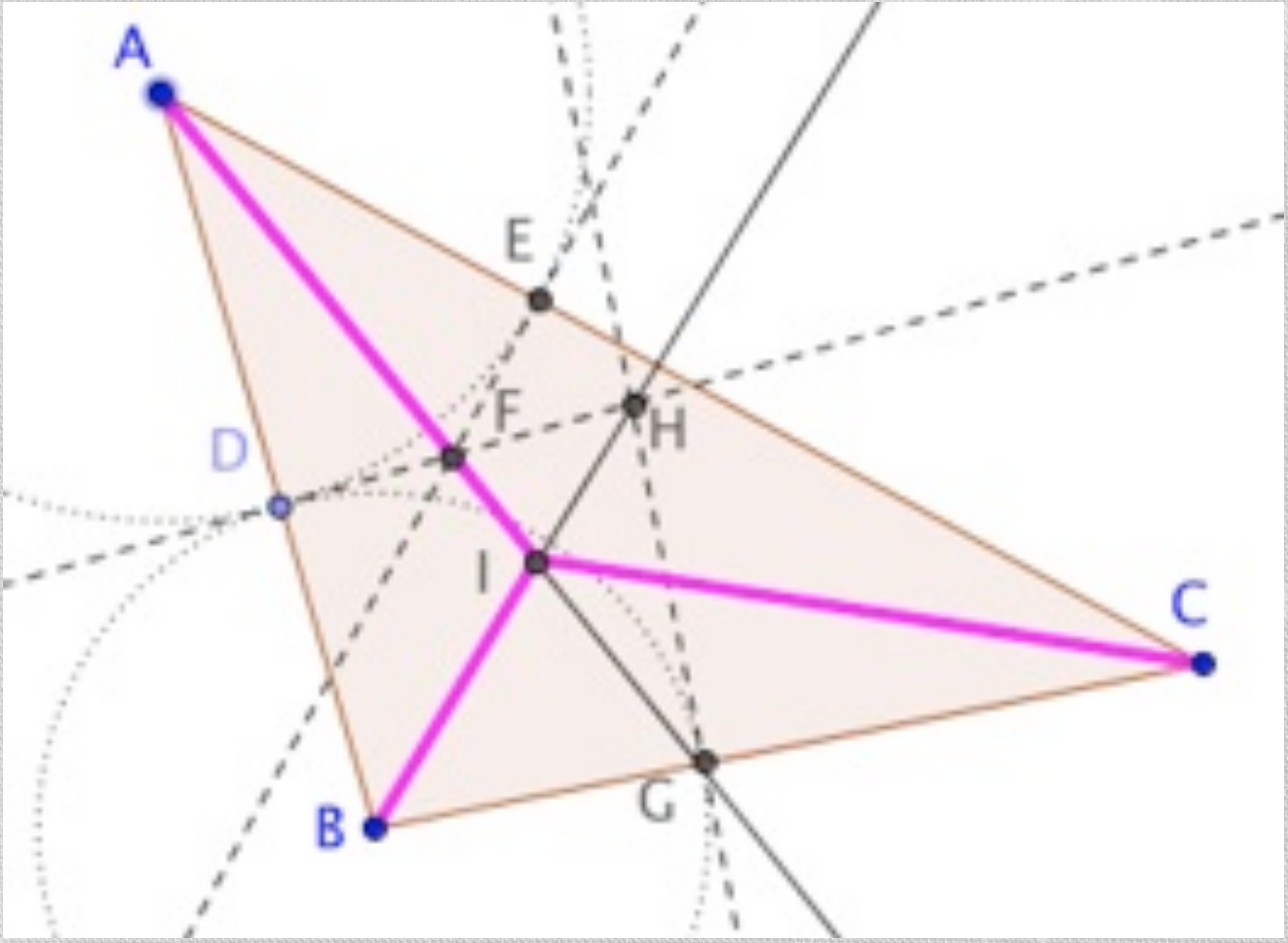


- **If we construct the incenter as the meeting point of the angle bisectors using a standard or custom tool for bisecting the interior angles of a triangle, the conjectured properties appear as a surprise.**

- **But if we just use the basic tools equivalent to straight-edge and compass, then we see how all these were constructed in the process of locating the incenter.**

- **We construct the angle bisectors by constructing a line whose points are equidistant from the angle sides.**

Constructing the angle bisectors for the incenter

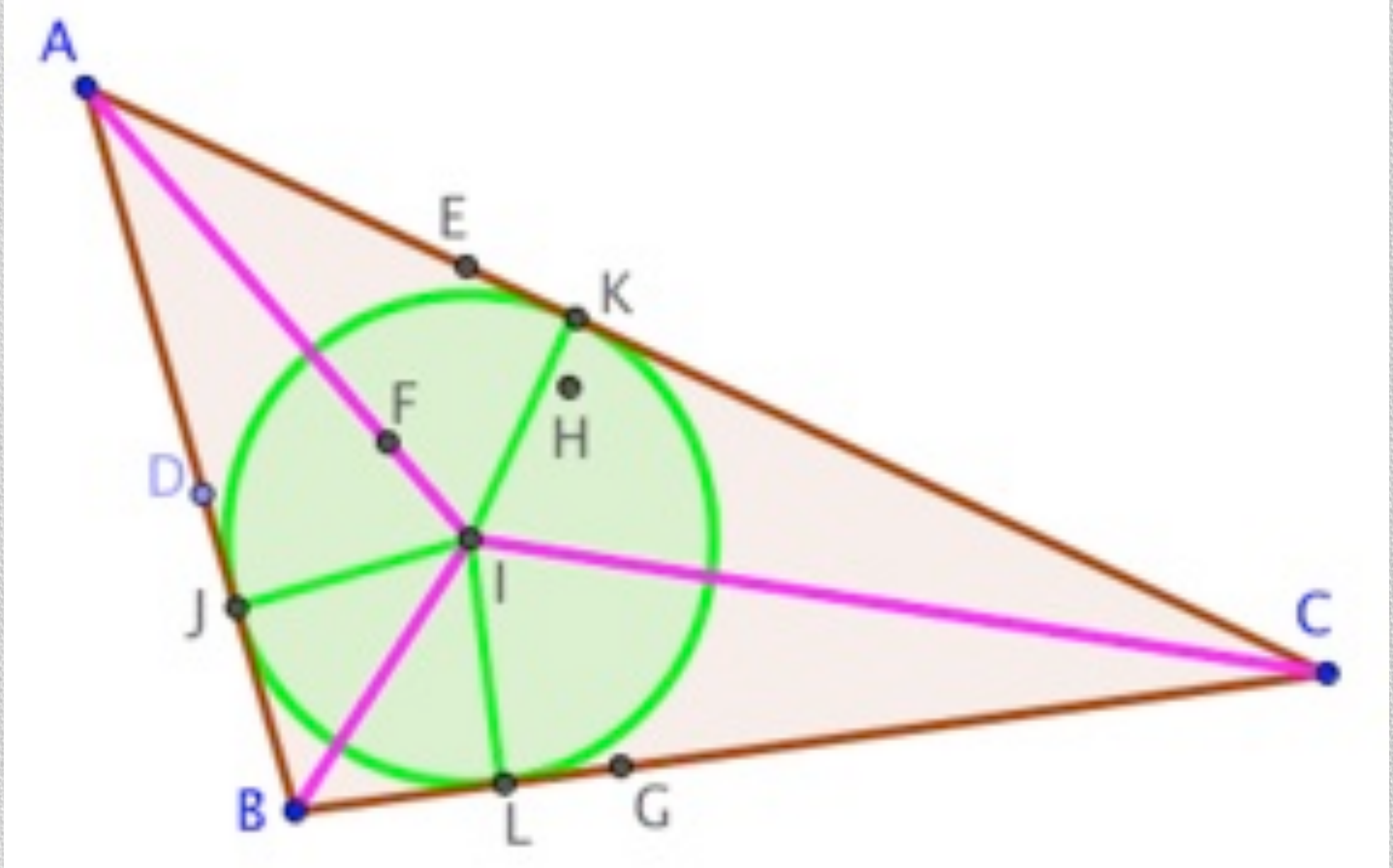


We construct point I at the intersection of the bisectors of A and B. Then we know that I is also on the bisector of C by constructing the perpendiculars to the sides from I. IJ=IK and IJ=IL so IL=IK.

All the properties of the conjecture follow from this:

- **We can inscribe a circle with radii IJ, IK, IL, etc.**

Inscribing the circle around the incenter



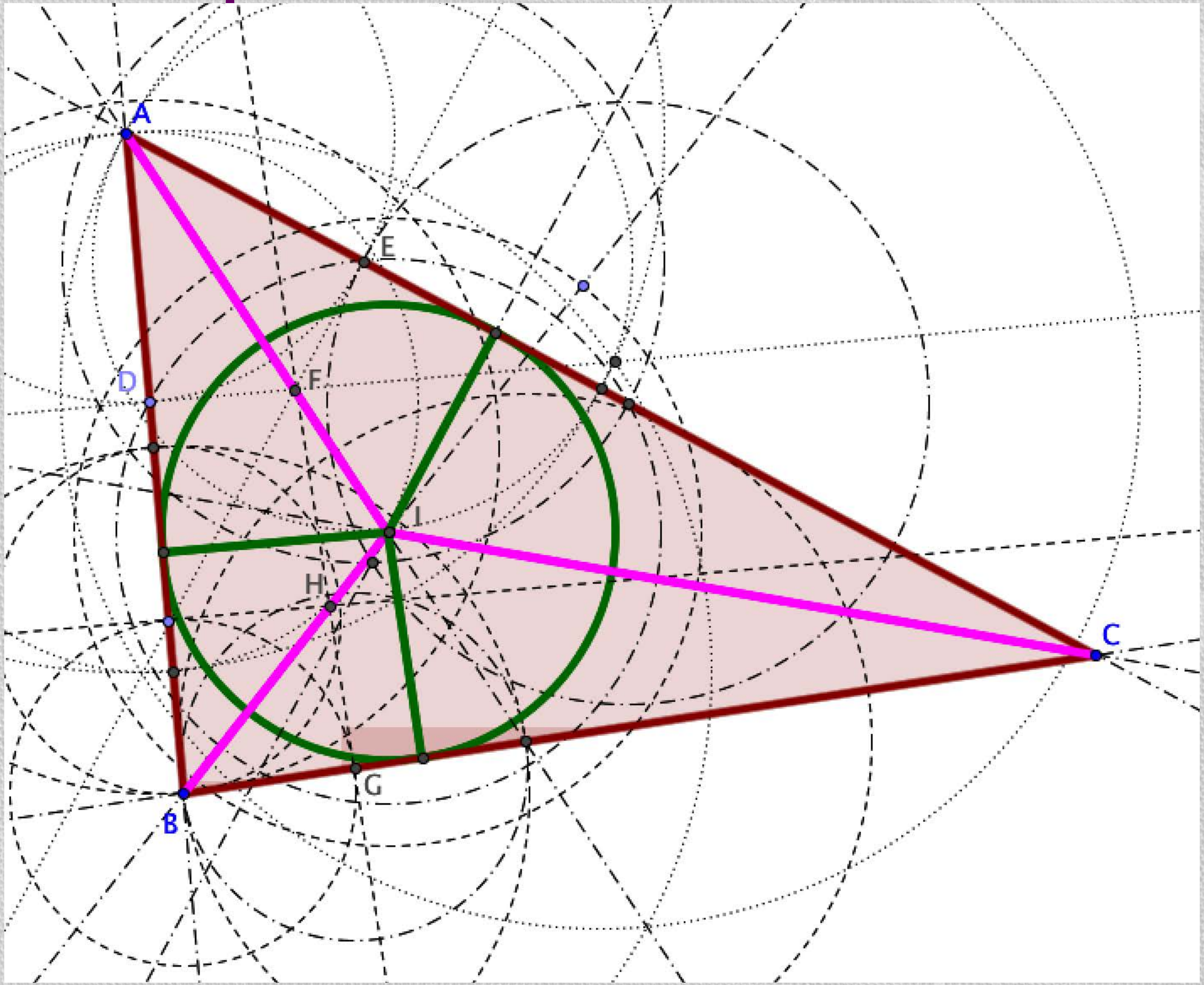
This construction involved the creation of 63 objects (points, lines and circles). It is becoming visually confusing. That is why it is often useful to package all of this in a special tool, which hides the underlying complexity.

GeoGebra has a tool for angle bisector.

We can define a custom tool for incenter.

It is wonderful to use these powerful tools, as long as you understand what dependencies are still active behind the visible drawing.

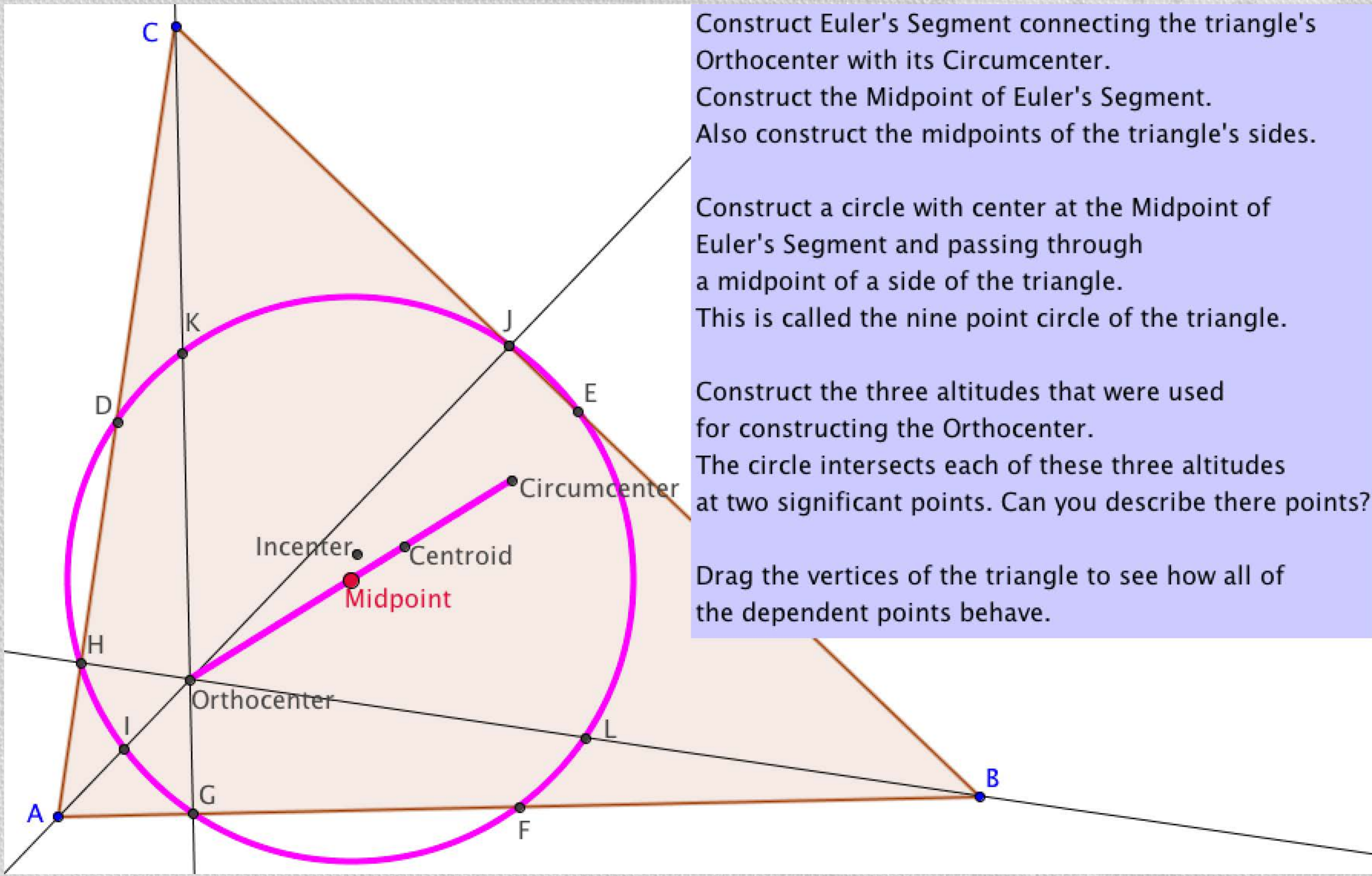
All the dependencies to construct an incenter



- **Euler's work sparked a revival of interest in geometry. His proof that the centers of a triangle have certain relationships surprised people**

- **E.g., the distance from the orthocenter (H, meeting point of the altitudes) to the centroid (G, meeting point of the medians) = 2GO (circumcenter, meeting point of the perpendicular bisectors).**

Explore the dependencies



Construct Euler's Segment connecting the triangle's Orthocenter with its Circumcenter.
Construct the Midpoint of Euler's Segment.
Also construct the midpoints of the triangle's sides.

Construct a circle with center at the Midpoint of Euler's Segment and passing through a midpoint of a side of the triangle.
This is called the nine point circle of the triangle.

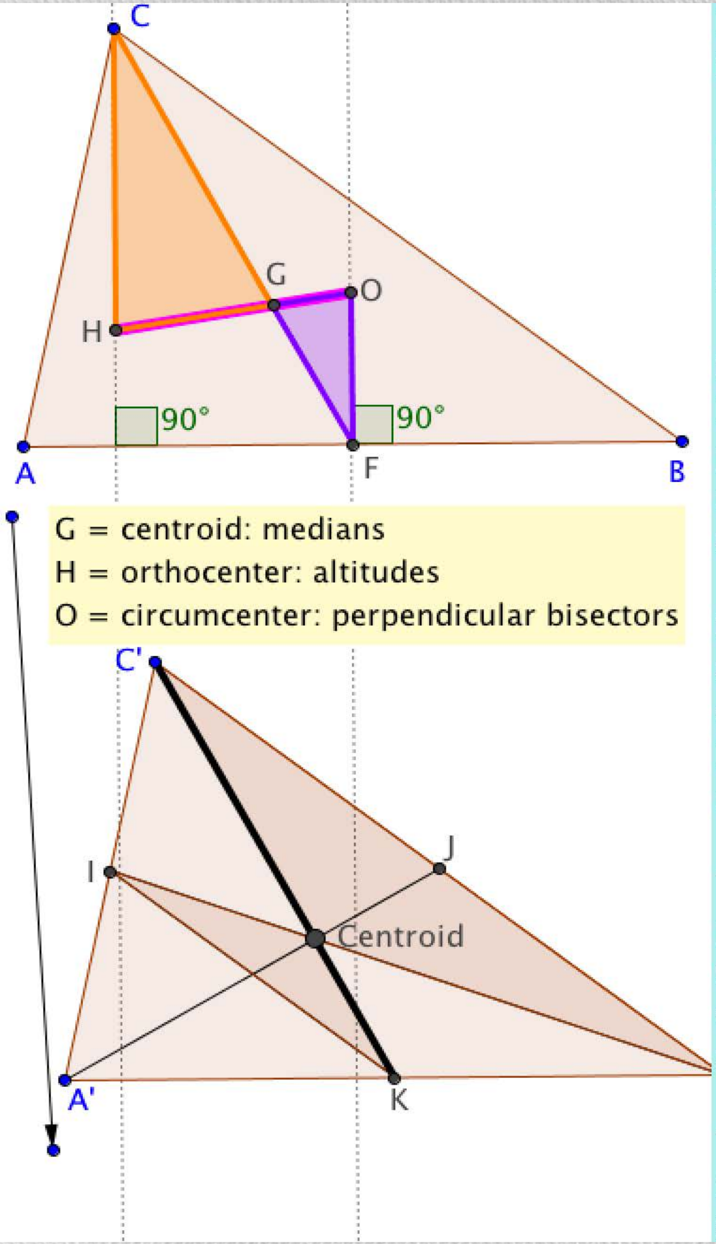
Construct the three altitudes that were used for constructing the Orthocenter.
The circle intersects each of these three altitudes at two significant points. Can you describe there points?

Drag the vertices of the triangle to see how all of the dependent points behave.

- **E.g., the distance from the orthocenter (H, meeting point of the altitudes) to the centroid (G, meeting point of the medians) = 2GO (circumcenter, meeting point of the perpendicular bisectors).**

- **But these are not innate properties of a simple triangle; they are the consequences of the constructions of the centers. They can be proven based on the constructed dependencies.**

Proof that GH=2*GO based on construction of dependencies



Proof that the centroid of a triangle is one-third of the way on the line from the circumcenter to the orthocenter.

First, the medians that construct the centroid are divided into segments with lengths of 1:2 ratio as seen by the two brown similar triangles below, whose corresponding parts are in the ratio 1:2. Therefore the median CF is divided by the centroid at G into segments with lengths of 1:2.

Second, the orthocenter is constructed on the altitudes from the vertices, so H is on an altitude from C. The circumcenter is constructed to be on the perpendicular bisectors, so FO is on a perpendicular to AB through the midpoint of AB at F. Thus, CH and FO are both perpendicular to AB and are parallel to each other.

Since CH and FO are parallel and are cut by CF, angle GCH and angle GFO are congruent, as are angle CHG and FOG and angles CGH and FGO. So the orange and purple triangles on the left are similar. Since the ratio of FG to CG is 1:2, the ratio of all corresponding parts are in the same ratio. This proves that GH = 2*GO.

This diagram used one median through G, one altitude through H and one perpendicular bisector through O. The equivalent diagrams could be drawn with the other lines, creating similar triangles in the two other orientations of triangle ABC, reflecting the construction of G, H and O as co-linear and at the concurrency of their three lines.

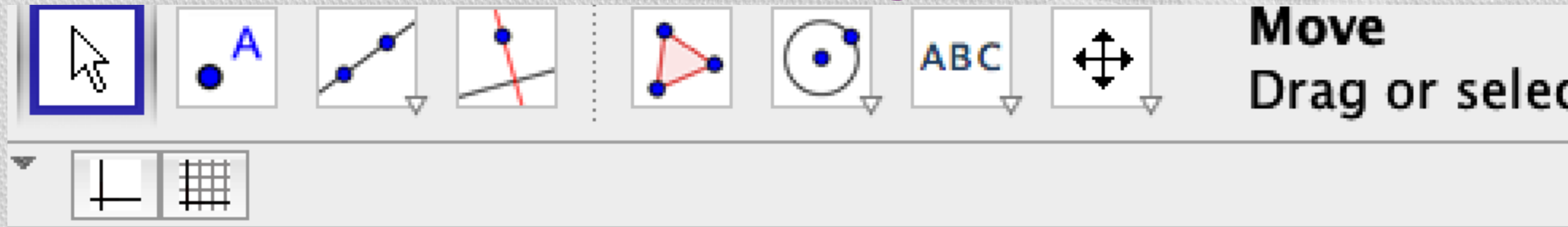
Thus, we have shown that the construction of the three centers determines that they are necessarily co-linear and at a distance of 1:2 from each other. QED

6. Creative-discovery as human-centered approach

- Let's move from math theory to math didactics.

- Here are two resources that illustrate the combination of dragging to discover and constructing to create.
- Given an existing dynamic figure, explore its dependencies by dragging its points.
- Then design a construction process to create such a figure.
- A typical team moves back and forth between such discovery and creation.

The inscribed triangles resource

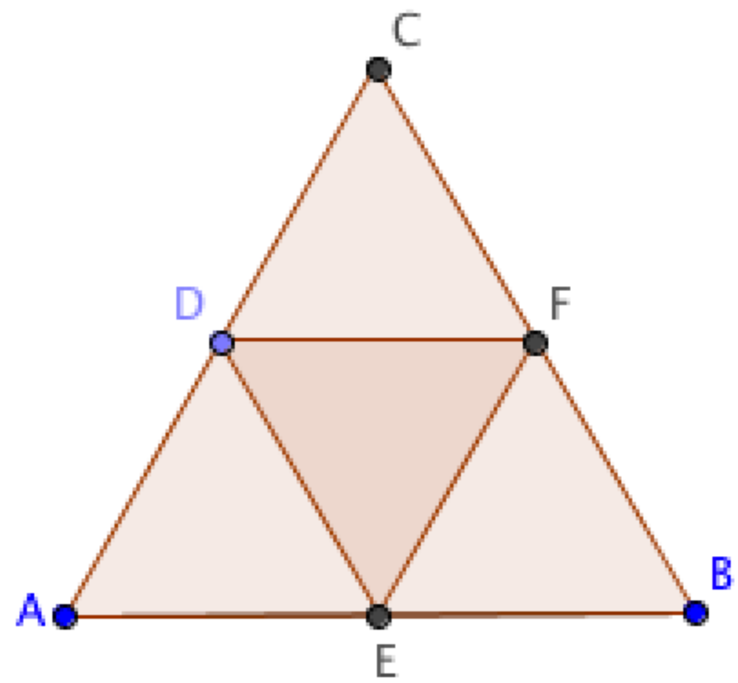


Take turns dragging vertex A of triangle ABC and vertex D of triangle DEF.

Chat about dependencies you notice and what you wonder about this figure.

Construct a triangle inscribed in a triangle that behaves the same as this one.
Chat about how you are constructing and why.

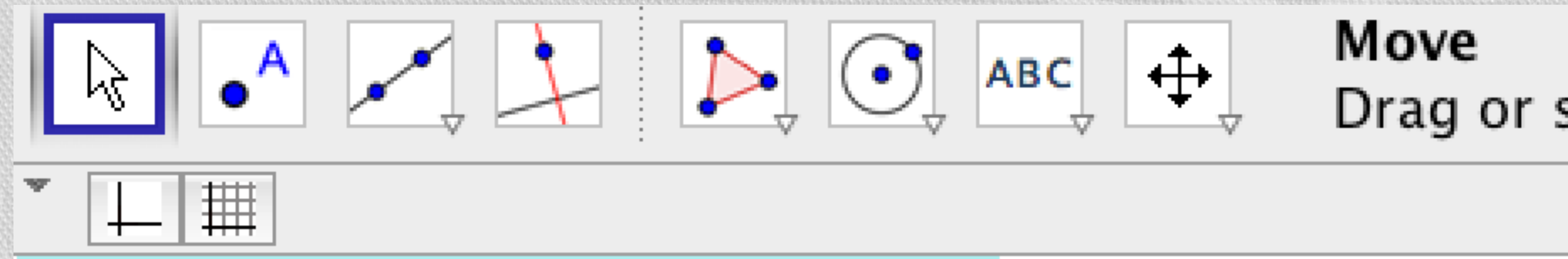
It might be helpful to look at the other tabs for this topic and think about them together.



Some generalization

- **When a team succeeds, they understand that they have discovered some interesting dependencies and they have learned how to create such dependencies and figures themselves.**

- **The dependencies of the inscribed triangles can be generalized to square, hexagon and even an n-sided polygon. The proof follows closely from the understanding of the dependencies**

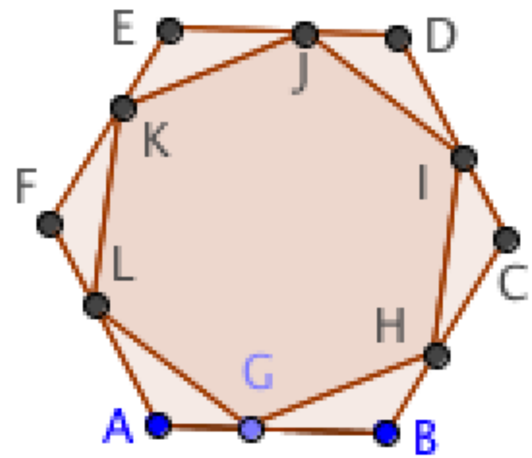


Take turns dragging vertex A of hexagon ABCDEF and vertex G of hexagon GHIJKL.

Chat about dependencies you notice and what you wonder about this figure.

Construct a hexagon inscribed in a hexagon that behaves the same as this one.

Notice that you can use the Regular Polygon Tool.
Explore how it works.
Chat about how you are constructing and why.



Can you make a conjecture about inscribing regular N-sided polygons?
Can you prove (or disprove) your conjecture?

7. Teaching this approach to teachers

- **We provide a professional development course to math teachers. In a semester, they have about 18 hour-long online collaborative sessions going through several GeoGebra tabs each session.**

- **They discuss some readings about dynamic geometry and collaborative math discourse.**
- **They reflect in their individual journals, in their group chat rooms and in the class discussion board about how their chat sessions went and how to present to their students.**

The approach

- Collaborative learning
- Emphasis on increasing significant math discourse
- Focus on design of dependencies
- Constructivist, hands-on, student-centered
- Creative-discovery: explore and design
- Full use of GeoGebra: construction, custom tools, not just illustrative pre-constructed apps
- Reflection on persistent record (chat room, logs, replayer)
- Tuned to level of teachers or students

8. Teaching this approach to students

- Last Spring, teachers organized 8 sessions for their own students, using VMT with GeoGebra.
- Some teachers organized after-school sessions, some integrated in classroom or school computer lab
- Used mainly topics supplied from beginning of “Topics” workbook.
- Teachers were not prepared to define their own topics successfully.

- It would be nice to have student teams across schools – or even across countries in English, like Facebook ☺ .

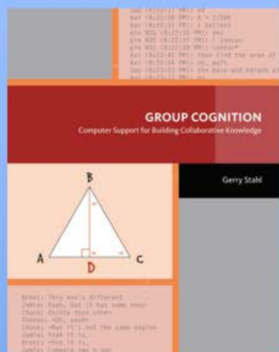
9. Analyzing how teachers and students learn this approach in their discourse and interaction

- **We have been analyzing online math teams for 10 years.**
- **Have only looked at one team from last Spring so far.**
- **Now have lots of interesting data.**

- **We have complete logging of interactions (chat and GeoGebra actions) and can replay sessions for analysis.**
- **More on course design and data analysis in coming days**

The VMT trilogy

Group Cognition (2006)



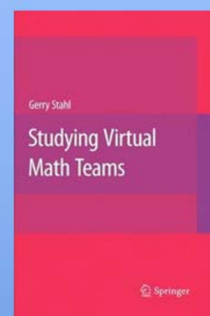
Computer Support for Building Collaborative Knowledge

MIT Press, 510 pages
Available for Kindle

The theory of group cognition emerges from several studies of CSCL and CSCW technologies. Analysis of interaction. Theory of CSCL.

www.GerryStahl.net/elibrary/gc

Studying Virtual Math Teams (2009)

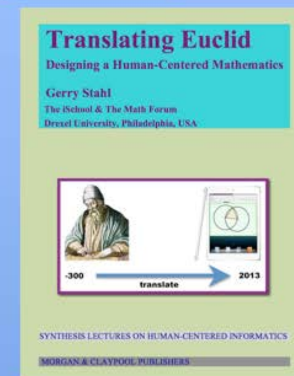


Springer Press, 626 pages
CSCL Book Series, paperback

Studies of the VMT Project technology, pedagogy, analysis, theory by team members and international collaborators

www.GerryStahl.net/elibrary/svmt

Translating Euclid (2013)



Creating a Human-Centered Mathematics

Morgan Claypool Publishers, 325 pages, e-book & paperback

Latest results of this design-based CSCL research from many perspectives.

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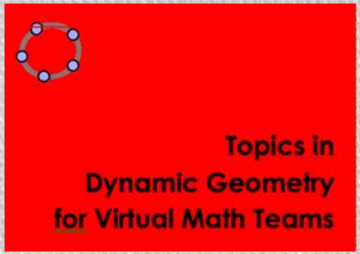
For further info…



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Topics in Dynamic Geometry for VMT:

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Slides:

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