Topics in Dynamic Geometry for Virtual Math Teams

Gerry Stahl

& The VMT Project Team

The Math Forum at Drexel University

Introduction

Topics in Dynamic-Geometry for Virtual Math Teams is a set of topic statements for use with the Virtual Math Teams with GeoGebra (VMTwG) collaboration software.[1]

Dynamic geometry is a new form of mathematics—and you can be a pioneer in it, exploring, discovering and creating new insights and tools. Dynamic geometry realizes some of the potential that has been hidden in geometry for thousands of years: by constructing dynamic-geometry figures that incorporate carefully designed mathematical relationships and dependencies, you can drag geometric objects around to investigate their general properties.

In addition, the approach of these topics allows you to take advantage of the power of *collaboration*, so that your team can accomplish more than any one of you could on your own—by chatting about what you are doing, and why, as well as discussing what you notice and wonder about the dynamic figures. Working in a team will prevent you from becoming stuck. If you do not understand a geometry term or a task description, someone else in the team may have a suggestion. If you cannot figure out the next step in a problem or a construction, discuss it with your team. Decide how to proceed as a team.

The topics have been designed for *everyone* interested in geometry. Students who have not yet studied any geometry can use the topics to prepare them for thinking geometrically. Students who are in the middle of a geometry course or have already completed one can use the topics to gain a deeper appreciation of geometry. Even experienced geometry teachers can use the topics to gain a new perspective on an ancient subject.

To harness the truly awesome power of *collaborative dynamic geometry* requires patience, playfulness and persistence. It will pay off by providing skills, tools and understanding that will be useful for a lifetime. You will need to learn how to construct complicated figures; this will be tricky and require practice. You will need to think about the hidden dependencies among dynamic points, which make geometry work; this may keep you up at night. You will even create your own custom construction tools to extend GeoGebra; this will put you in control of mathematics.

Collaborative problem solving is central to this set of topics. The topics have been selected to offer you the knowledge, skills and hands-on experience to solve typical geometry problems, to explain your solution to others and to think more like a mathematician.

These topics present a special *approach* to dynamic geometry, which may be quite different from approaches to learning geometry that you are accustomed to.[2] They stress the importance of understanding how dependencies are constructed into geometric figures. Dragging points around in pre-constructed GeoGebra apps can help you to discover and visualize relationships within a figure. However, such apps also hide the dependencies that maintain the relationships. You should know how to construct those dependencies yourself. This will give you a deeper understanding of geometry as a mathematical system. This approach may take more time and thought, but it will also be more fun and more rewarding.

The approach in this document is built around a set of 13 *core topics* that provide a coherent experience of collaborative dynamic geometry. In addition, there is an introductory topic for individuals to do on

---

[1] The latest version of ***Topics in Dynamic Geometry for Virtual Math Teams*** is always available at: www.GerryStahl.net/elibrary/topics/topics.pdf

[2] The approach is discussed in detail from various perspectives in ***Translating Euclid: Designing a Human-Centered Mathematics***. See www.GerryStahl.net/elibrary/euclid

[2] The approach is discussed in detail from various perspectives in ***Translating Euclid: Designing a Human-Centered Mathematics***. See www.GerryStahl.net/elibrary/euclid

[3] The standard single-user version of GeoGebra is available for download at: www.GeoGebra.org The GeoGebra

their own to get started and then a transition topic for individuals to do at the end to start to explore the rest of GeoGebra. There are also extensions to some of the core topics for individuals or groups to explore farther what they are interested in. Finally, several open-ended advanced topics present new areas of mathematics beyond the core topics.

Each topic is designed for an online team to work on together for about one hour. The sequence of topics introduces student teams to GeoGebra and guides them in the exploration of dynamic geometry, including triangles, quadrilaterals and transformations. Ideally, everyone should spend 13 hour-long, online, synchronous, collaborative sessions working on the core topics. An instructor might want to select which of these topics to use and which to skip or make optional if it is not possible to do all 13 or if there is time available to do more than 13. A virtual math team might choose which topics they want to explore.

Several "tours" are included at the end of this document. They provide *tutorials* in important features of the VMT and GeoGebra software. Teachers and students can take the tours when they want more information on using the software. To work on topics outside of a team, individual teachers or students can download the single-user desktop version of GeoGebra and then download selected .ggb files. [3]

The pages of this document can be distributed to students as *worksheets* to keep their notes on and in case the instructions in the chat room tabs become erased. The whole document can be distributed to team members to serve as a journal for students to take notes on their sequence of topics and to provide access to the tutorials. It is helpful if everyone has a printed or electronic version of this booklet on their physical or digital desktop when they are working on the topics.

To get started, it is important that everyone do the *Individual Warm-Up Topic* on the computer that they will be using—well before the first collaborative session. This checks that your computer is properly set up and that you are able to enter VMT chat rooms. It also provides a valuable introduction to GeoGebra.

The topics and tours have been designed for a *broad range of users*. Students who have not previously studied much geometry should focus on the main points of each topic and make sure that all team members understand the constructions. Teams of experienced math teachers may engage in more in-depth discussion on implications, conceptualizations and pedagogy of the dynamic-geometry topics. If something is unclear in the topic instructions, discuss it in your team and decide as a group how to proceed. Pace your team to try to complete all the core parts of a topic in the time you have together.

Here is a *general procedure* you might want to follow: Before the time assigned for a group session, read the topic description in this document. It might suggest watching a brief video or taking a tour at the end of this document as important preparation. Think about the topic on your own. Then, in the group session, discuss the topic and work together on the various tasks. Discuss what you are doing in the chat and respond to the questions posed in the topic. Mark down in your copy of this document what you noticed that surprised you and what you wondered about that you want to think more about later. Do not just rush though the topic; discuss what is important in it. The point is to learn about collaborative dynamic geometry, not just to get through the topic steps. When the session is over, try to work on your own on any parts that your team did not get to. Report to your team what you discover. Maybe the team can get together for an extra session on the rest of the topic.

This set of topics offers a unique opportunity to experience deeply topics that have fascinated people since the beginning of civilization. Take advantage. Be creative. Collaborate. Explore. Chat. Enjoy!

---

[3] The standard single-user version of GeoGebra is available for download at: www.GeoGebra.org The GeoGebra files to be loaded into GeoGebra tabs in VMT chat rooms can be downloaded. For instance, download the .ggb file for Topic 7.3 at: www.GerryStahl.net/vmt/topics/7c.ggb as well as from the "VMT channel" of GeoGebraTube at: www.geogebratube.org/collection/show/id/4531

Table of Contents

Topic 1: Individual Warm-up Activity

You can start on this Warm-up activity by yourself at any time to explore the Virtual Math Teams with GeoGebra (VMTwG) environment for discovering collaborative dynamic geometry.

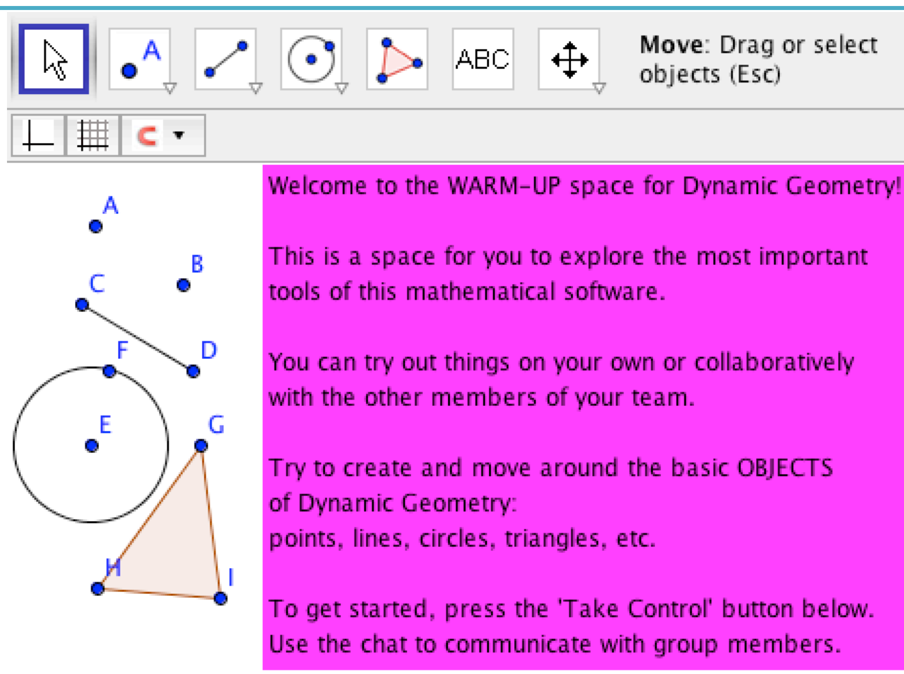
First, you can get a good introduction to using the GeoGebra software from this **YouTube clip**: http://www.youtube.com/watch?v=2NqblDIPl38. It gives a clear view of how to use GeoGebra. It also provides tips on GeoGebra tools that are equivalent to traditional geometry construction with a physical straightedge (for making line segments) and compass (for making circles).

Then, read **"Tour 1: Joining a Virtual Math Team."** This will introduce you to using the VMT software to register, login, find a chat room, etc. The Tours are all found toward the end of this booklet, after all the Topics.

Also look through **"Tour 2: GeoGebra for Dynamic Math."** This will provide more details about the GeoGebra system for dynamic geometry. For instance, it will describe many of the buttons shown in the Tool Bar. You can always go back to these tours if you become stuck using VMT or GeoGebra.

Section 1.01 Welcome!

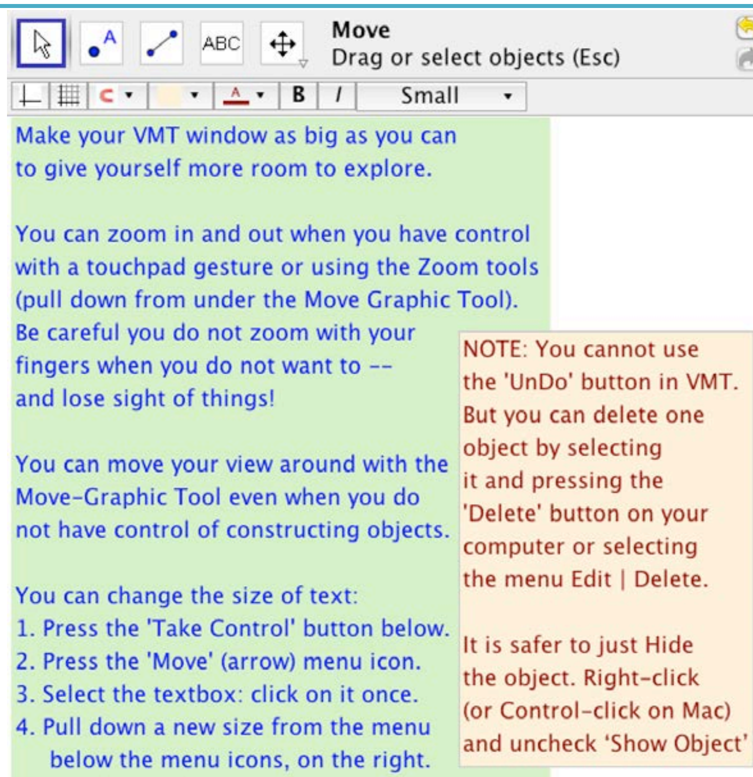
Start in the Welcome tab (shown below) of the Warm-Up "Topic 01" chat room by reading the instructions and then dragging the objects shown (press the "Take Control" button at the bottom, select the "Move Tool" arrow and click on a point to drag it). Then create your own similar objects using the buttons in the Tool Bar (near the top). You can change to one of the other tabs, like "#1 Team Member" to have an empty GeoGebra workspace to create your own points and lines. Try out each of the tools in the Tool Bar.



If there is not an unused tab available to create your points and lines in, you can create a new tab with the "+" button in the upper right corner of the VMTwG window. You can use the "ABC" text tool to construct a text box with your name or comments in it: Select the text tool and then click in the tab about where you want to have the text box. A form will open for you to enter your text.

Section 1.02 Helpful Hints

Here are some hints for taking advantage of the limited space on your computer. Read these hints and adjust the size of your VMT window and its GeoGebra tab.



There are a number of Zoom Tools, which can be pulled down from the Move Graphic Tool (the crossed arrows on the right end of the Tool Bar).

If you are using a touchpad on your computer, you can zoom with a two-finger touchpad gesture. **Caution**: It is easy to move things around without wanting to as your fingers move on the touchpad. Things can quickly zoom out of sight. This can happen even when you do not have control of construction. Then you will not see the same instructions and figures that other people on your team see.

Note: Most Zoom Tools will only effect what **you** see on your computer screen, **not** what your teammates see on theirs. It is possible for you to create points on an area of your screen that your teammates do not see. If this happens, ask everyone if they want you to adjust everyone's screen to the same zoom level and then select the menu item "GeoGebra" | "Share Your View."

When you work in a team, it is important to have everyone agree before changes are made that affect everyone. If you want to delete a point or other object – especially one that you did not create – be sure that everyone agrees it should be deleted.

Be very careful when **deleting** points or lines. If any objects are dependent on them, those objects will also be deleted. It is easy to unintentionally delete a lot of the group's work.

Instead of deleting objects, you should usually **hide** them. Then the objects that are dependent on them will still be there and the dependencies will still be in effect. Use control-click (on a Mac) or right-click (in Windows) with the cursor on a point or line to bring up the context-menu. Select "Show Object" to unclick that option and hide the object. You can also use this context-menu to hide the object's label, rename it, etc.

You may want to **save** the current state of your GeoGebra tab to a .ggb file on your desktop sometimes so you can load it back if things are deleted. Use the menu "File" | "Save" to save your work periodically. Use the menu "File" | "Open" to load the latest saved version back into the current tab. Check with everyone in your team because this will change the content of the tab for everyone.

If the instructions in a tab are somehow erased, you can look them up in this document. You can also scroll back in the history of the tab to see what it looked like in the past. Finally, you can have your whole team go to a different room if one is available in the VMT Lobby that is not assigned to another team.

If you have **technical problems** with the chat or the figures in the tabs not showing properly, you should probably close your VMT window and go back to the VMT Lobby to open the room again.

You should use this booklet to **keep notes** on what you learn or wonder, either on paper or on your computer. Here is a space for notes on Topic 01:

Notes:

Topic 2: Messing Around with Dynamic Geometry

Dynamic geometry is an innovative form of mathematics that is only possible using computers. It is based on traditional Euclidean geometry, but has interesting objects, tools, techniques, characteristics and behaviors of its own. Understanding dynamic geometry will help you think about other forms of geometry and mathematics.

In this topic, you will practice some basic skills in dynamic geometry. There are several tabs to work through; try to do them all with your team. Pace yourselves. If the team becomes stuck on one tab during a session, move on and come back to it later, maybe on your own after the team session.

Make sure that everyone in your team understands the important ideas in a tab and then have everyone move to the next tab.

Section 2.01 Dynamic Points, Lines & Circles

Geometry begins with a simple point. A point is just the designation of a particular location. In dynamic geometry, a point can be dragged to another location.

Everything in geometry is built up from simple points. For instance, a line segment is made up of all the points (the "locus") along the shortest (direct, straight) path between two points (the endpoints of the segment). A circle is all the points ("circumference," "locus") that are a certain distance ("radius") from one point ("center").

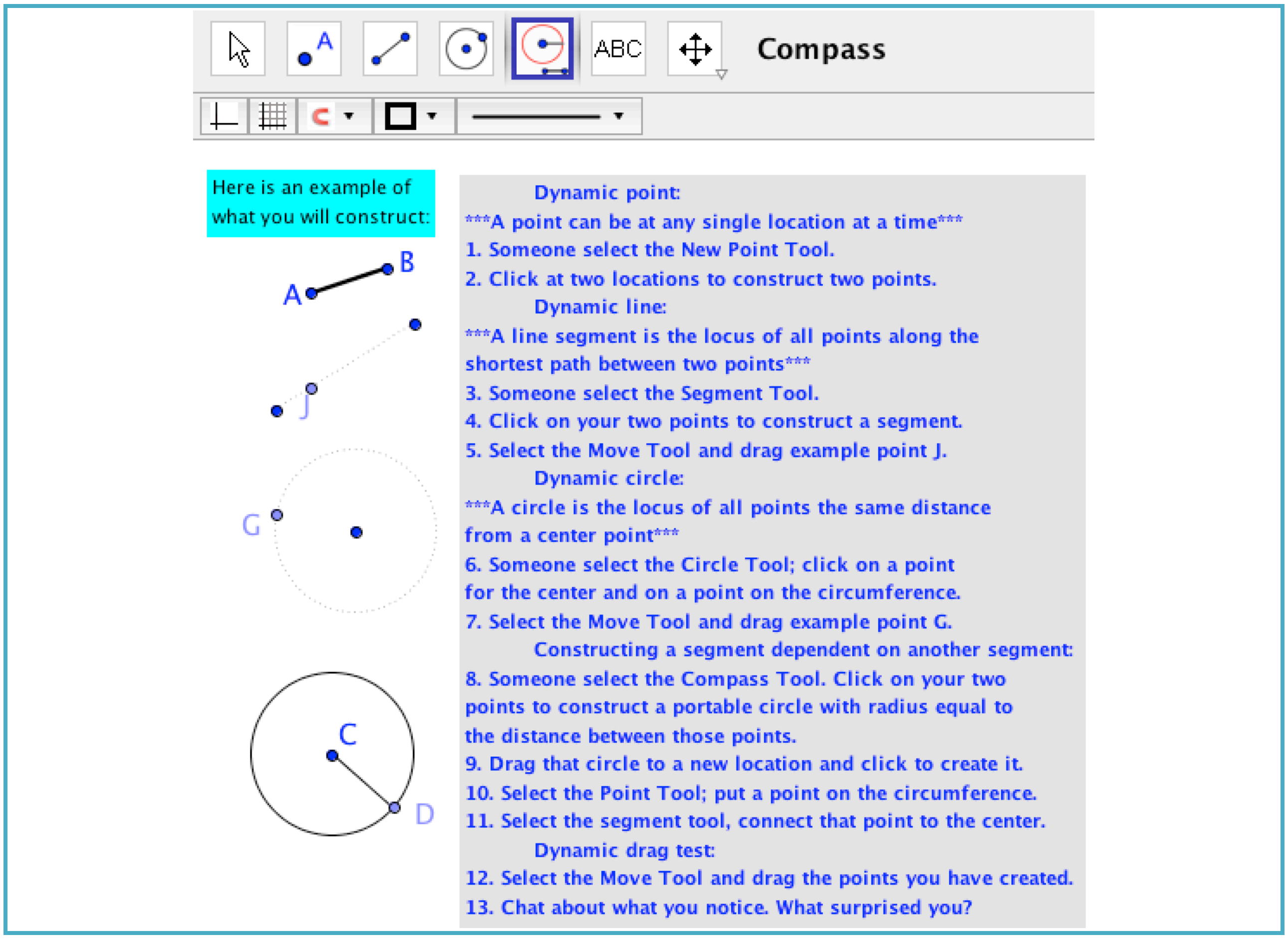
In this tab, create some basic dynamic-geometry objects and drag them to observe their behavior. Take turns taking control and creating objects like the ones you see.

When you drag point J between the two points, the locus of a line segment will be colored in. (This locus will only appear on your computer screen, so everyone in the team has to try it themselves).

The same for dragging point G around the locus of the circle.

With these simple constructions, you are starting to build up, explore and understand the system of dynamic geometry.

Don't forget that you have to press the "Take Control" button to do actions in GeoGebra. Chat about who should take control for each step. Be sure you "Release Control" when you are done so someone else can take control.

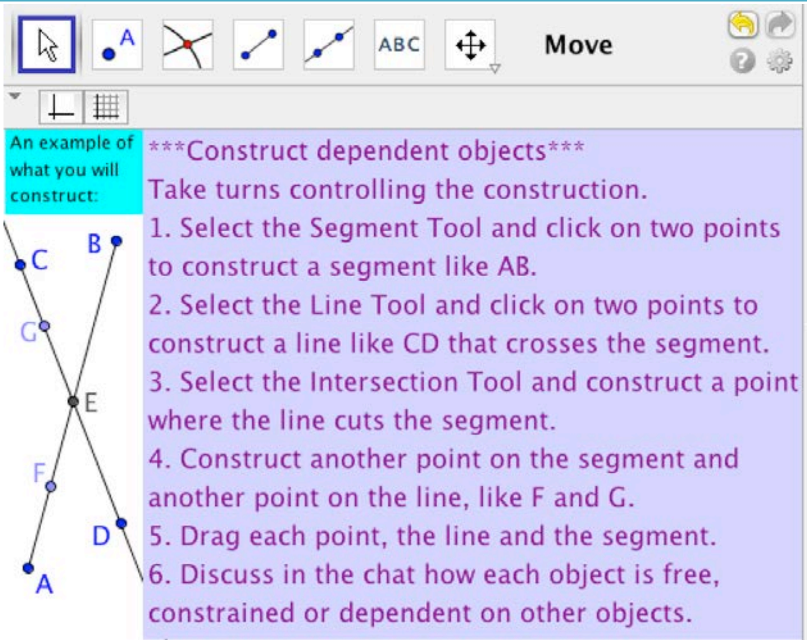


Note: You can change the "properties" of a dynamic-geometry object by first Taking Control and then control-clicking (on a Mac computer: hold down the "control" key and click) or right-clicking (on a Windows computer) on the object. You will get a pop-up menu. You can turn the Trace (locus) on/off, show/hide the object (but its constraints still remain), show/hide its label information, change its name or alter its other properties (like color and line style). Try these different options.

Section 2.02 Dynamic Dragging

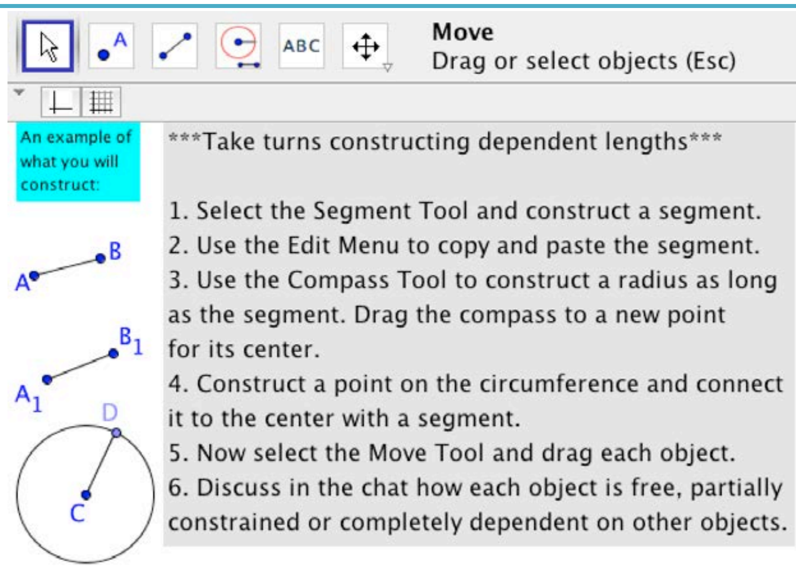
When you construct a point to be on a line (or segment, or ray, or circle) in dynamic geometry, it is constrained to stay on that line; its location is dependent upon the location of that line, which can be dragged to a new location. Use the "drag test" to check if a point really is constrained to the line: select the Move tool (the first tool in the Tool Bar, with the arrow icon), click on your point and try to drag it; see if it stays on the line.

Take control and construct some lines and segments with some points on them, like in the example shown in the tab. Notice how some points can be dragged freely, some can only be dragged in certain ways (we say they are partially "constrained") and others cannot be dragged directly at all (we say they are fully "dependent").



Section 2.03 Constructing Segment Lengths

Constructing one segment to be the same length as another segment is different from just copying the segment. The compass tool can be used to construct a segment whose length is dependent on the length of another segment. To use the compass tool, first define its radius and then locate a center for it. Drag point A to change the length of segment AB. Does the copy made with copy-and-paste change its length automatically? Does the radius CD change its length automatically when AB is changed? Why do you think this happens?



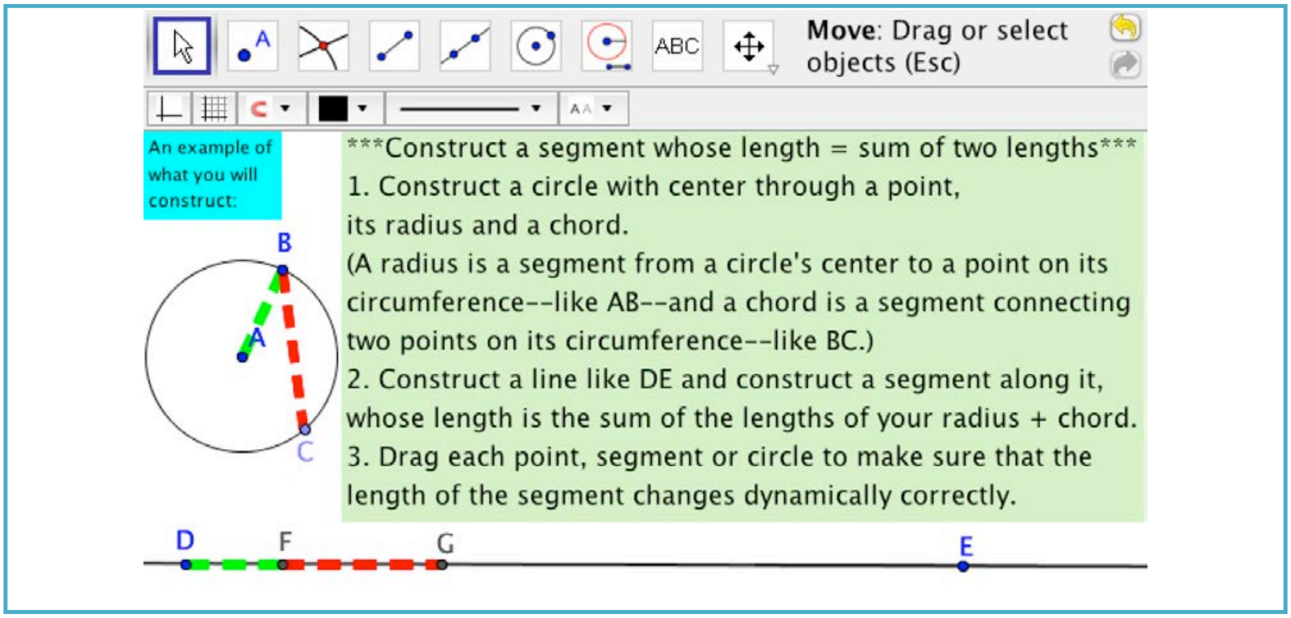
The compass tool can be tricky to use. Here is a brief YouTube video that shows how to use it:

http://www.youtube.com/watch?v=AdBNfEOEVco

Section 2.04 Adding Segment Lengths

In dynamic geometry, you can construct figures that have complicated dependencies of some objects on other objects. Here you will construct a segment whose length is dependent on the length of two other segments. Copying a length like this so that the length of the copy is always the same as the length of the original (even when the original is dragged to a different length) is one of the most important operations in dynamic geometry. Make sure that everyone in your team understands how to do this.

Use the compass tool to copy the lengths of the line segments. Using the compass tool requires practice. Creating line segment length DG = AB + BC provides a good visual image when you drag point B.



Discuss with your team how to do each step, especially step 2. Do you see how you can use the compass tool to lay out segments with given lengths (like AB and BC) along a given line (like DE)? Explain to your teammates the difference between the circle tool and the compass tool and let them add their ideas.

Enter in your (paper or digital) copy of this document a summary of what you and your team noticed and wondered during your work on this Topic 02. This will be a valuable record of your work. You may want to come back and think more about these entries later.

What we noticed:

What we wondered:

Topic 3: Visualizing the World's Oldest Theorems

Scientific thinking in the Western world began with the ancient Greeks and their proofs of theorems in geometry. Thales lived about 2,600 years ago (c. 624–546 BCE). He is often considered the first philosopher (pre-Socratic), scientist (predicted an eclipse) and mathematician (the first person we know of to prove a mathematical theorem deductively). Pythagoras came 30 years later and Euclid (who collected many theorems of geometry and published them in his geometry book called *Elements*) came 300 years later. Thales took the practical, arithmetical knowledge of early civilizations—like Egypt and Babylonia—and introduced a new level of theoretical inquiry into it. With dynamic-geometry software, you can take the classic Greek ideas to yet another level.

> Before working on this topic with your team, it would be good to read **Tour 3: "VMT to Learn Together Online"** near the end of this booklet. It will show you how to use the Referencing Tool and other communication techniques.

Section 3.01 Visualize the Theorem of Thales

Thales took a "conjecture" (a mathematical guess or suspicion) about an angle inscribed in a semi-circle and he proved why it was true. You can use dynamic geometry to *see* that it is true for all angles all along the semi-circle. Then you can *prove* that it is always true.

Construction Process

Take control and follow these steps to construct an angle inscribed in a semi-circle like the one already in the tab. The letter labels on your team's new figure will be different than the letter labels listed below. You will be able to move the angle dynamically and see how things change.

Step 1. Construct a ray like AB.

Step 2. Construct a circle with center at point B and going through point A.

Step 3. Construct a point like point C at the intersection of the line and the circle, forming the diameter of the circle, AC.

Step 5. Construct a point like D anywhere on the circumference of the circle.

Step 6. Create triangle ADC with the polygon tool.

Step 7. Create the interior angles of triangle ADC. (Always click on the three points forming the angle in clockwise order—otherwise you will get the measure of the outside angle.) In geometry, we still use the Greek alphabet to label angles: α, β, γ are the first three letters (like a, b, c), called "alpha," "beta," and "gamma."

Step 8. Drag point D along the circle. What do you notice? Are you surprised? Why do you think the angle at point D always has that measure?

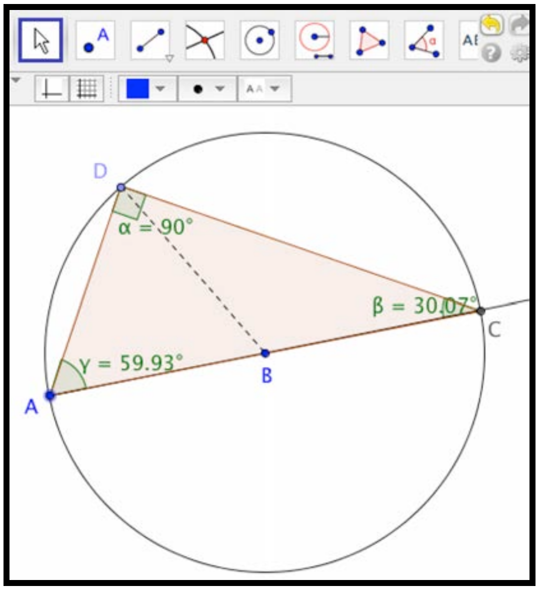
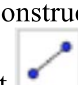


Figure 3-1. The Theorem of Thales.

Challenge

Try to come up with a proof for this theorem.

Hint: To solve a problem or construct a proof in geometry, it is often helpful to construct certain extra lines, which bring out interesting relationships. Construct the radius BD as a segment .

Thales had already proven two theorems previously:

(1) The base angles of an isosceles triangle are equal. (An "isosceles" triangle is defined as having at least two equal sides.)

(2) The sum of the angles $\alpha+\beta+\gamma=180°$ in any triangle.

Can you see why $\alpha=\beta+\gamma$ in the figure, no matter how you drag point D? (Remember that all radii of a circle are equal by definition of a circle.) That means that $(\beta+\gamma)+\beta+\gamma=180°$. So, what does α have to be?

Section 3.02 Visualization #1 of Pythagoras' Theorem

Pythagoras' Theorem is probably the most famous and useful theorem in geometry. It says that the length of the hypotenuse of a right triangle (side **c**, opposite the right angle) has the following relationship to the lengths of the other two sides, **a** and **b**:

$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2$$

The figures in the rest of the tabs of this topic show ways to visualize this relationship. They involve transforming squares built on the three sides of the triangle to show that the sum of the areas of the two smaller squares is equal to the area of the larger square. The area of a square is equal to the length of its side squared, so a square whose side is **c** has an area equal to $\mathbf{c}^2$.

Explain what you see in these two visualizations. Can you see how the area of the $\mathbf{c}^2$ square is rearranged into the areas $\mathbf{a}^2$ and $\mathbf{b}^2$ or vice versa?

Notice that these are geometric proofs. They do not use numbers for the lengths of sides or areas of triangles. This way they are valid for any size triangles. In the GeoGebra tab, you can change the size and orientation of triangle ABC and all the relationships remain valid. Geometers always made their proofs valid for any sizes, but with dynamic geometry, you can actually change the sizes and see how the proof is still valid (as long as the construction is made with the necessary dependencies).

It is sometimes helpful to see the measures of sides, angles and areas to help you make a conjecture about relationships in a geometric figure. However, these numbers never really prove anything in geometry. To prove something, you have to explain why the relationships exist. In dynamic geometry, this has to do with how a figure was constructed—how specific dependencies were built into the figure. In this figure, for instance, it is important that the four triangles all remain right triangles and that they have their corresponding sides the same lengths (**a**, **b**, and **c**). If these lengths change in one triangle, they must change exactly the same way in the others. Can you tell what the side length of the square in the center has to be?

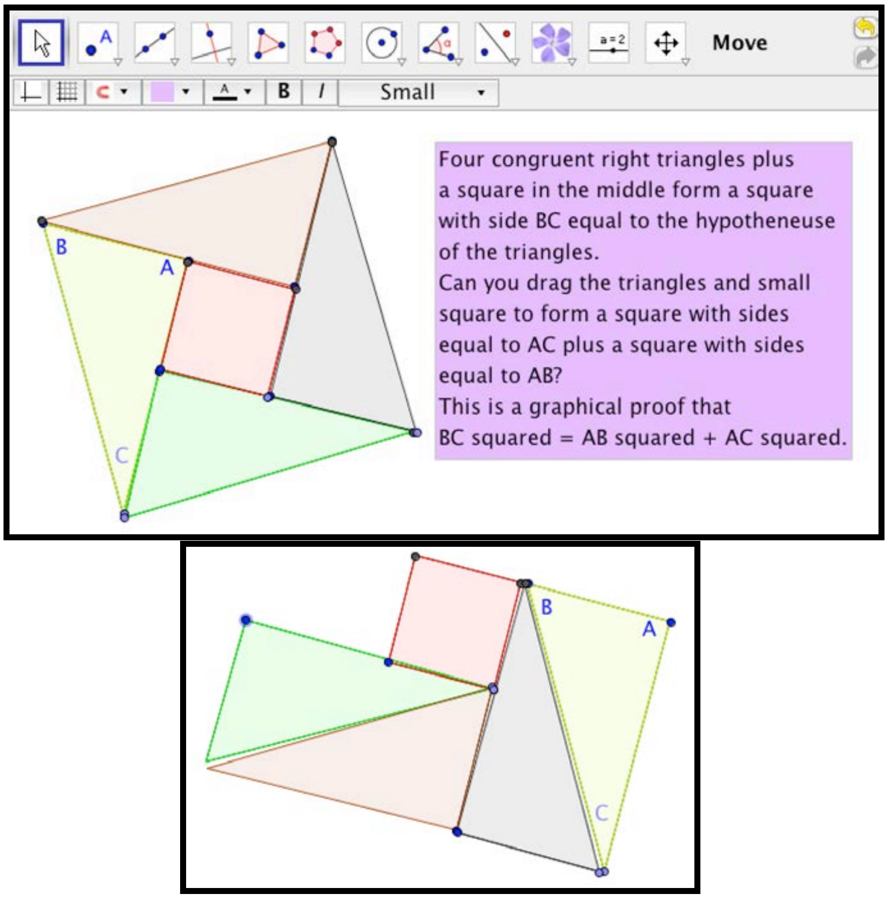


Figure 3-2. Visualization #1 of Pythagoras' Theorem.

Section 3.03 Visualization #2 of Pythagoras’ Theorem

The next figure automates the same proof of Pythagoras’ Theorem with GeoGebra sliders. Try it out. Do not forget, you have to “Take Control” before you can move the sliders. Move the sliders for α and s to see what they change.

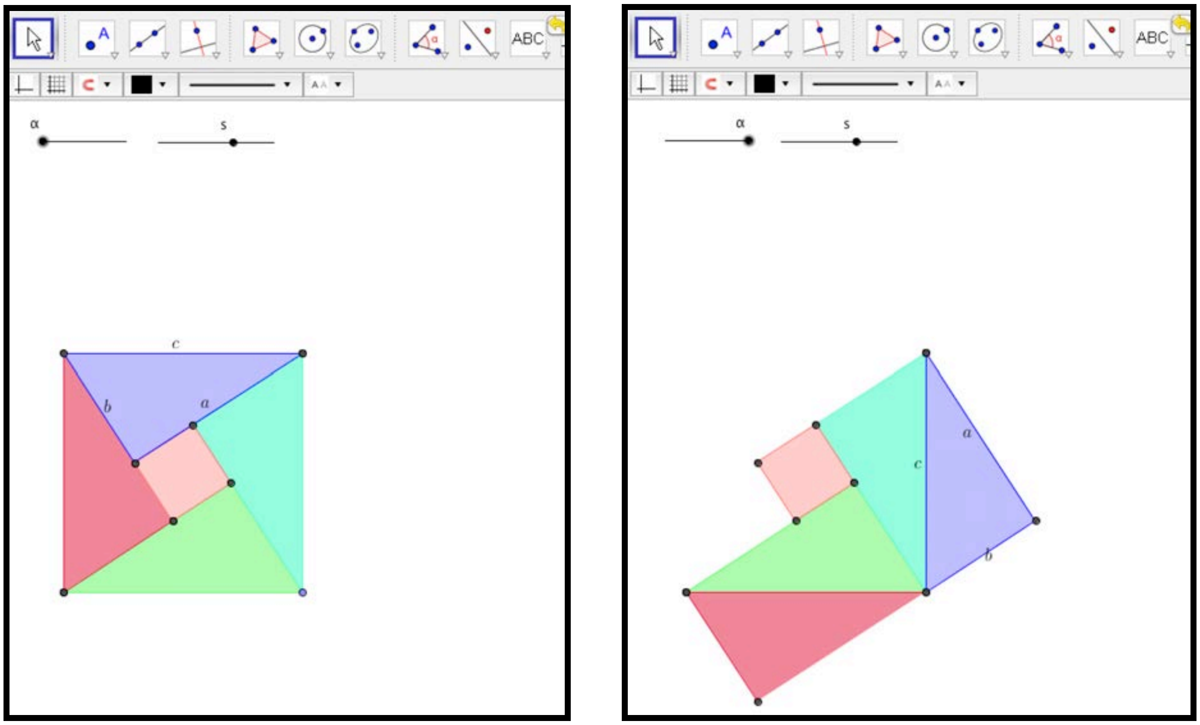


Figure 3-3. Visualization #2 of Pythagoras' Theorem.

Section 3.04 Visualization #3 of Pythagoras' Theorem

The next figure shows another way to visualize the proof of Pythagoras' Theorem. Slide the slider. Is it convincing?

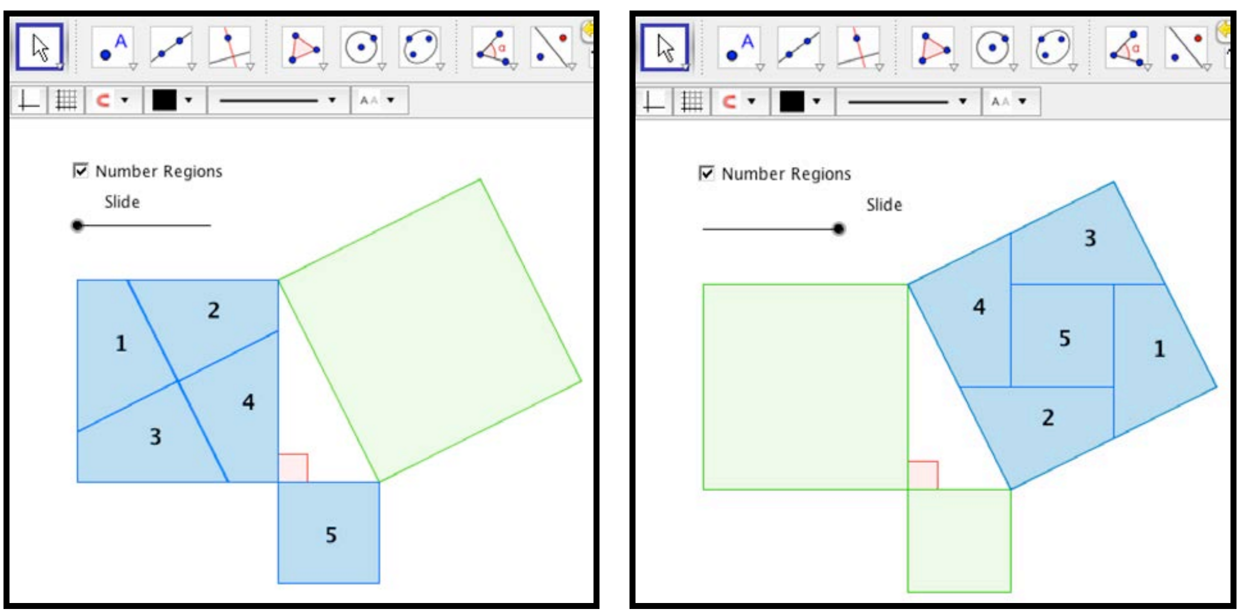


Figure 3-4. Visualization #3 of Pythagoras' Theorem.

Section 3.05 Visualization #4 of Pythagoras' Theorem

The next figure shows an interesting extension of the proof of Pythagoras' Theorem:

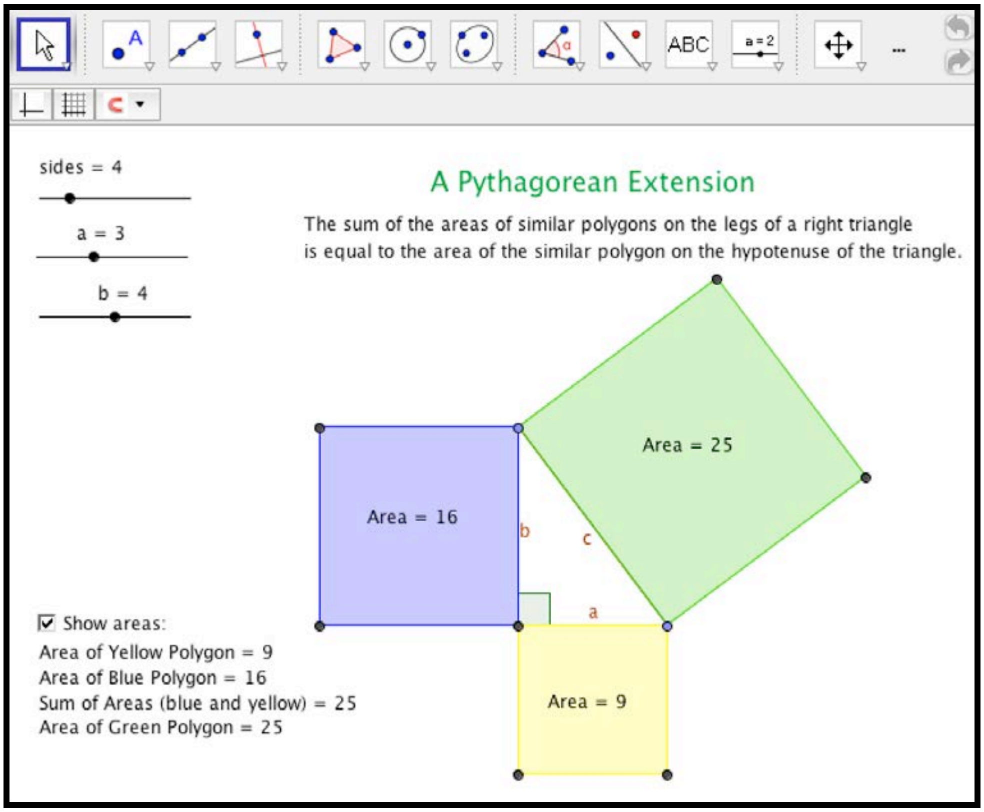


Figure 3-5. Visualization #4 of Pythagoras' Theorem.

Can you explain to each other why it works for all regular polygons if it works for triangles?

Section 3.06 Visualization #5 of Pythagoras' Theorem

Finally, here is Euclid's own proof of Pythagoras' Theorem in his 47th proposition. It depends on some relationships of quadrilaterals, which you may understand better after completing the topics in this workbook. Drag the sliders in this GeoGebra figure slowly and watch how the areas are transformed.

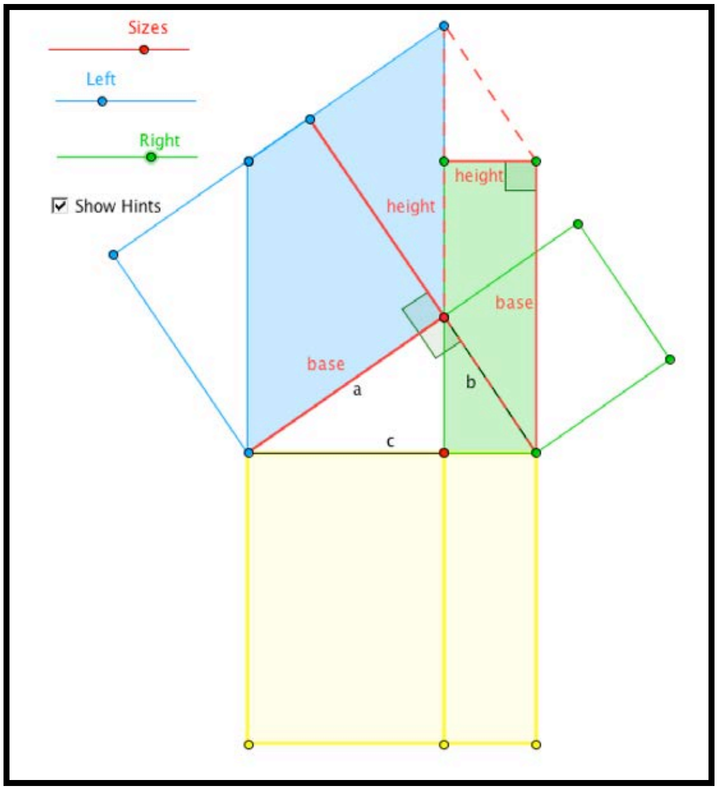


Figure 3-6. Visualization #5 of Pythagoras' Theorem.

There are many other visual, geometric and algebraic proofs of this famous theorem. Which do you find most elegant of the ones you have explored here?

What we noticed:

What we wondered:

Topic 4: Constructing Triangles

A triangle is a relatively simple geometric construction: simply join three segments at their endpoints. Yet, there are many surprising and complex relationships inside triangles.

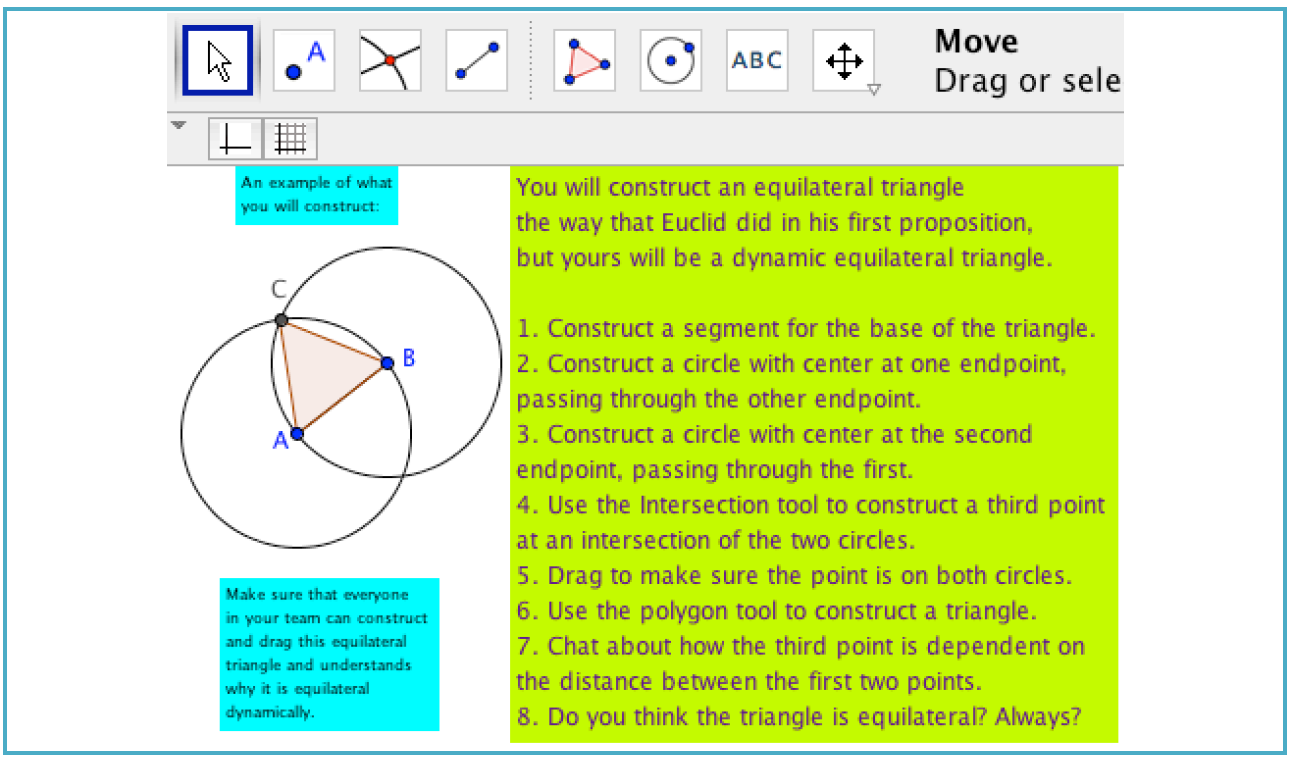
> Before working on this topic with your team, it could be very helpful to watch two brief **YouTube clips** that show clearly how to copy a segment and to construct an equilateral triangle:
>
> http://www.youtube.com/watch?v=AdBNfEOEVco
>
> http://www.youtube.com/watch?v=ORIaWNQSM_E

Section 4.01 Constructing an Equilateral Triangle

When Euclid organized ideas and techniques of geometry 2,300 years ago, he started with this construction of an equilateral triangle, whose three sides are constrained to always be the same lengths as each other. This construction can be considered the starting point of Euclidean and dynamic geometry.

Have everyone in your team work on the first tab for this topic. It shows—in one simple but beautiful example—the most important features of dynamic geometry. Using just a few points, segments and circles (strategically related), it constructs a triangle whose sides are always equal no matter how the points, segments or circles are dragged. Using just the basic definitions of geometry—like the points of a circle are all the same distance from the center—it proves that the triangle must be equilateral (without even measuring the sides).

Your team should construct an equilateral triangle like the one already in the tab. Drag the one that is there first to see how it works. Take turns controlling the GeoGebra tools.

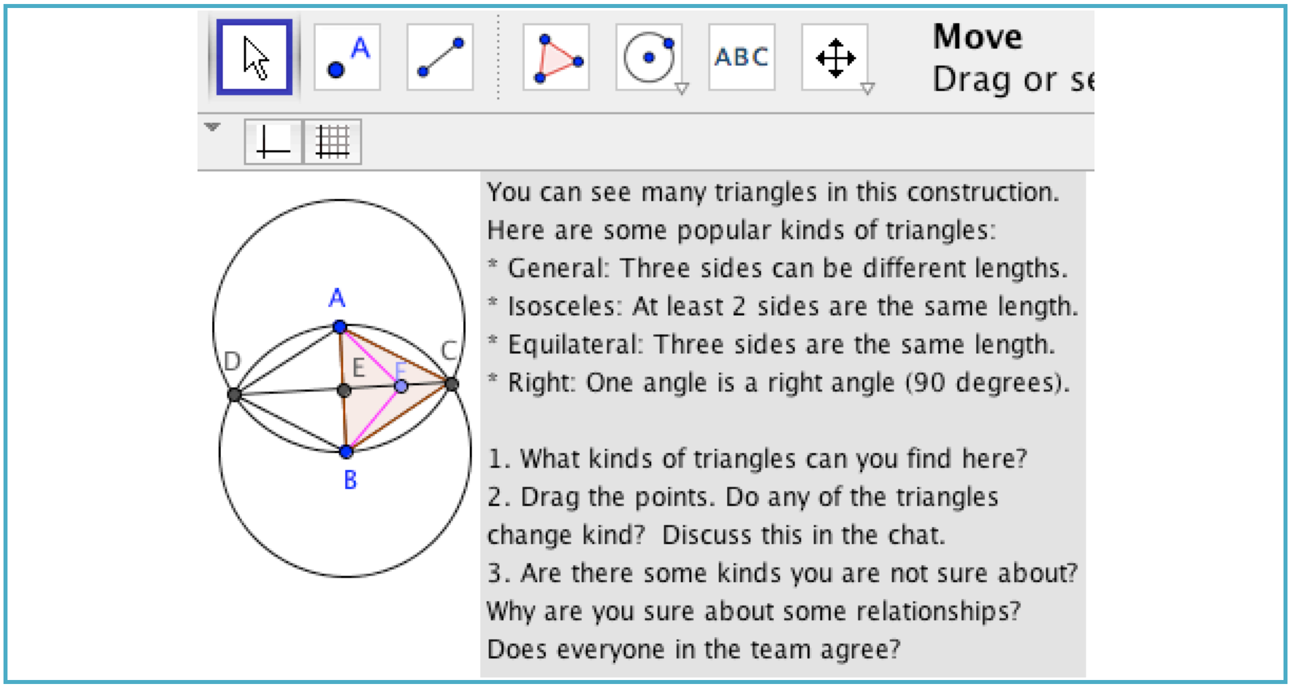


Euclid argued that both of the circles around centers A and B have the same radius, namely AB. The three sides of triangle ABC are all radii of these two circles. Therefore, they all have the same length. Do you agree with this argument (proof)? Are you convinced that the three sides of ABC have equal lengths – *without having to measure them*? If you drag A, B, or C and change the lengths of the sides are they always still equal?

Section 4.02 Where's Waldo?

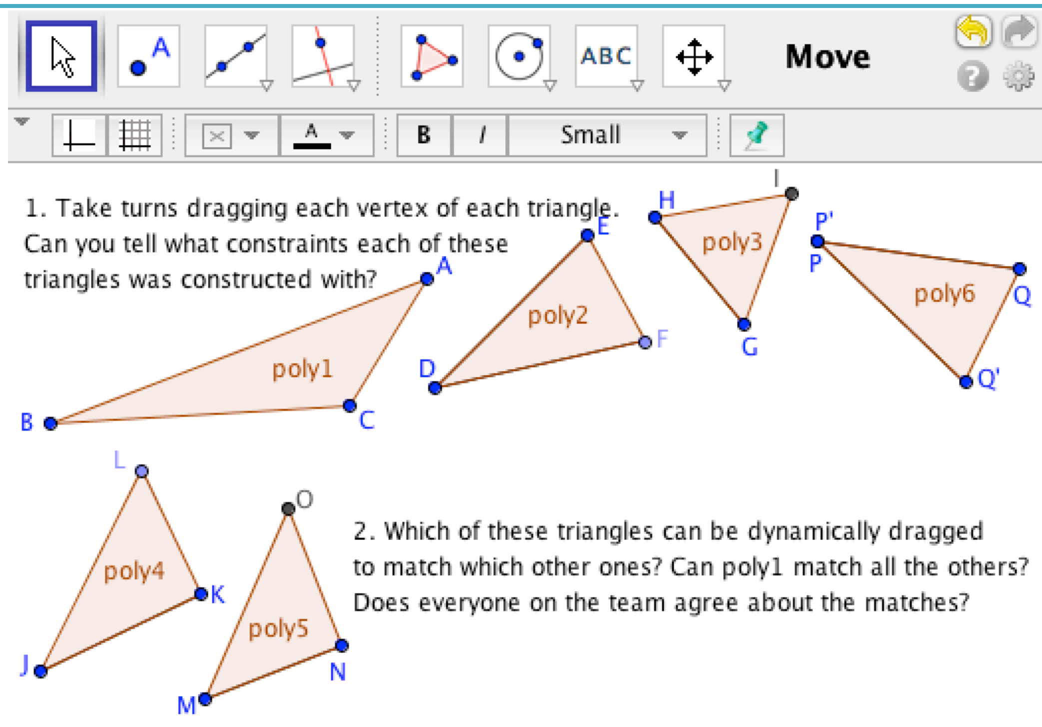
Let us look more closely at the relationships that are created in the construction of the equilateral triangle. In this tab, more lines are drawn in. Explore some of the relationships that are created among line segments in this more complicated figure. What line segments do you think are equal length – without having to measure them? What angles do you think are equal without having to measure them? Try dragging different points; do these equalities and relationships stay dynamically? Can you see how the construction of the figure made these segments or angles equal?

Can you find different kinds of triangles in this construction? If a triangle always has a certain number of sides or angles equal, then it is a special kind of triangle. We know the construction of this figure defined an equilateral triangle, ABC. What other kinds of triangles did it define?



Section 4.03 Exploring Different Triangles

Here are some triangles constructed with different dynamic constraints. See if your team can figure out which ones were constructed to always have a certain number of equal sides, a certain number of equal angles or a right angle. Which triangles can be dragged to appear the same as which other triangles?



What we noticed:

What we wondered:

Topic 5: Programming Custom Tools

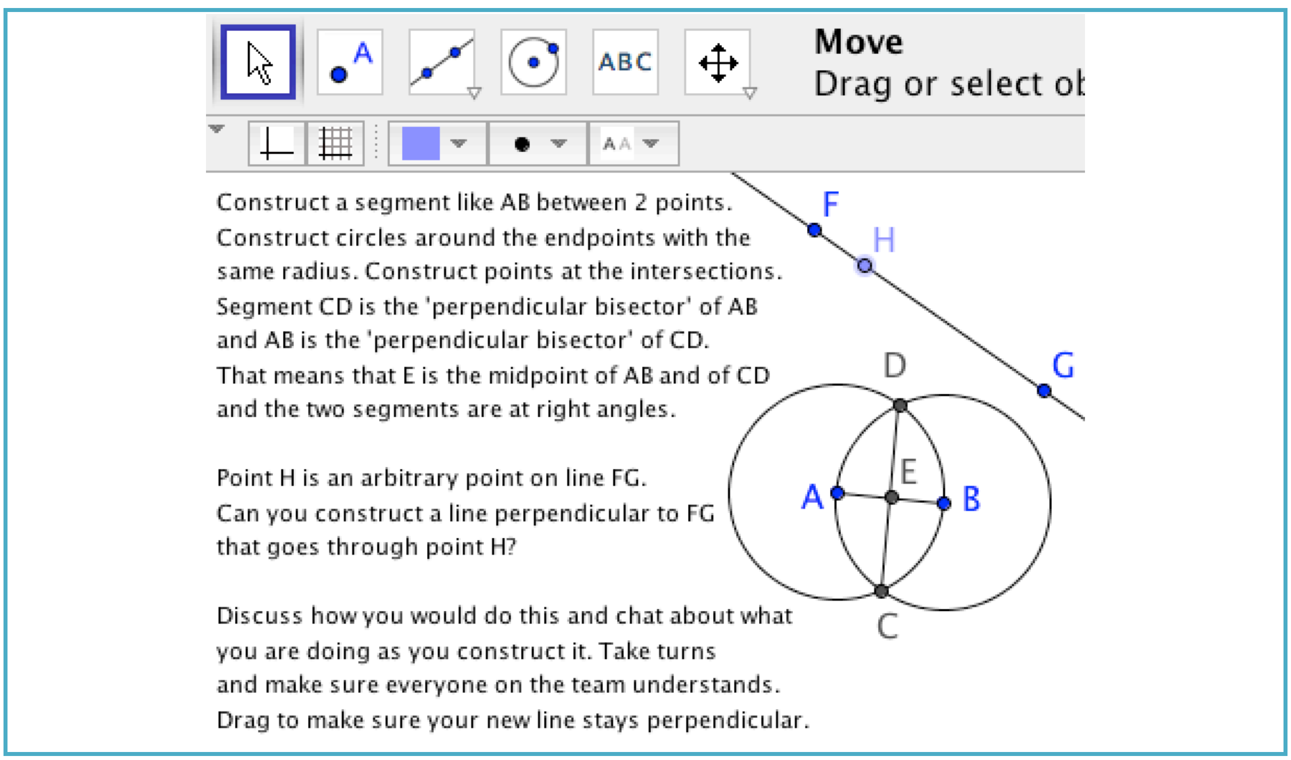
For constructing geometric figures and for solving typical problems in geometry, it is useful to have tools that do things for you, like construct midpoints of segments, perpendiculars to lines, and parallel sets of lines. GeoGebra offers about 100 tools that you can use from the tool bar, input bar or menu. However, you can also create your own custom tools to do additional things—like copy an angle, construct an isosceles triangle or locate a center of a triangle. Then you can build your own mini-geometry using a set of your own custom tools—like defining a "nine-point circle" using custom tools for several centers of a triangle and for an "Euler segment" connecting them (see Section 6.10). Programming your own tools can be fun once you get the hang of it. Furthermore, it gives you a good idea about how GeoGebra's standard tools were created and why they work.

Section 5.01 Constructing a Perpendicular Bisector

The procedure used to construct an equilateral triangle can be used to locate the midpoint of a segment and to construct a perpendicular to that segment, passing through the midpoint.

As you already saw, the construction process for an equilateral triangle creates a number of interesting relationships among different points and segments. In this tab, points A, B and C form an equilateral triangle. Segment AB crosses segment CD at the exact midpoint of CD and the angles between theses two segments are all right angles (90 Degrees). We say that AB is the "perpendicular bisector" of CD—meaning that AB cuts CD at its midpoint, evenly in two sectors, and that AB is perpendicular (meaning, at a right angle) to CD.

For many geometry constructions, it is necessary to construct a new line perpendicular to an existing line (like line FG). In particular, you may need to have the perpendicular go through the line at a certain point (like H). Can your team figure out how to do that?



Section 5.02 Creating a Perpendicular Tool

Now that you know how to construct a perpendicular, you can automate this process to save you work next time you need a perpendicular line. Create a custom tool to automatically construct a perpendicular to a given line through a given point by following the directions in the tab.

Members of the team should each create their own custom perpendicular tool. One person could create a tool to construct a perpendicular bisector through the midpoint of a given line (no third point would be needed as an input for this one). Another person could create a perpendicular through a given point on the line. A third person could create a perpendicular through a given point that is not on the line. Everyone should be able to use everyone else's custom tool in this tab. Do the three tools have to be different? Does GeoGebra have three different tools for this? Do your custom tools work just like the GeoGebra standard perpendicular tools? Are there other cases for constructing perpendiculars?

GeoGebra makes programming a tool easy. However, it takes some practice to get used to the procedure. To program a tool, you have to define the Outputs you want (the points, lines, etc. that will be created by the custom tool) and then the Inputs that will be needed (the points, lines, etc. that a person will have to create to use the tool). For instance, to create a perpendicular to line AB through point C on it using this custom tool, a person would first select the custom tool as the active tool. Next, they would construct or select three points to define A, B and C (the line and a point on it as inputs to the custom tool). Then the perpendicular line would appear automatically (as the output of the custom tool).

Hint: You can identify the output objects by selecting them with your cursor before or after you go to the menu "Tools" | "Create New Tool …". Hold down the Command key (on a Mac) or the Control key (in Windows) to select more than one object. You can also identify the output objects from the pull-down list in the Output Objects tab. Similarly, you can identify input objects in the Input Objects tab by selecting them with the cursor or from the pull-down list. GeoGebra might identify most of the necessary input objects automatically. Give the tool a name that will help you to find it later and check the "Show in Toolbar" box so your tool will be included on the toolbar.

To be able to use your custom tool in another tab or another chat room later, you have to save it now. Save your custom tool as a **.ggt** file on your desktop. Take control and use the GeoGebra menu "Tools" | "Manage Tools…" | "Save As." Save your custom tool to your computer desktop, to a memory stick, or somewhere that you can find it later and give it a name like "Maria's_Perpendicular.ggt" so you will know what it is. When you want to use your custom tool in another tab or topic, take control, use the GeoGebra menu "File" | "Open…" then find and open the .ggt file that you previously saved. You should then be able to select your custom tool from the toolbar or from the menu "Tools" | "Manage Tools." When your custom tool is available to you, it will also be available to your teammates when they are in that tab.

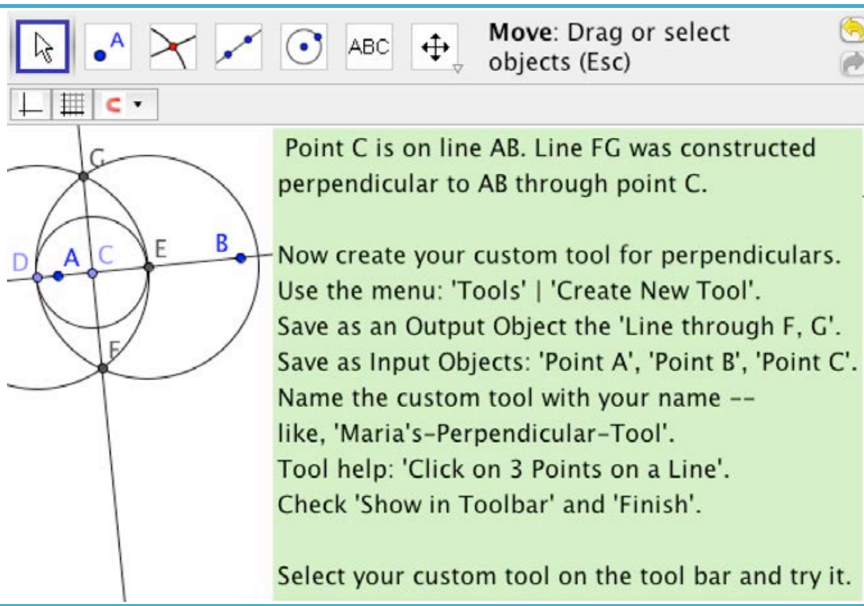
Hint: If a custom tool does not appear on your tool bar when you think it should be available, use the menu "Tools" | "Customize Toolbar", find the custom tool in the list of Tools, and insert it on the toolbar list where you want it (highlight the group or the tool you want it to be listed after).

Note: The custom tools in GeoGebra have some limitations, unfortunately. As noted in the help page, Outputs of custom tools are not moveable, even if they are defined as `Point[<Path>]`; if you need moveable output, you can define a list of commands and use it with Execute Command. For instance, if you define a right-triangle tool, you will not be able to freely drag the new vertex of the right triangles that are created with this tool; they will not be dynamic.

Hint: The GeoGebra help page for custom tools at http://wiki.geogebra.org/en/Custom_Tools has more information about defining and using custom tools. Most of that information applies when using VMT-with-GeoGebra as well.

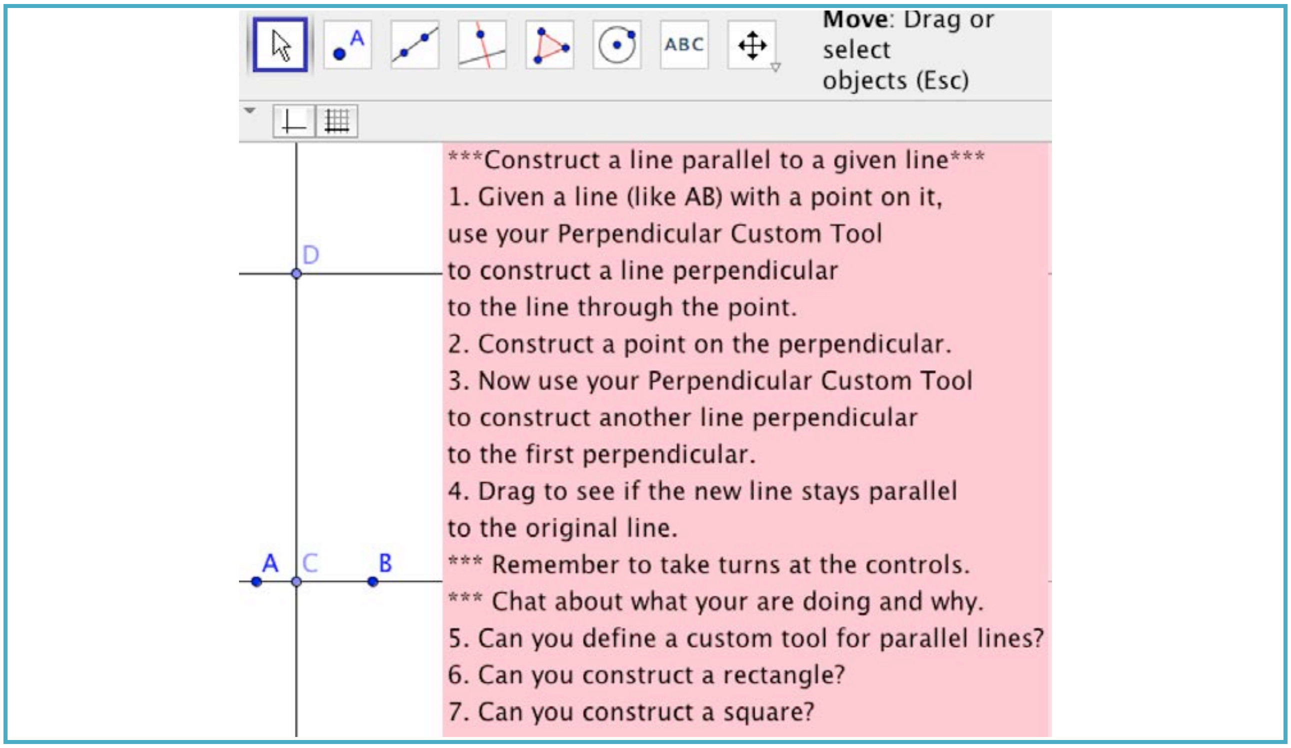
GeoGebra has a perpendicular tool that works like your custom tool. If you just used the standard tool, you would not be aware of the hidden circles that determine the dependencies to keep the lines perpendicular during dragging. Now that you understand these dependencies, you can use either the standard tool or your custom tool. You will not see the hidden circles maintaining the dependencies of lines that are dynamically perpendicular, but you will know they are there, working in the background.

The tools of GeoGebra extend the power of dynamic geometry while maintaining the underlying dependencies. By defining your own custom tools, you learn how dynamic geometry works "under the hood." In addition, you can extend its power yourself in new ways that you and your team think of.



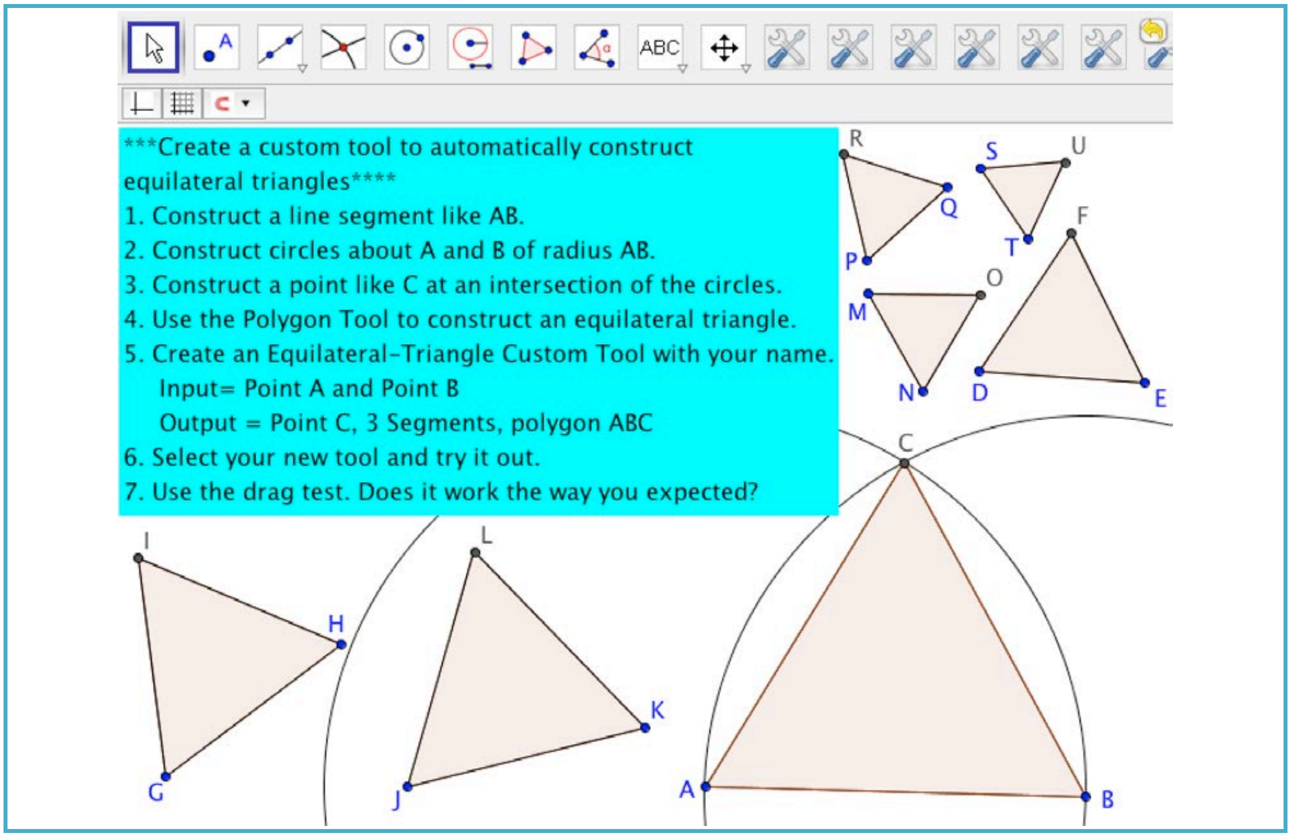
Section 5.03 Creating a Parallel Line Tool

One member of the group should create a custom perpendicular tool (like you did in the previous tab) in this tab. Now use this custom perpendicular tool (or the standard perpendicular tool) to create a custom parallel tool. See how tools can build on each other to create a whole system of new possible activities.



Section 5.04 Creating an Equilateral-Triangle Tool

Create a specialized custom tool for quickly generating equilateral triangles.



What we noticed:

What we wondered:

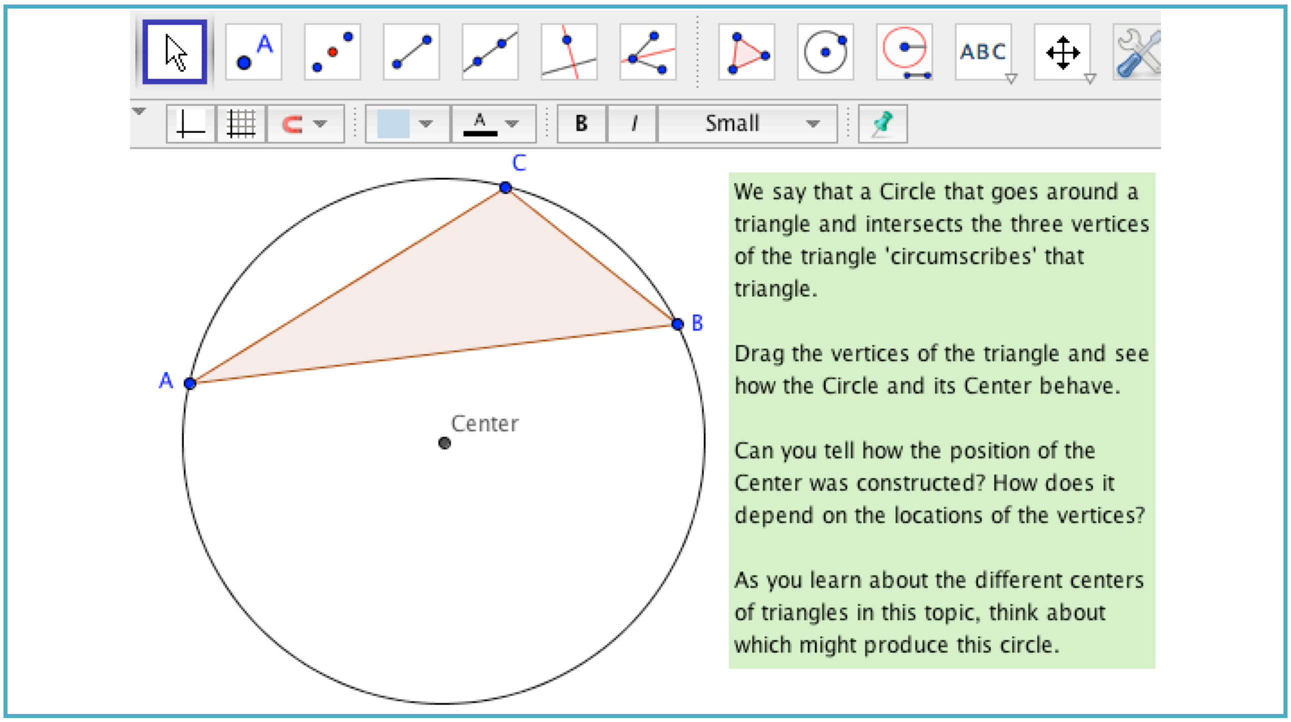
Topic 6: Finding Centers of Triangles

There are a number of interesting ways to define the "center" of a triangle, each with its own interesting properties.

(**Note**: *This is a long topic and may take a couple of sessions to complete. Or you can work on it more on your own after your team session works on the first several tabs.*)

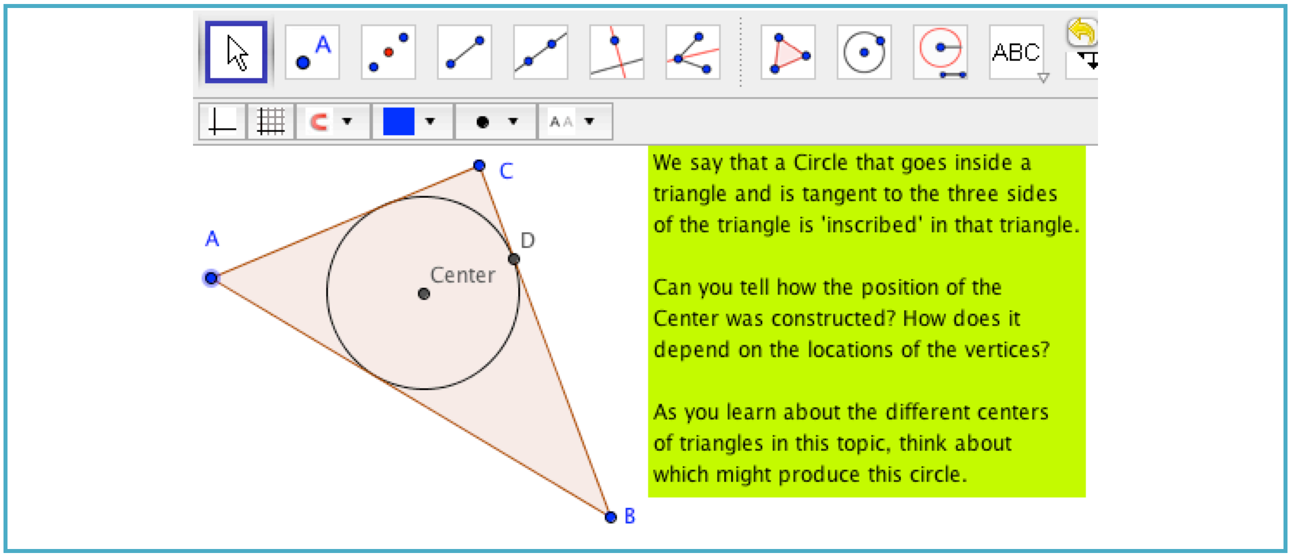
Section 6.01 The Center for Circumscribing a Triangle

One "center" of a triangle is the point that is the center of a circle that circumscribes the triangle. Given a triangle ABC, how would you find and construct the center of a circle that goes through all three vertices of the triangle? When is this center inside of the triangle?



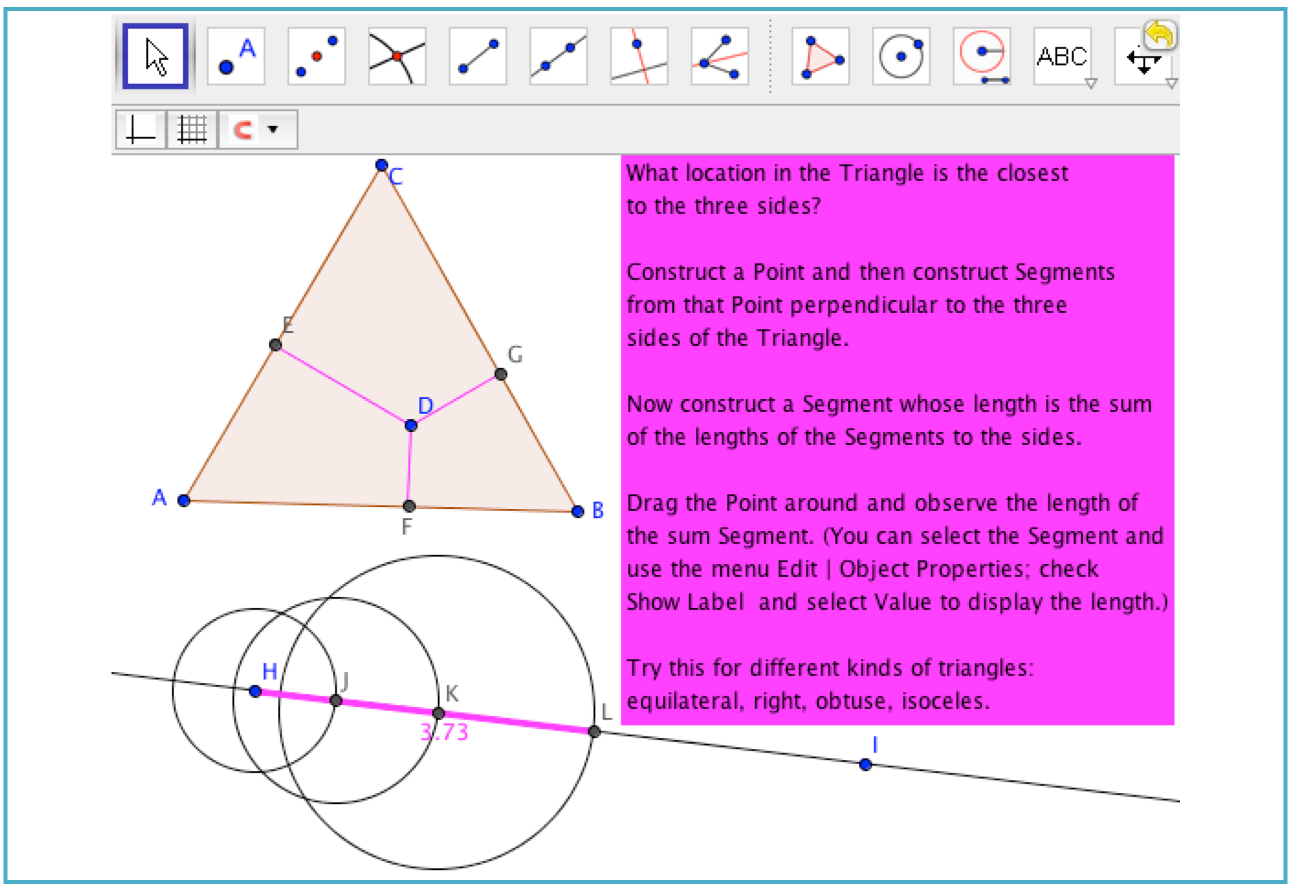
Section 6.02 The Center for Inscribing a Triangle

Another center of a triangle is the point that is the center of a circle that is inscribed in the triangle. Drag the triangle in this tab to try to figure out how this center was constructed. When is this center inside of the triangle?



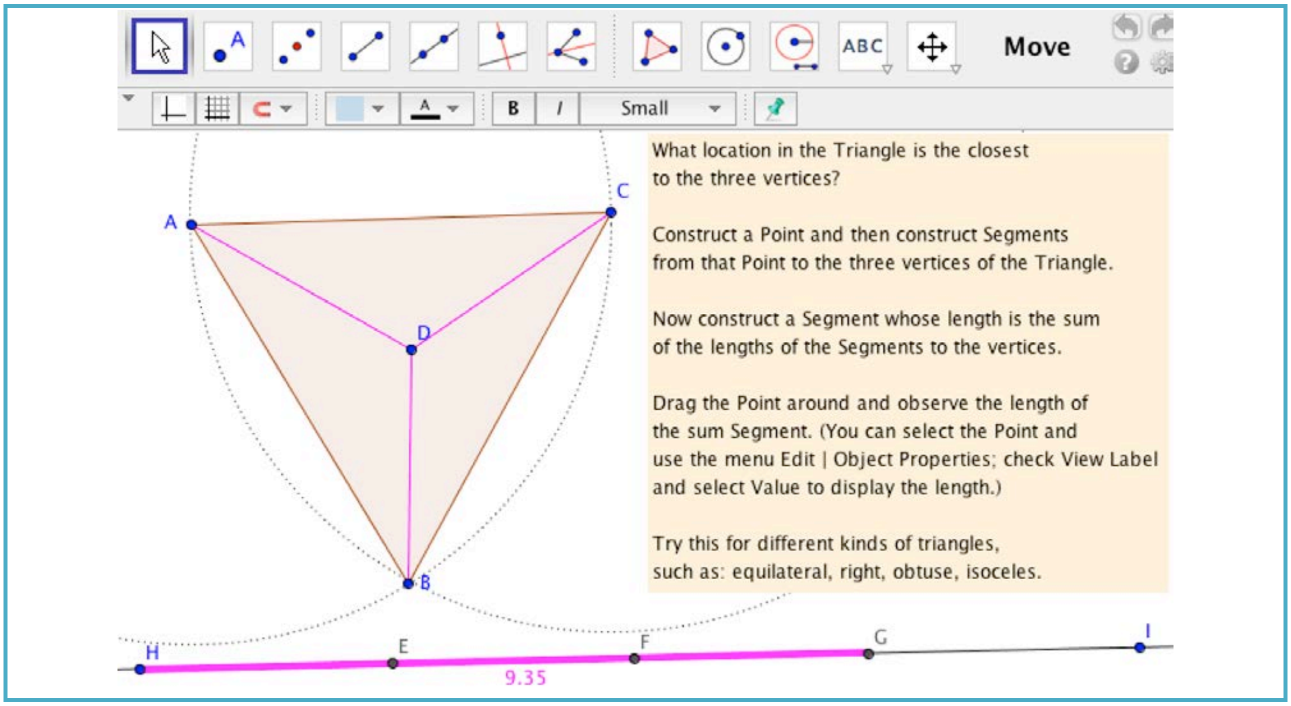
Section 6.03 The Center Closest to The Sides of a Triangle

If you want to be as close as possible to all three sides of a triangle, where should you locate the center such that the sum of the distances to the three sides is as small as possible? Do you understand how line segment HL was constructed to display the sum of the lengths from the center to the sides? Are the segments DE, DF and DG perpendicular to the triangle sides? Why?



Section 6.04 The Center Closest to the Vertices of a Triangle

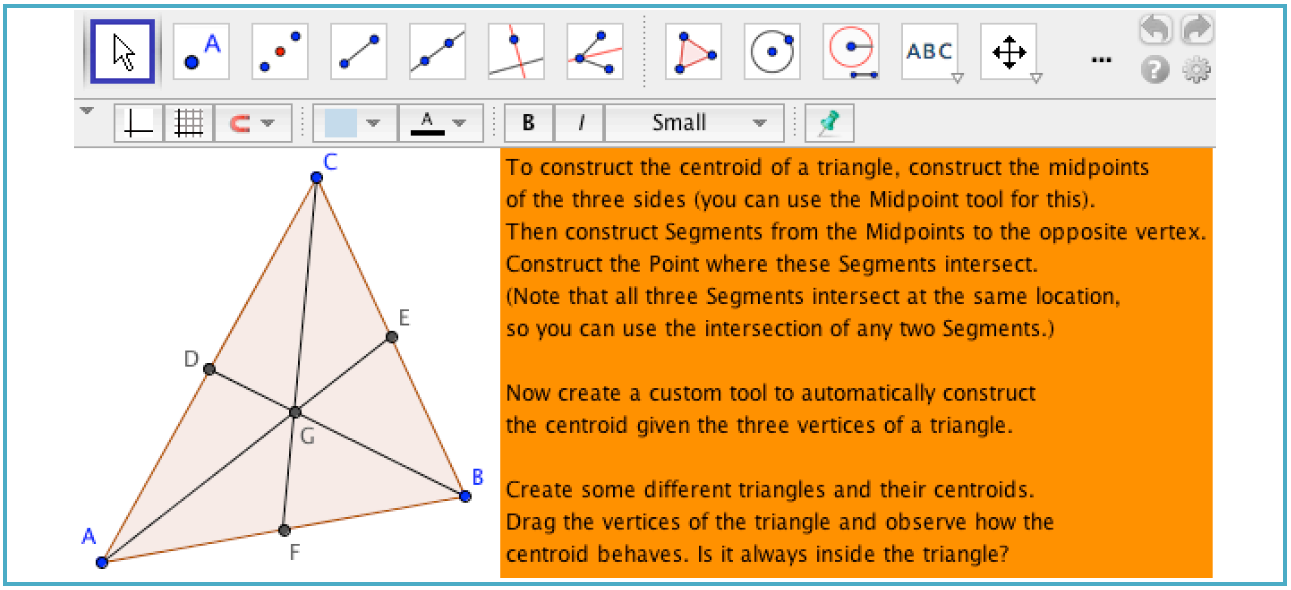
If you want to be as close as possible to all three vertices of a triangle, where should you locate the center such that the sum of the distances to the three vertices is as small as possible? Do you understand how line segment GH was constructed? Chat about this to make sure everyone in your team understands.



Section 6.05 The Centroid of a Triangle

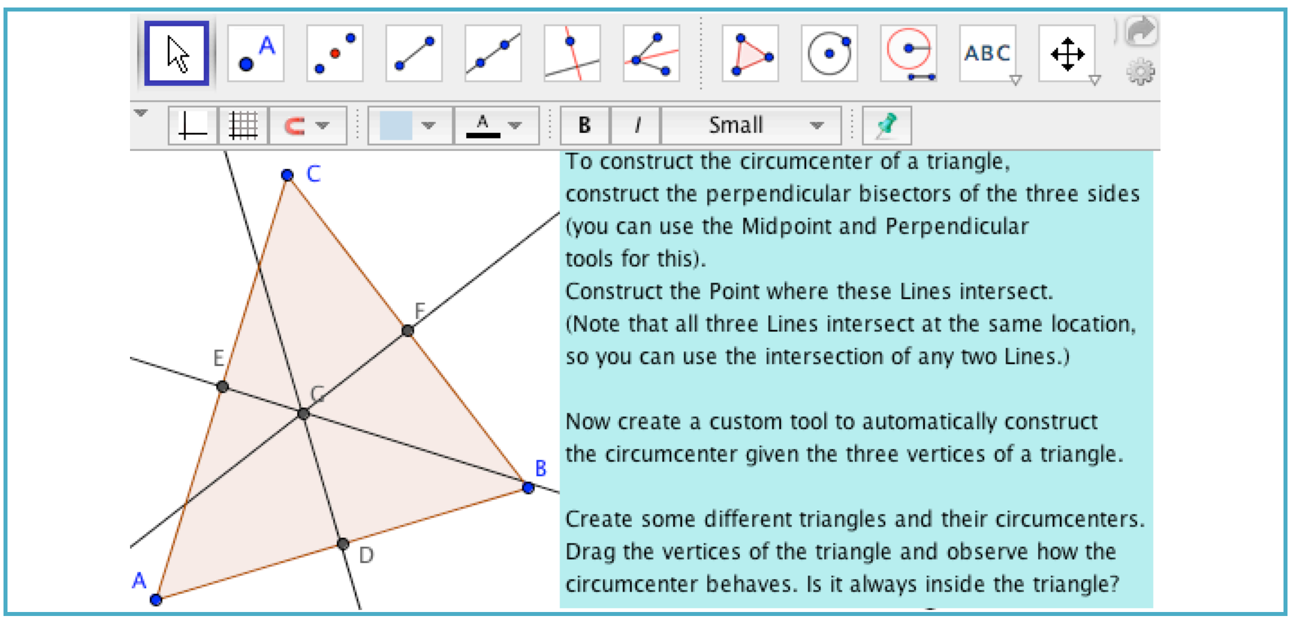
The "centroid" of a triangle is the meeting point of the three lines from the midpoints of the triangle's sides to the opposite vertex. Create a custom centroid tool.

Take control and use the GeoGebra menu "Tools" | "Manage Tools…" | "Save As." Save your custom tool to your computer desktop or somewhere that you can find it later and give it a name like "Tanya's_Centroid.ggt" so you will know what it is. When you want to use your custom tool in another tab or topic, take control, use the GeoGebra menu "File" | "Open…" then find and open the .ggt file that you saved. You should then be able to select your custom tool from the menu "Tools" | "Manage Tools." When your custom tool is available to you, it will also be available to your teammates when they are in that tab.



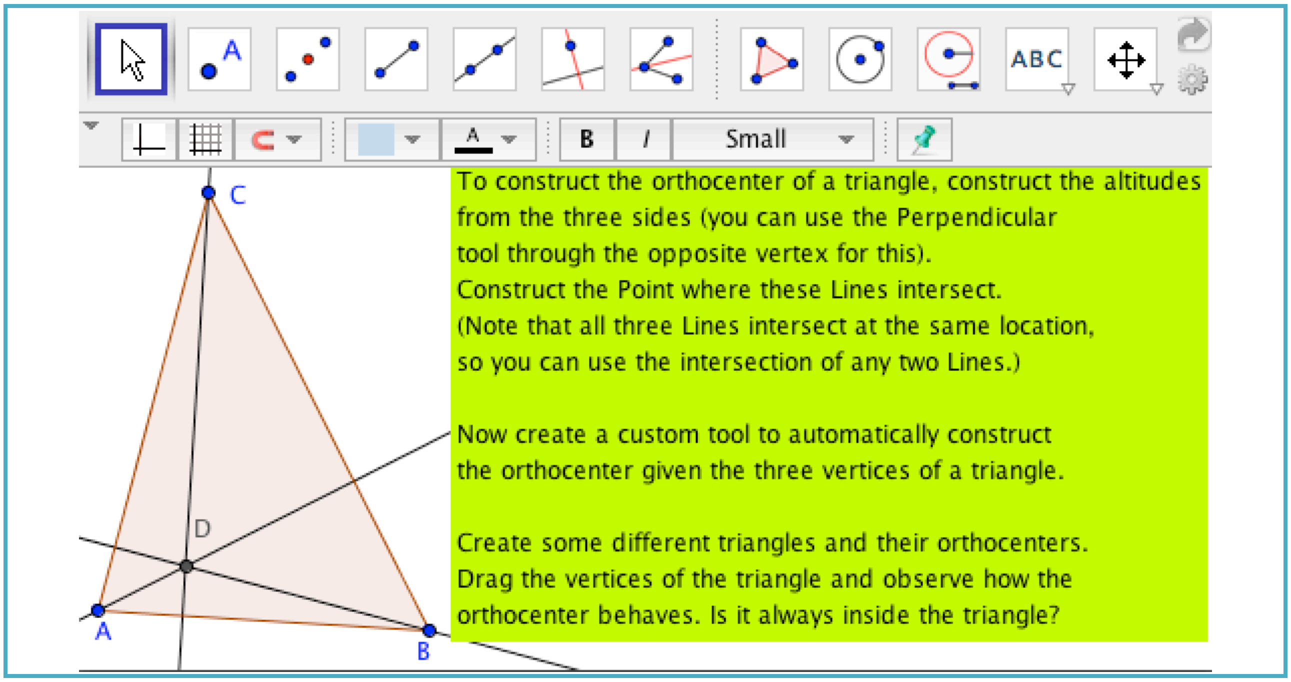
Section 6.06 The Circumcenter of a Triangle

The "circumcenter" of a triangle is the meeting point of the three perpendicular bisectors of the sides of the triangle. Create a custom circumcenter tool and save it.



Section 6.07 The Orthocenter of a Triangle

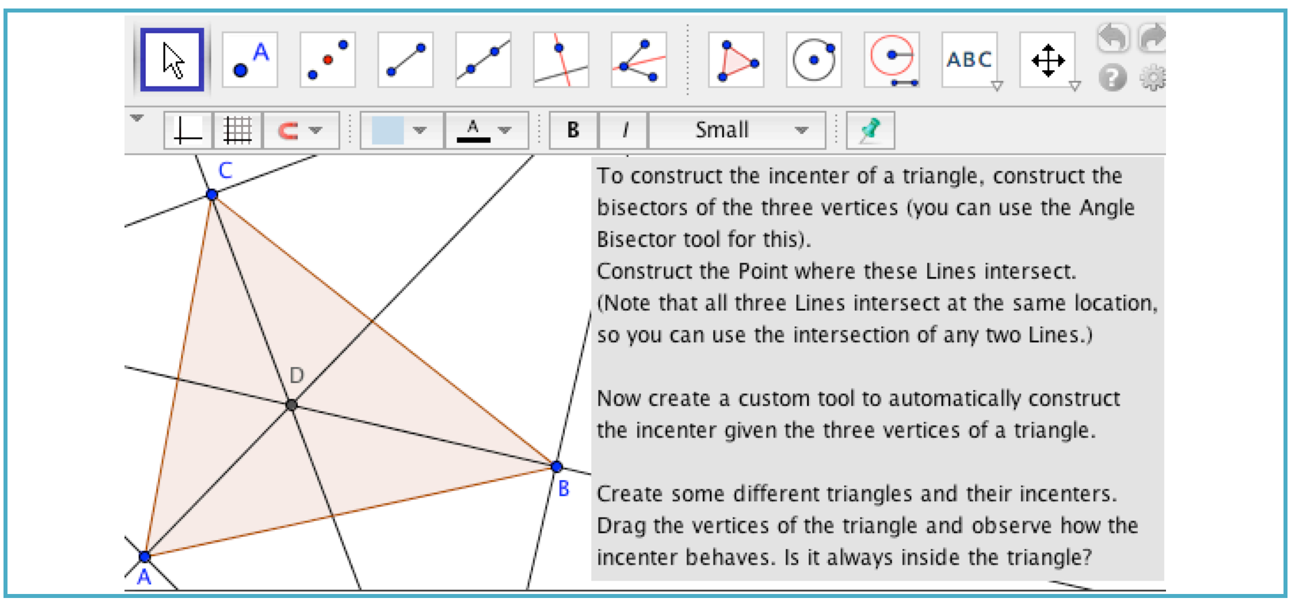
The "orthocenter" of a triangle is the meeting point of the three altitudes of the triangle. An "altitude" of a triangle is the segment that is perpendicular to a side and goes to the opposite vertex. Create a custom orthocenter tool and save it.



Section 6.08 The Incenter of a Triangle

The “incenter” of a triangle is the meeting point of the three angle bisectors of the angles at the triangle’s vertices. Create a custom incenter tool and save it.

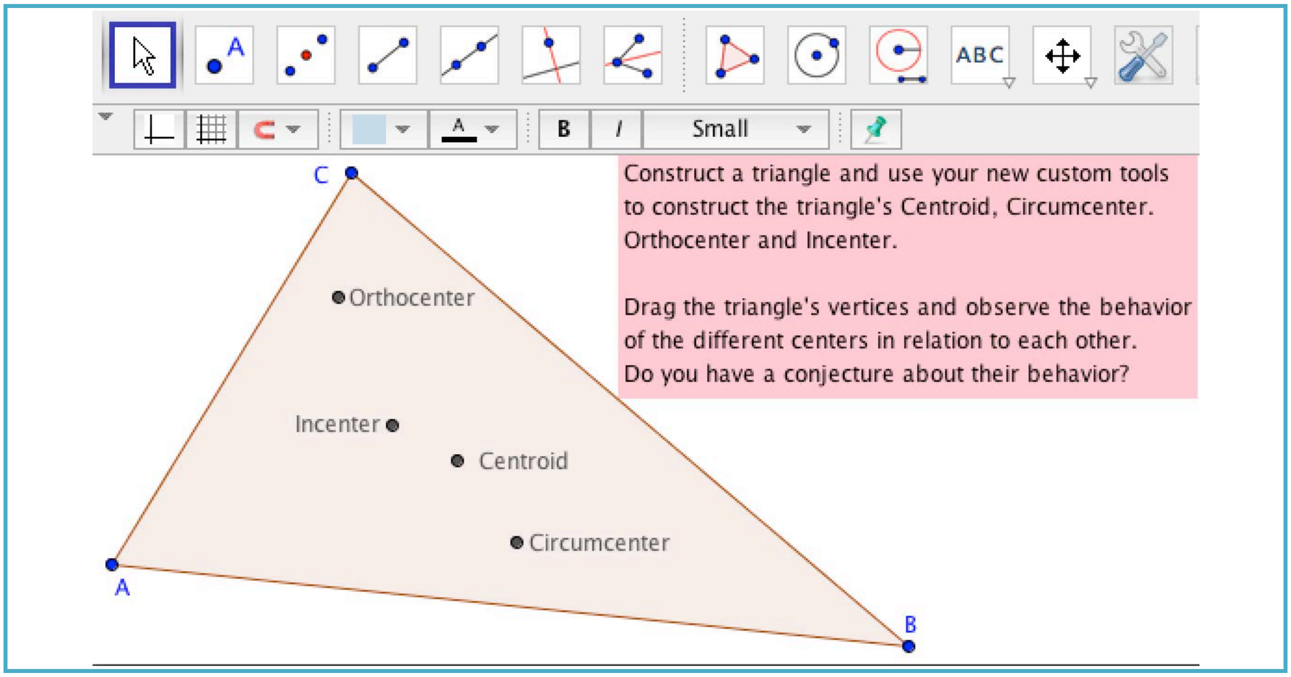
Note: The incenter of a triangle is the center of a circle inscribed in the triangle. A radius of the inscribed circle is tangent to each side of the triangle, so you can construct a perpendicular from the incenter to a side to find the inscribed circle’s point of tangency – and then use this point to construct the inscribed circle.



Section 6.09 The Euler Segment of a Triangle

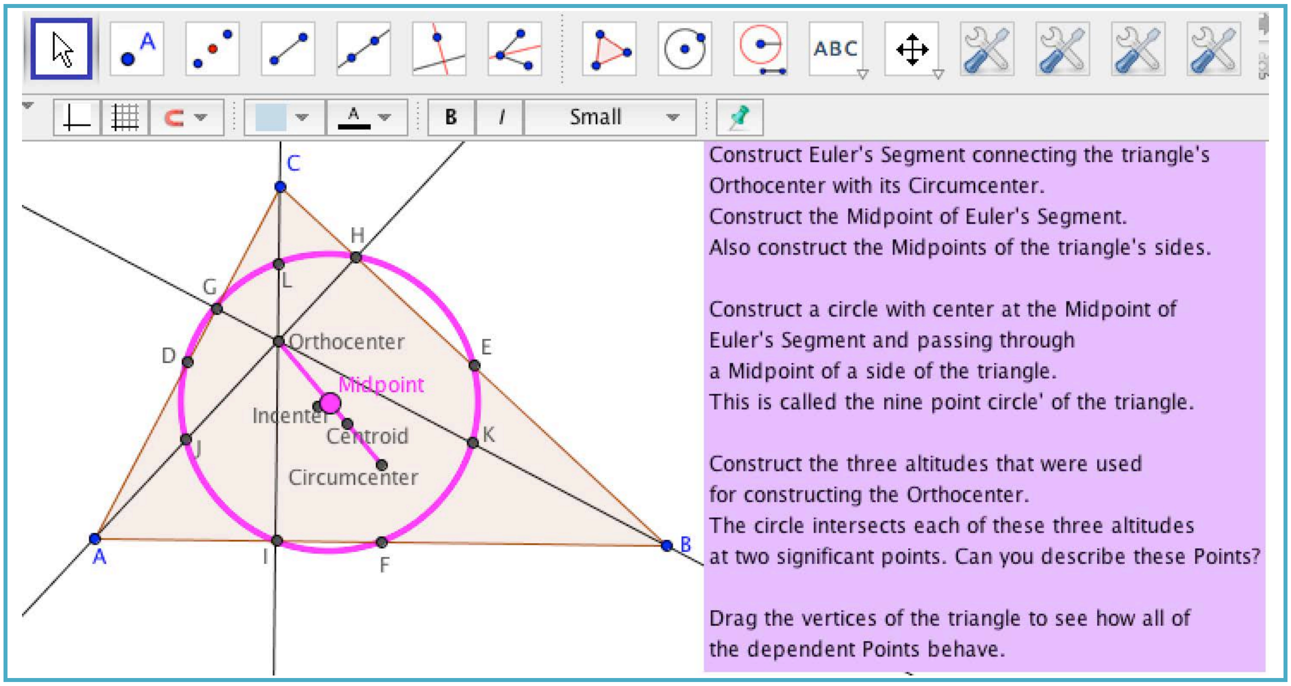
A Swiss mathematician named Euler discovered a relationship among three of the centers that you created custom tools for. Can you discover what he did? He did this in the 1700s—without dynamic-geometry tools. Euler's work renewed interest in geometry and led to many discoveries beyond Euclid.

Note: Take turns to re-create a custom tool for each of the triangle's special points: centroid, circumcenter, orthocenter and incenter, as done in the previous tabs. Or else, load the custom tools you created before using the GeoGebra menu "File" | "Open…" then find and open the .ggt files that you saved. You should then be able to select your custom tools from the menu "Tools" | "Manage Tools." When your custom tools are available to you, they will also be available to your teammates in that tab.



Section 6.10 The Nine-Point Circle of a Triangle

You can construct a circle that passes through a number of special points in a triangle. First construct custom tools for the four kinds of centers or open them from your .ggt files that you saved in previous tabs of this topic. Connect the orthocenter to the circumcenter: this is "Euler's Segment." The Centroid lies on this segment. A number of centers and related points of a triangle are all closely related by Euler's Segment and its Nine-Point Circle for any triangle. Create an Euler Segment and its related Nine-Point Circle, whose center is the midpoint of the Euler Segment.



You can watch a six-minute video of this segment and circle at: www.khanacademy.org/math/geometry/triangle-properties/triangle_property_review/v/euler-line.

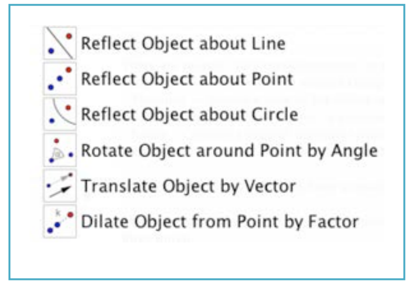
The video shows a hand-drawn figure, but you can drag your dynamic figure to explore the relationships more accurately and dynamically.

Are you amazed at the complex relationships that this figure has? How can a simple generic triangle have all these special points with such complex relationships? Could these result from the dependencies that get constructed when you define the different centers in your custom tools?

What we noticed:

What we wondered:

Topic 7: Transforming Triangles

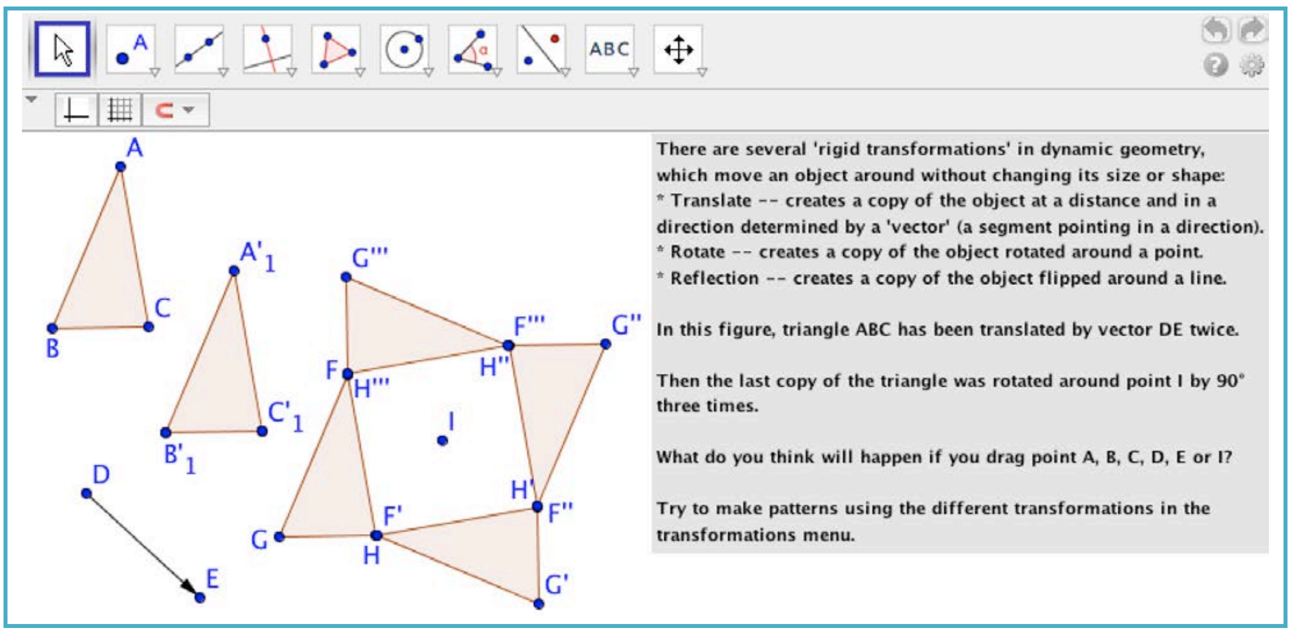


There are a number of "transformations" defined in dynamic geometry. They are useful for exploring and proving relationships among dynamic-geometry objects.

GeoGebra has a special menu of transformations. This topic introduces rotation, reflection and translation. Whole fields of mathematics could be based on transformation operations.

Section 7.01 Rigid Transformations

There are "rigid transformations," which maintain the size and shape of any dynamic-geometry object, but translate, rotate or reflect the object from its original location. Drag the figures in this tab to discover how the dependencies work. Think up some new patterns and figure out how to construct them using GeoGebra's transformation tools.

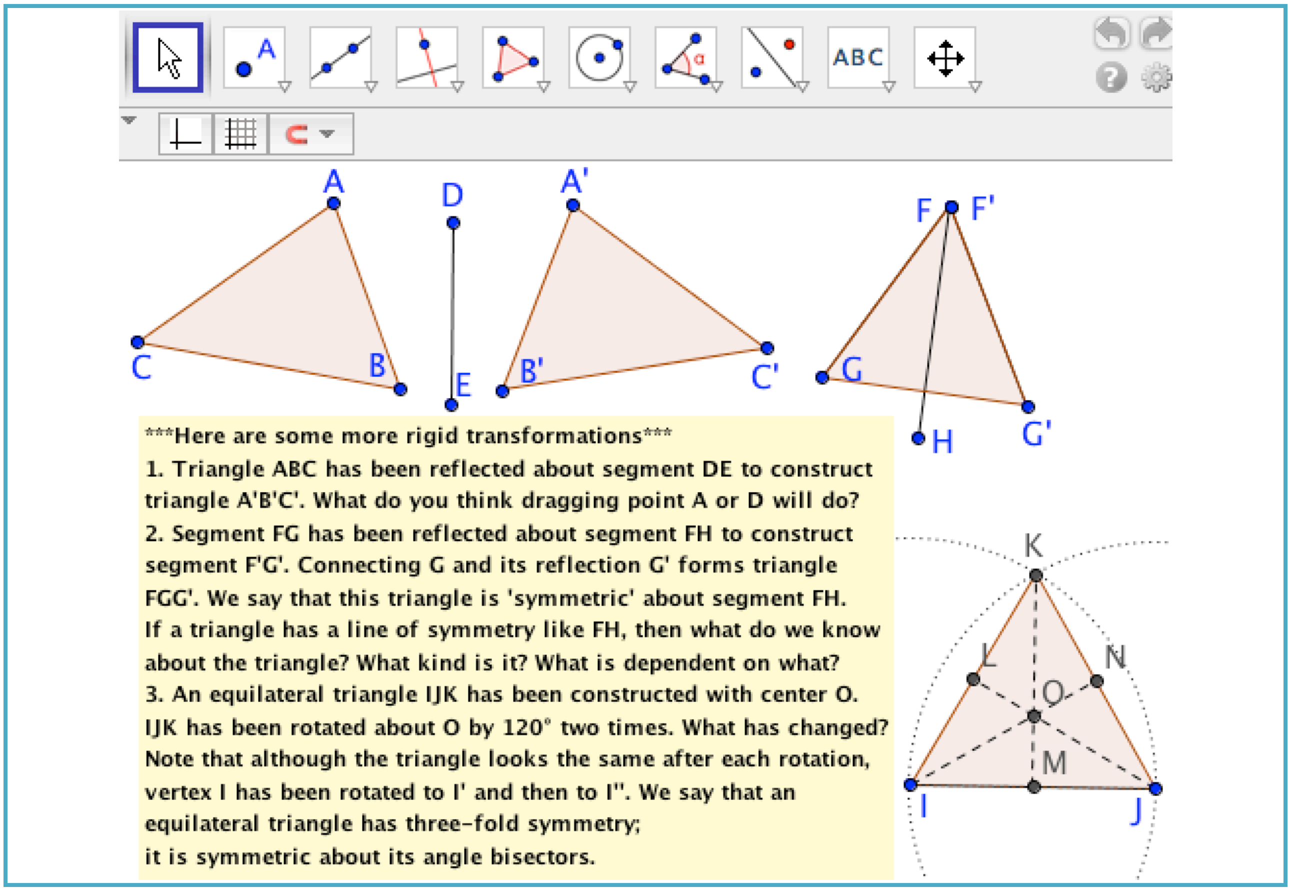


Section 7.02 Angles of Symmetry

Here are some examples of symmetry in triangles.

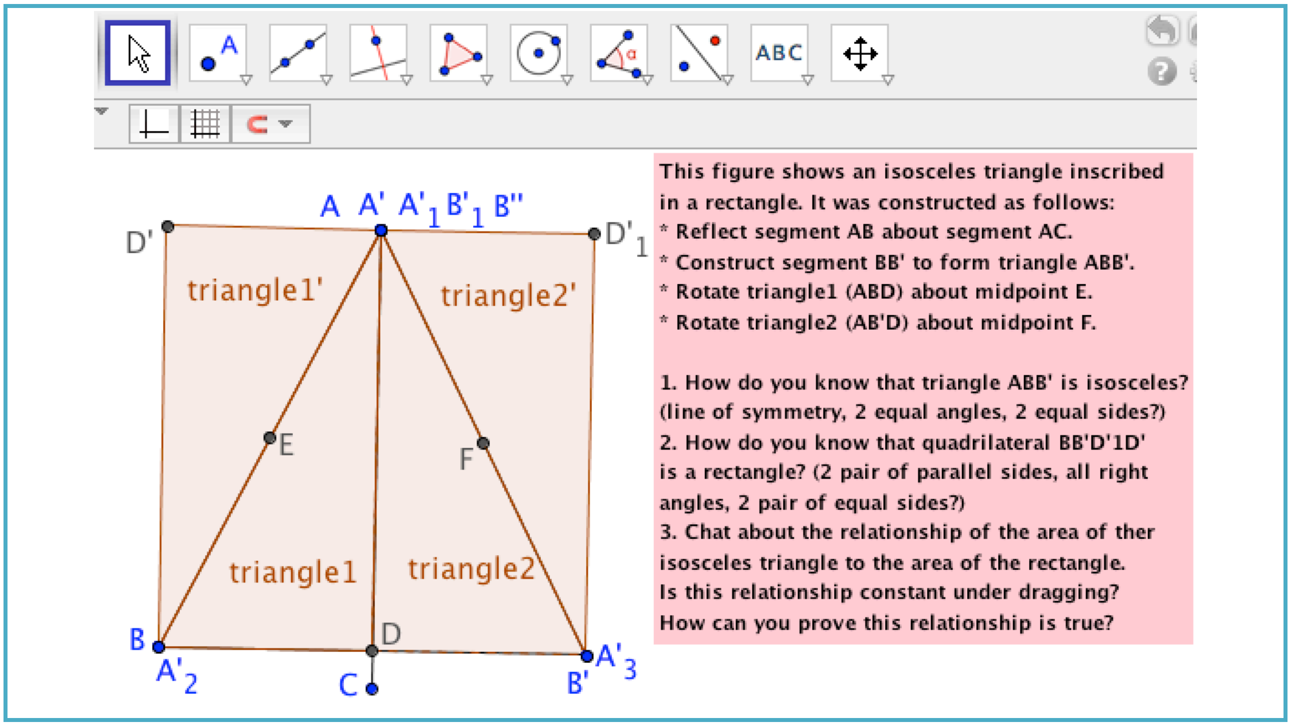
Construct an equilateral triangle like IJK with its angle bisectors. Try rotating the triangle around its center 120 degrees at a time. Try reflecting it about its angle bisectors. Symmetry means that certain transformations leave it looking the same.

What symmetries do you think a square has?



Section 7.03 Areas of Triangles

You can use rigid transformations to demonstrate relationships among areas of dynamic-geometry objects.



What we noticed:

What we wondered:

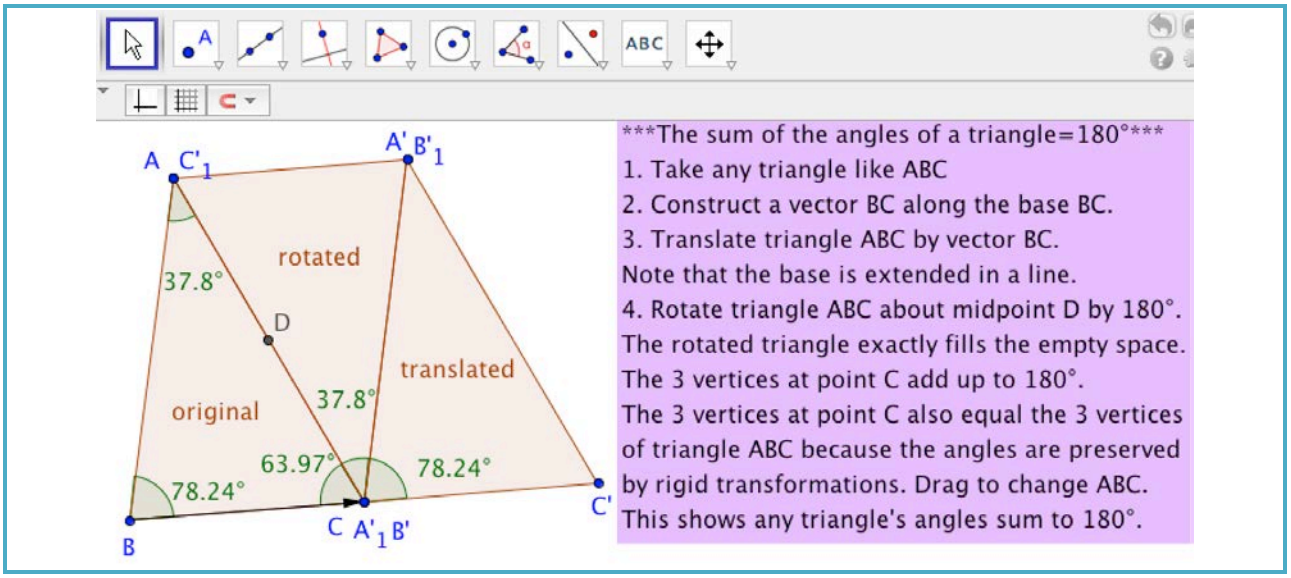
Topic 8: Exploring Angles of Triangles

There are a number of important theorems, propositions or proven facts about angles in triangles. They are basic theorems of geometry and are used often in solving typical geometry problems. Dynamic-geometry constructions can help you to understand why they are true. Constructing and dragging them will help you to remember, use and enjoy these facts. You will not have to memorize them, because you will understand them and be able to figure them out again if your forget them.

(**Note**: *This is a long topic. Try to explore some of the tabs on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session*.)

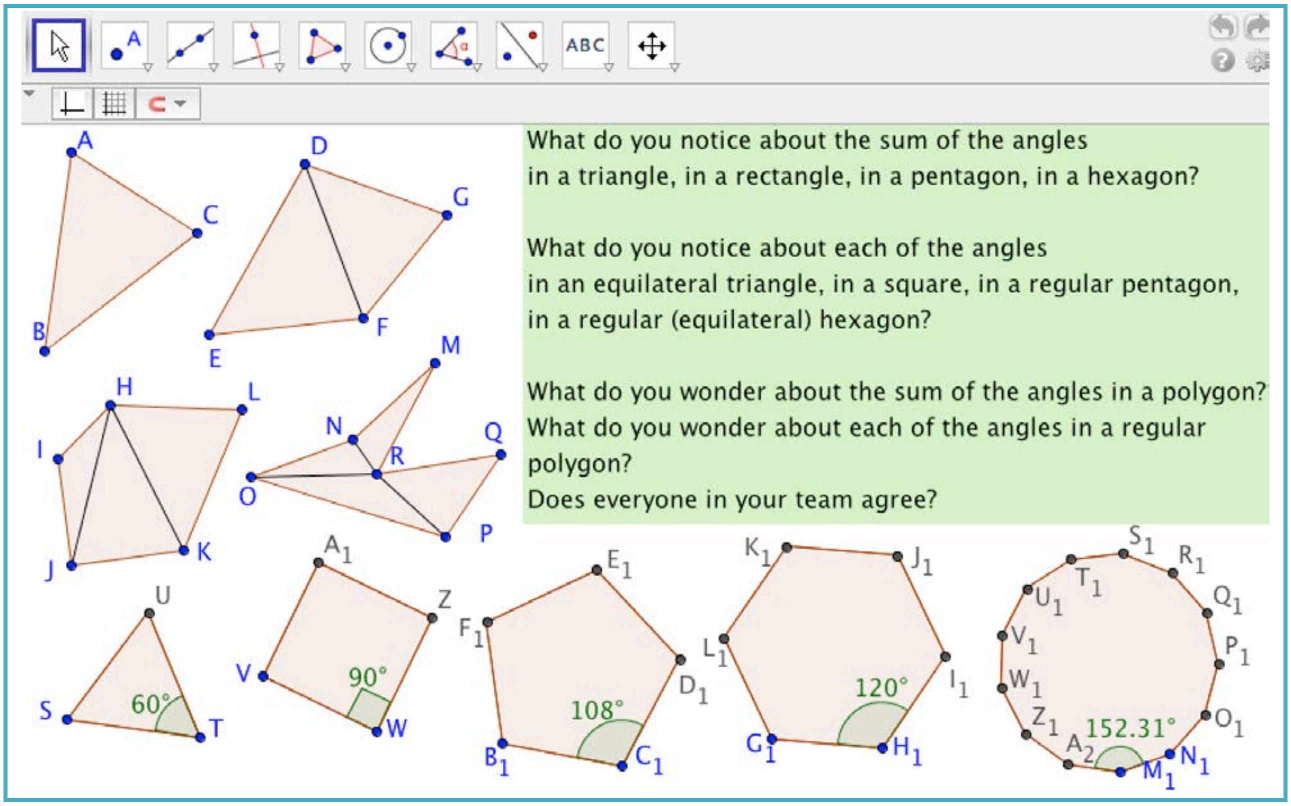
Section 8.01 Sum of the Angles in a Triangle

The angle around a complete circle can be defined (arbitrarily, based on the ancient Babylonian system of mathematics) as 360 degrees. That makes the number of degrees in an angle that forms a straight line 180 degrees and the number of degrees in a right angle (formed by a perpendicular to a line) 90 degrees. You may have heard that the three angles in any triangle add up to 180 degrees. Why is that so?



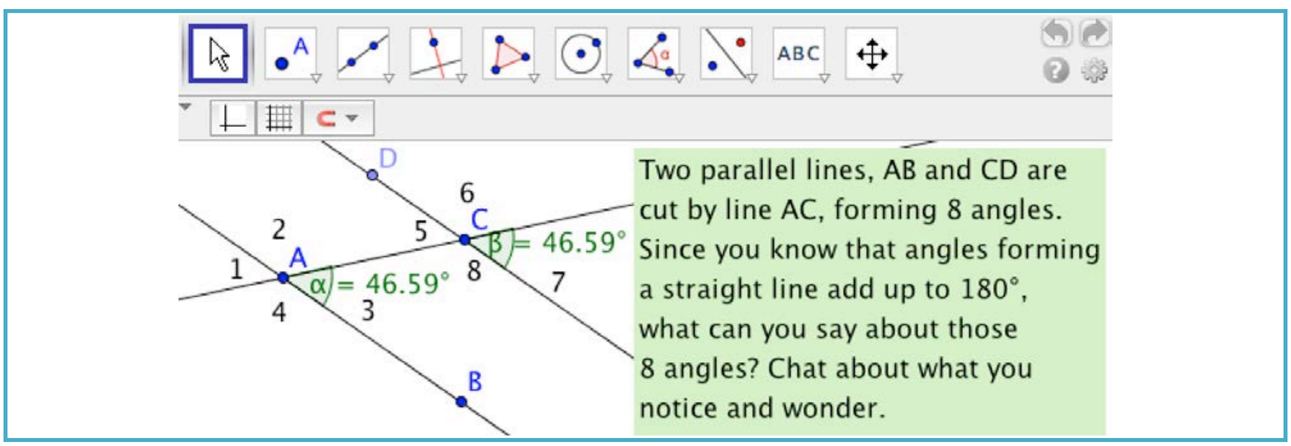
Section 8.02 Sum of the Angles in a Polygon

Now that you know there are 180 degrees in any triangle, can you figure out how many are in other polygons?



Section 8.03 Corresponding Angles

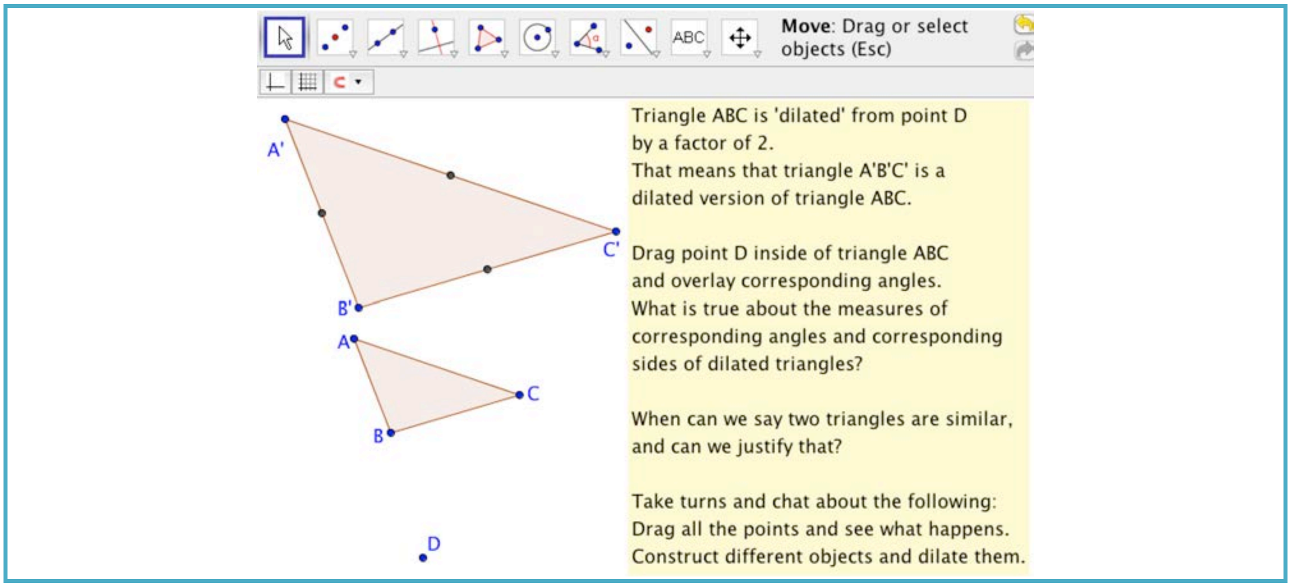
It is often useful when solving a geometry problem or applying geometry in the world to know which angles are equal to each other. Drag the lines in this tab around. Which angles stay equal to each other? Which pairs of angles always add up to 180 degrees?



Section 8.04 Dilation of Triangles

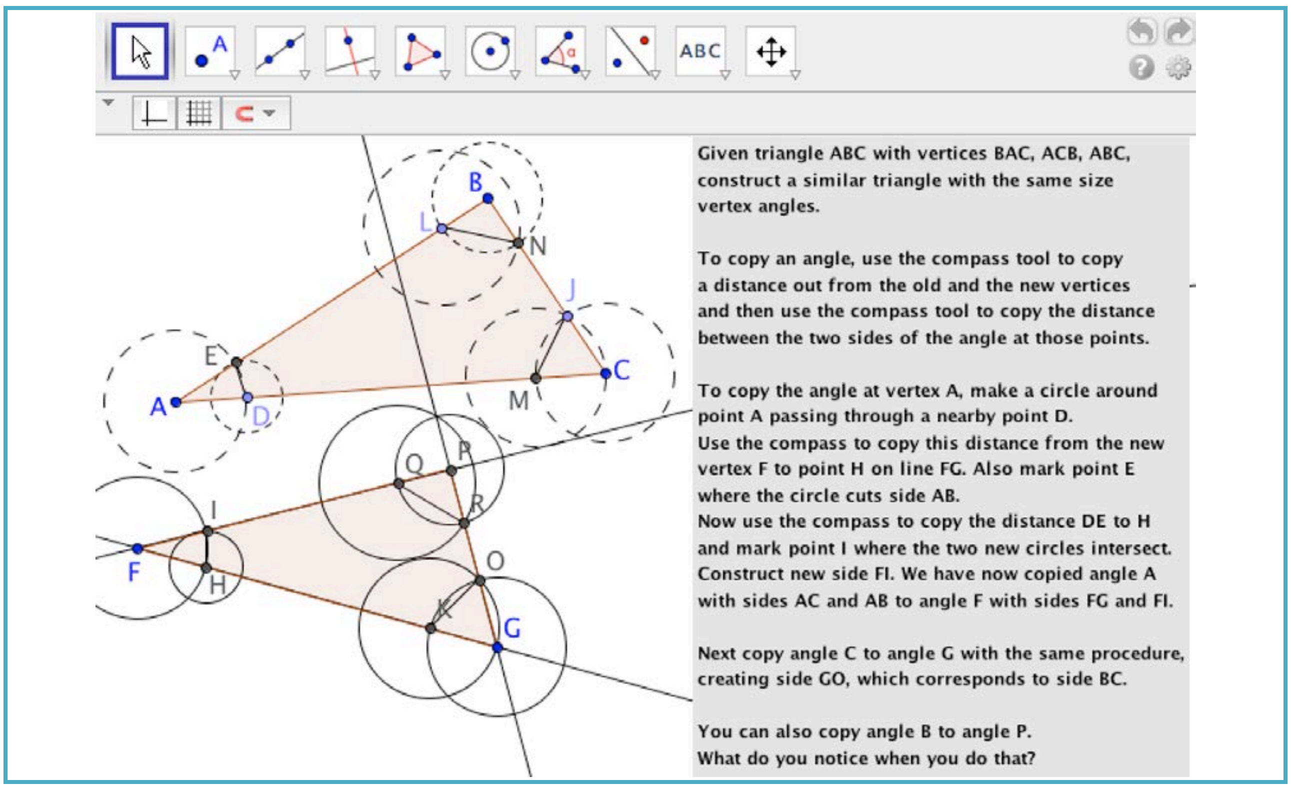
There is a transformation in dynamic geometry that is not rigid: "dilation." That means the object that is transformed can change size while remaining the same shape.

The term "similar" is a technical term in geometry. Two triangles are "similar" in dynamic geometry if they always have the same shape (i.e., all the same corresponding angles) as each other, even if they are different sizes.



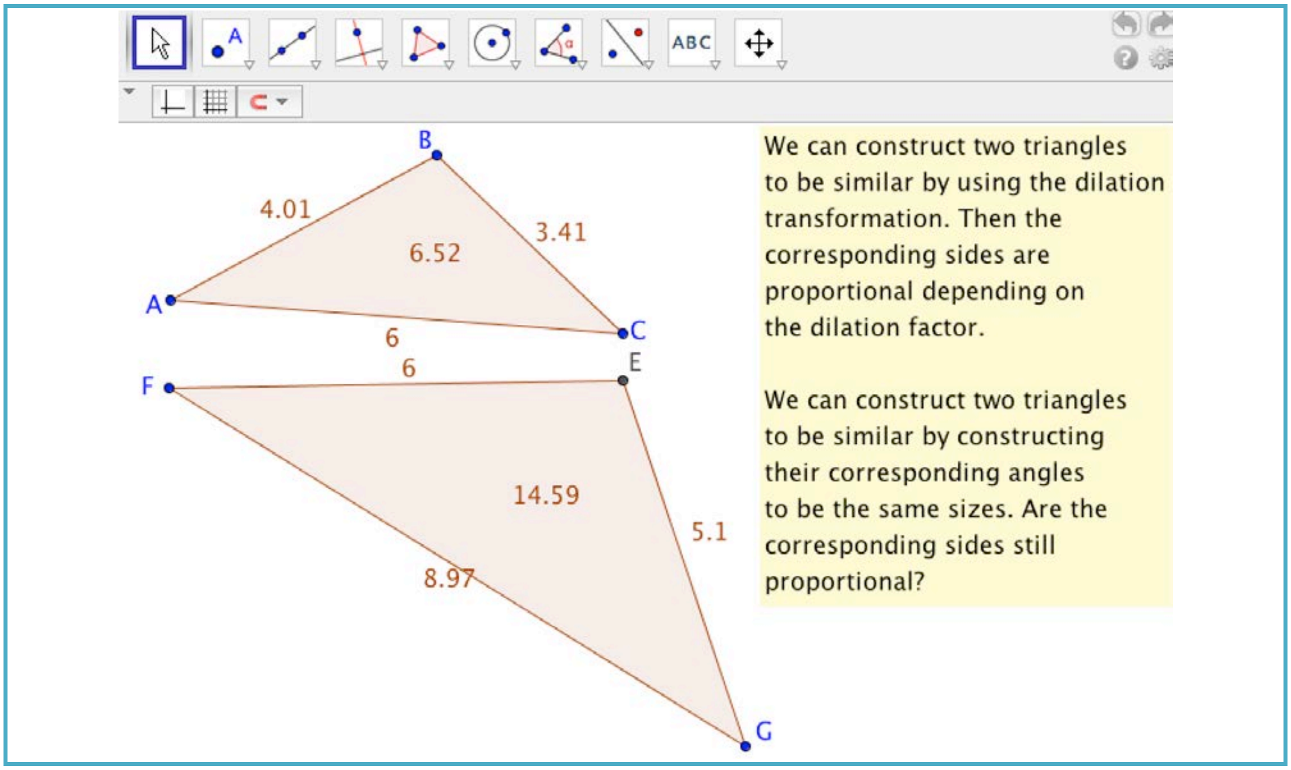
Section 8.05 Angles of Similar Triangles (AAA)

Given a dynamic triangle ABC, if we construct another triangle that is constrained to have 1, 2 or 3 angles the same size as corresponding angles in ABC, will the triangles be similar? This is the Angle-Angle-Angle (or AAA) rule for similar triangles.



Section 8.06 Sides of Similar Triangles

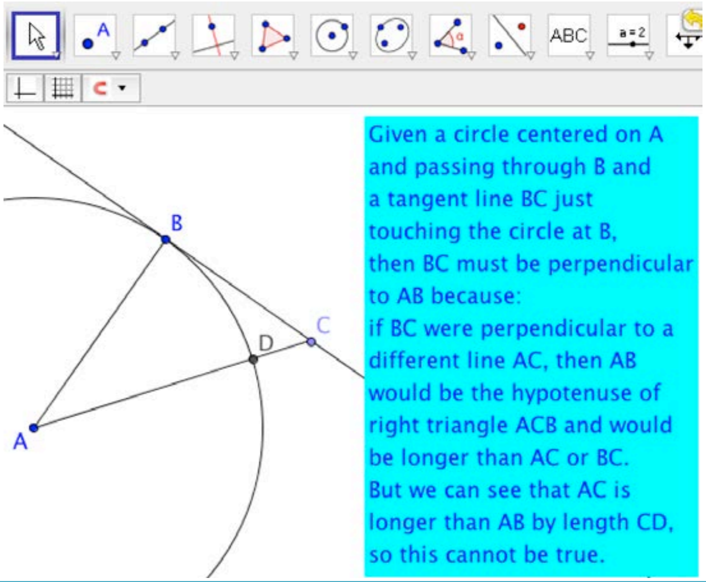
Given two similar triangles, what are the relationships among the corresponding sides of the triangles?



Section 8.07 Tangent to a Circle

The tangent to a circle is perpendicular to the radius at the point of tangency. You can prove this using an interesting kind of "proof by contradiction" that Euclid often used. First, you assume that the conjecture is false – i.e., that there might be some other line that is perpendicular to the tangent. Then you show that would lead to a contradiction of the given conditions. Therefore, the assumption must be wrong and the conjecture must be right.

Explain each step in the proof in the chat. Does this make sense to everyone in the team?



What we noticed:

What we wondered:

Topic 9: Visualizing Congruent Triangles

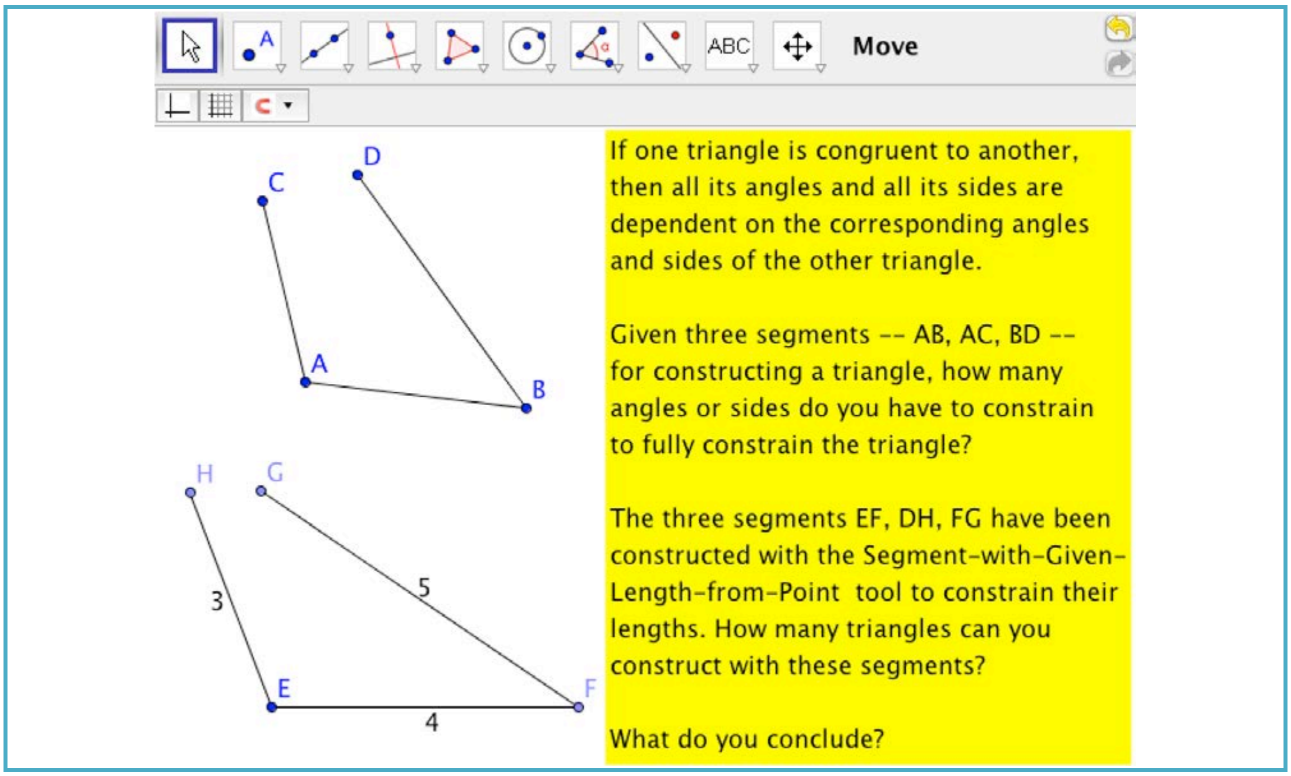
Before working on this topic with your team, you should read **Tour 5: "VMT Logs and Replayer for Reflection"** near the end of this booklet. It will show you how to download logs of your chat and to replay your session. Then you can review your discussions and include excerpts of the chat log or screen shots of the session in your journal or reports.

(**Note**: *This is a long topic. Try to explore some of the tabs on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session.*)

All the corresponding angles and sides of congruent triangles are equal. However, you can constrain two triangles to be congruent by just constraining 3 of their corresponding parts to be equal – for certain combinations of 3 parts. Dynamic geometry helps you to visualize, to understand and to remember these different combinations.

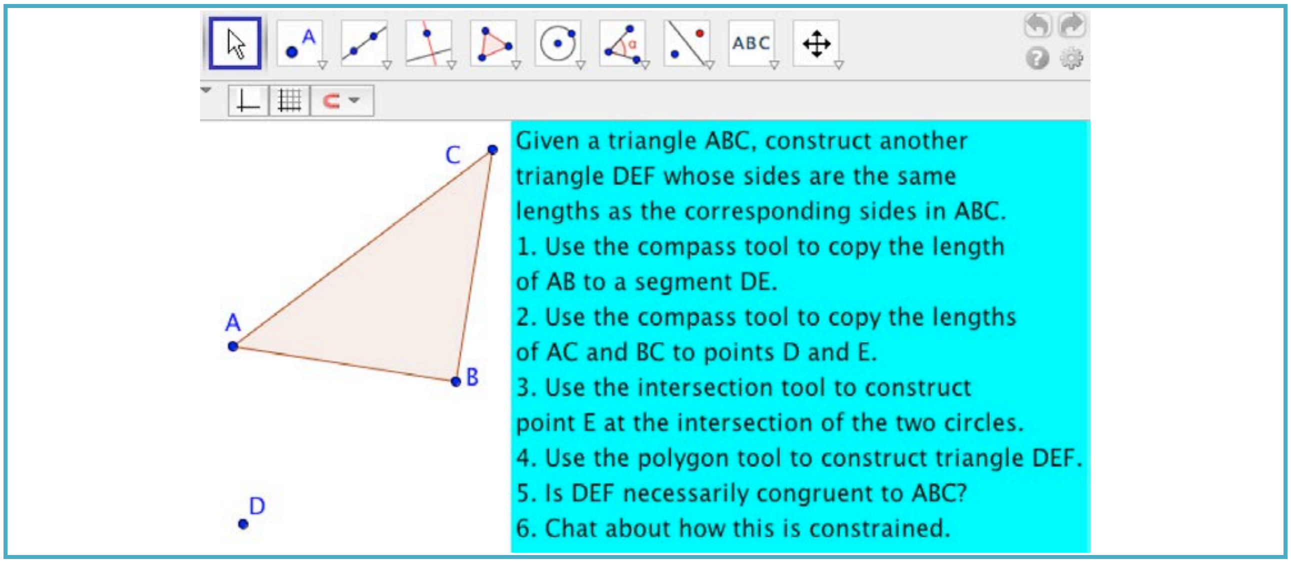
Section 9.01 Corresponding Sides and Angles of Congruent Triangles

What constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle?



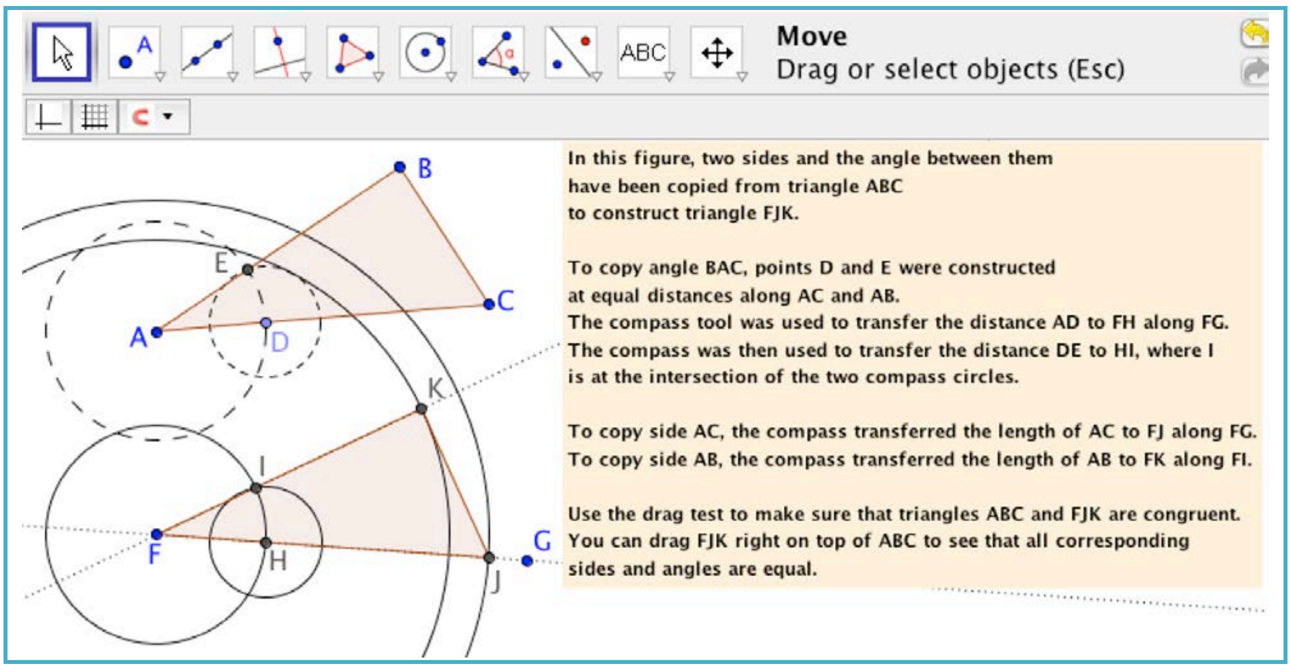
Section 9.02 Side-Side-Side (SSS)

If all three sides of one triangle are equal to the corresponding sides of another triangle, then the two triangles are congruent. This is called the "Side-Side-Side" (or "SSS") rule.



Section 9.03 Side-Angle-Side (SAS)

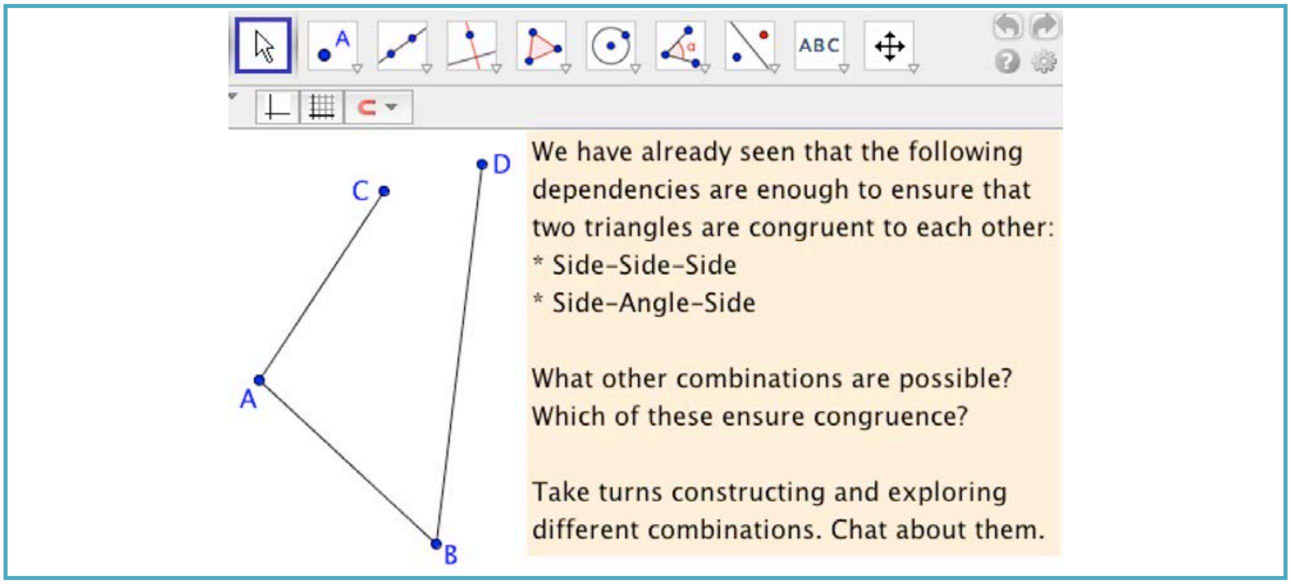
If two sides and the angle **between them** of one triangle are equal to the corresponding sides and angle **between them** of another triangle, then the two triangles are congruent.



Section 9.04 Combinations of Sides and Angles

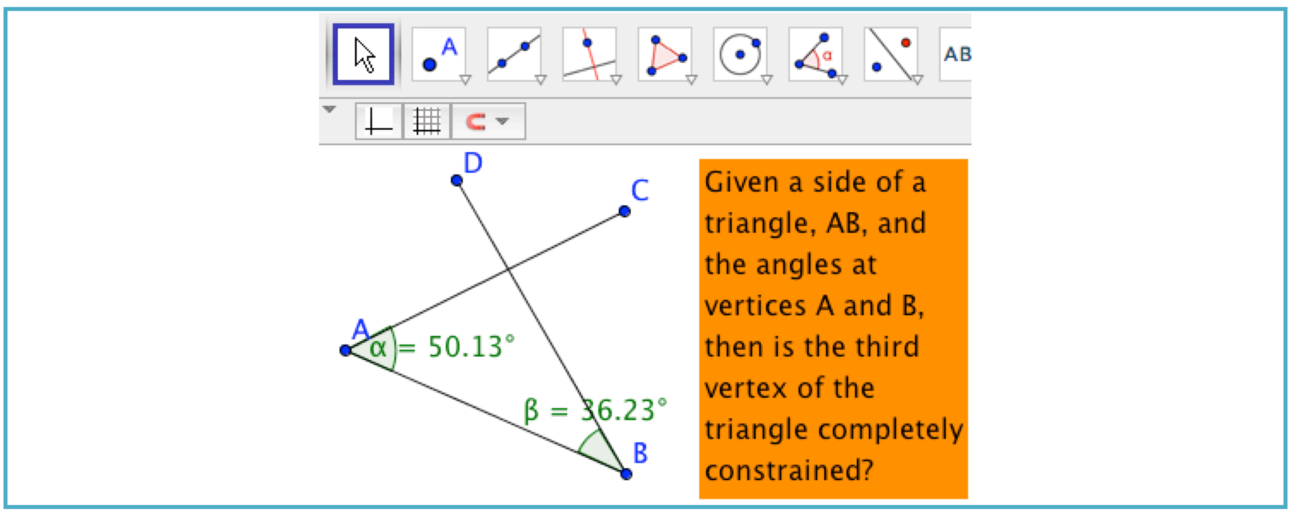
You can constrain two dynamic triangles to be congruent using a number of different combinations of equal corresponding sides and/or angles.

What combinations of constraints of sides and angles are necessary and sufficient to constrain the size and shape of a triangle?



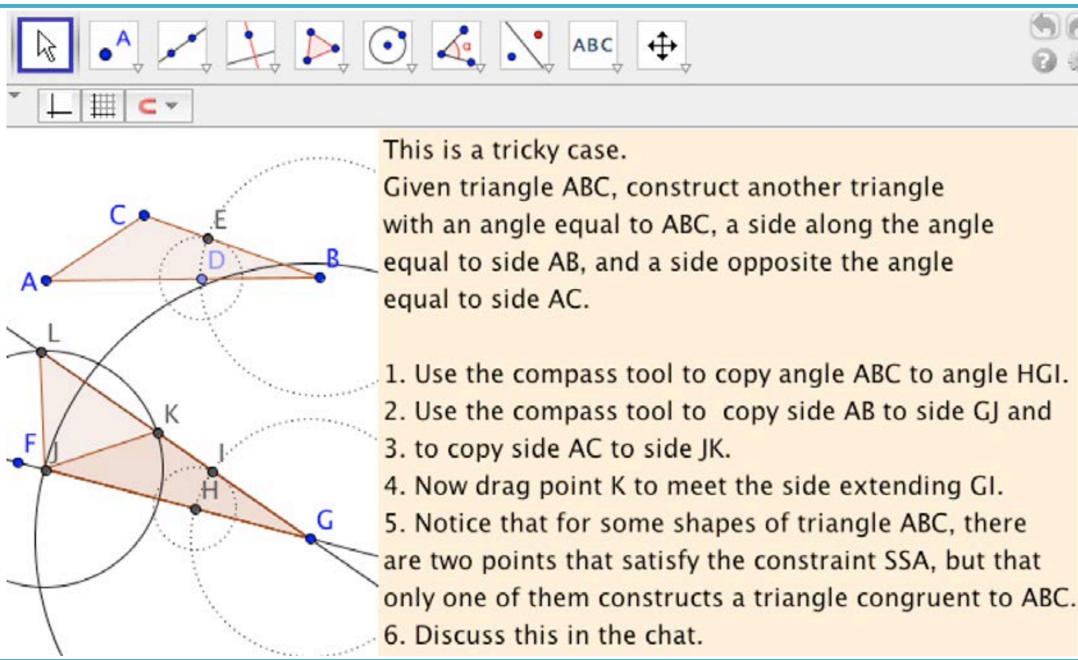
Section 9.05 Angle-Side-Angle (ASA)

If two angles and the side included **between them** of one triangle are equal to the corresponding two angles and side **between them** of another triangle, then the two triangles are congruent. This is called the Angle-Side-Angle or ASA rule.



Section 9.06 Side-Side-Angle (SSA)

What if two corresponding sides and an angle are equal, but it is not the angle included between the two sides?



What we noticed:

What we wondered:

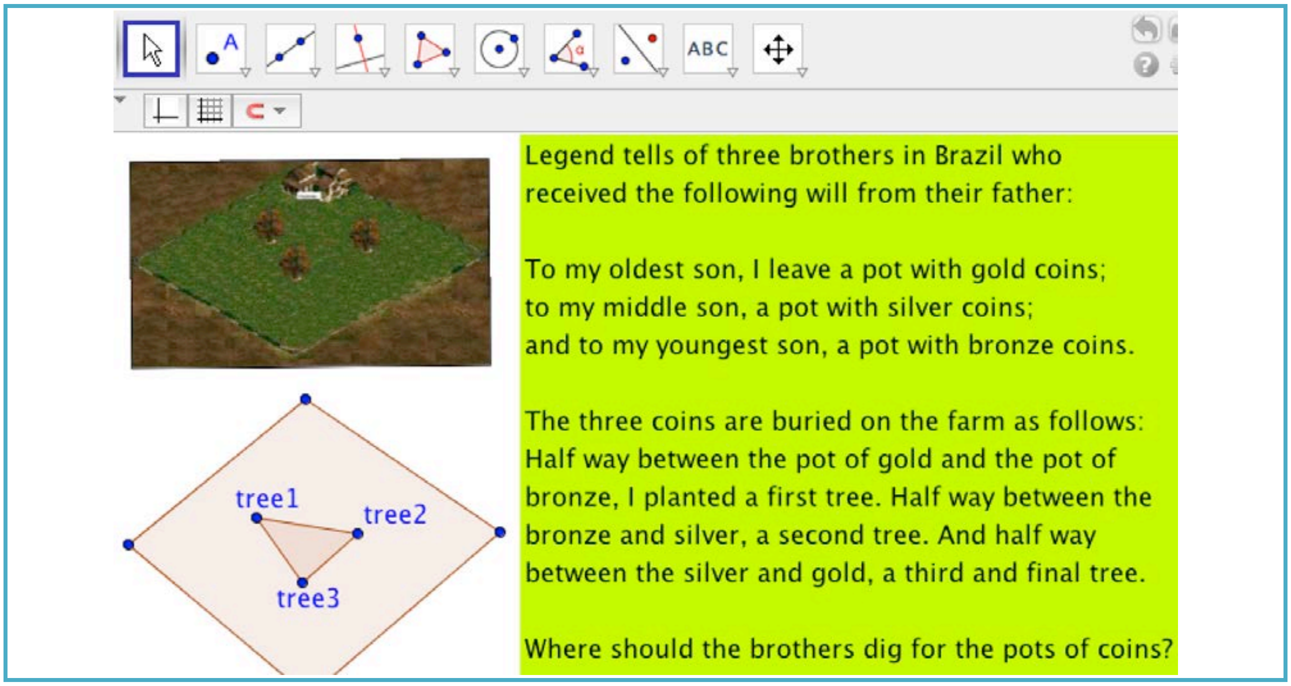
Topic 10: Solving Geometry Problems

Here is a set of challenge problems for your team.

If the team does not solve them during its session, try to solve them on your own and report your findings in the next team session.

Section 10.01 Treasure Hunt

Can you discover the pot of gold in this tale told by Thales de Lelis Martins Pereira? You might want to construct some extra lines in the tab.



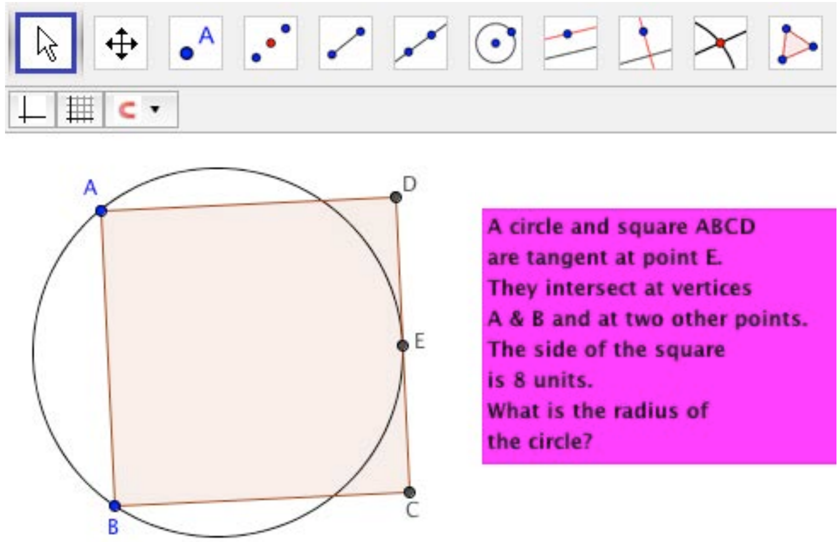
What we noticed:

What we wondered:

Section 10.02 Square and Circle

Can you determine the radius of the circle? You should not have to measure. What if the side of the square is “s” rather than “8”?

Hint: To solve this kind of problem, it is usually useful to construct some extra lines and explore triangles and relationships that are created. If you know basic algebra, you might set up some equations based on the relationships in the figure.



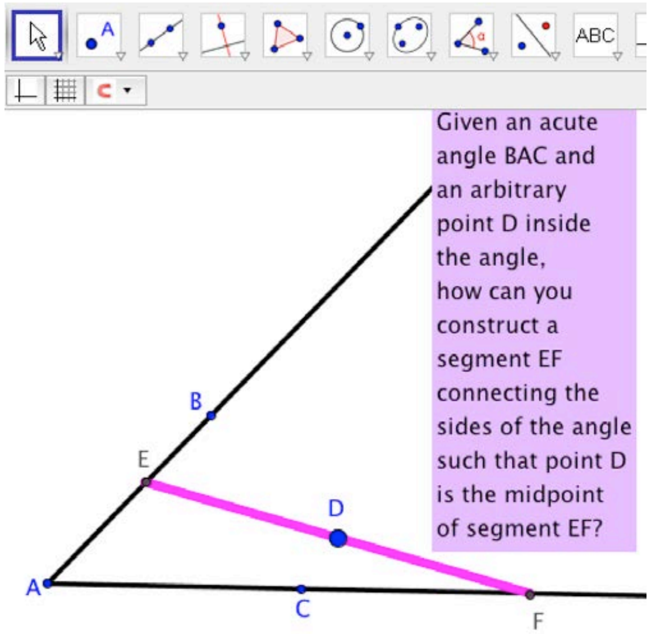
What we noticed:

What we wondered:

Section 10.03 Crossing an Angle

Can you construct the segment crossing the angle with the given midpoint?

Hint: This is a challenging problem. Try to add some strategic lines and drag the figure in the tab to see what would help to construct the segment EF at the right place.



What we noticed:

What we wondered:

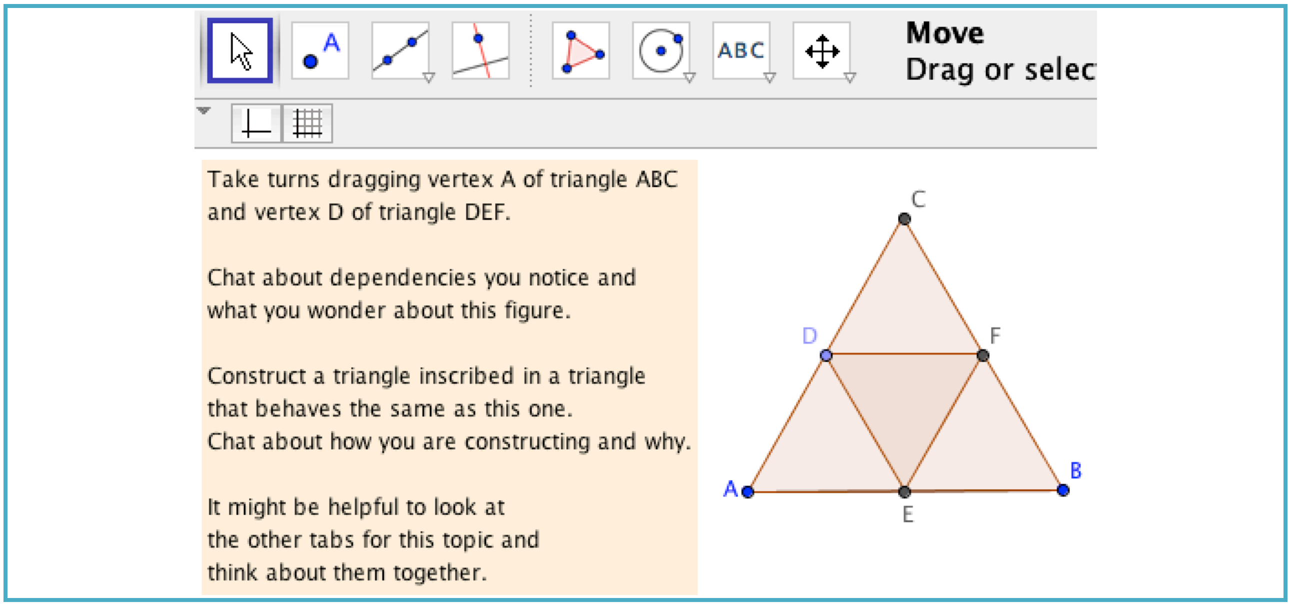
Topic 11: Inscribing Polygons

The following three tabs present an interesting set of figures. They illustrate how you can discover dependencies through dragging and create dependencies through construction. There are important similarities in the three tabs, suggesting a generalization.

If the team does not solve the three related problems during its session, try to solve them on your own and report your findings in the next team session.

Section 11.01 Inscribed Triangles

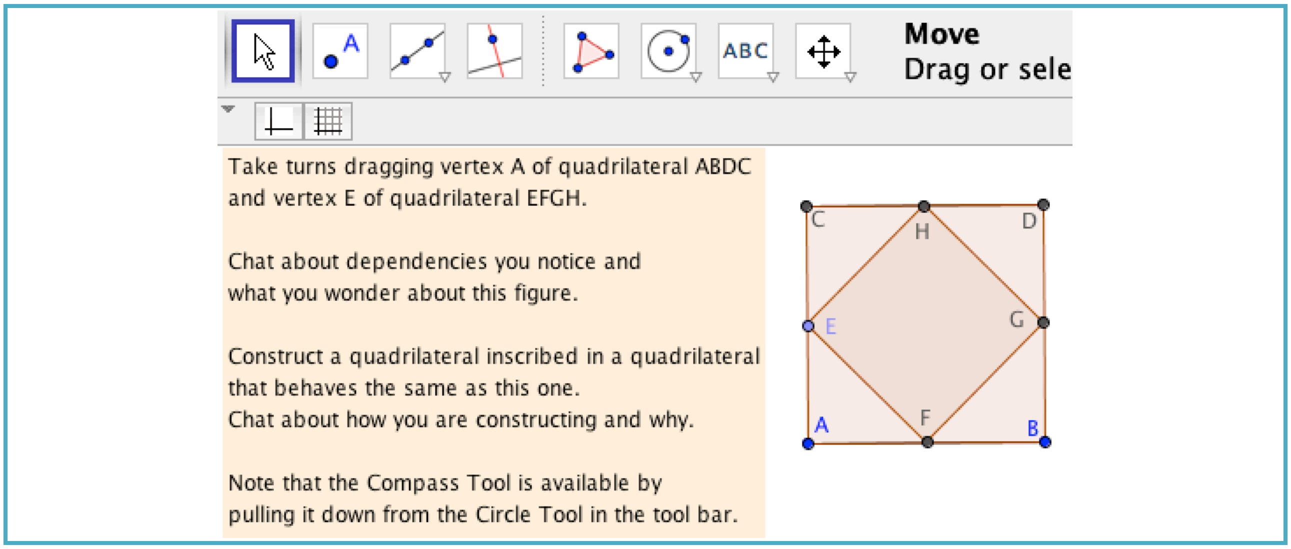
First, try to construct a pair of inscribed triangles.



If you solved this, did you construct the lengths of the sides of the outer triangle to be directly dependent upon each other? Did you construct the lengths of the sides of the inner triangle to be directly dependent upon each other? If not, do you think that the triangles are equilateral?

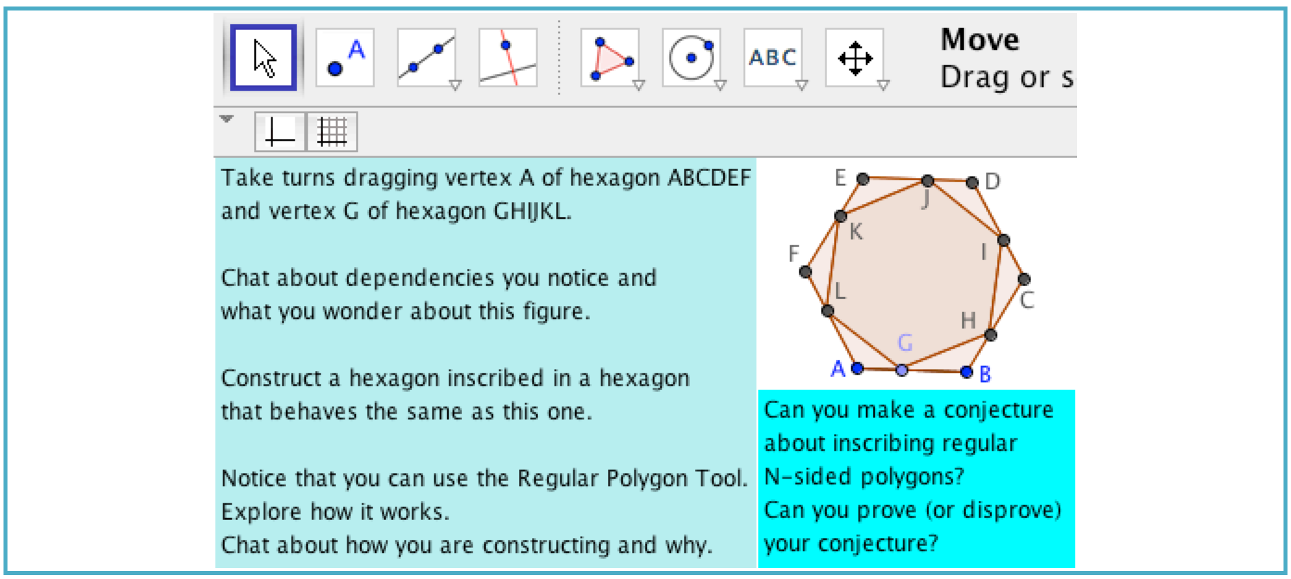
Section 11.02 Inscribed Quadrilaterals

Can inscribed quadrilaterals behave the same way?



Section 11.03 Inscribed Hexagons

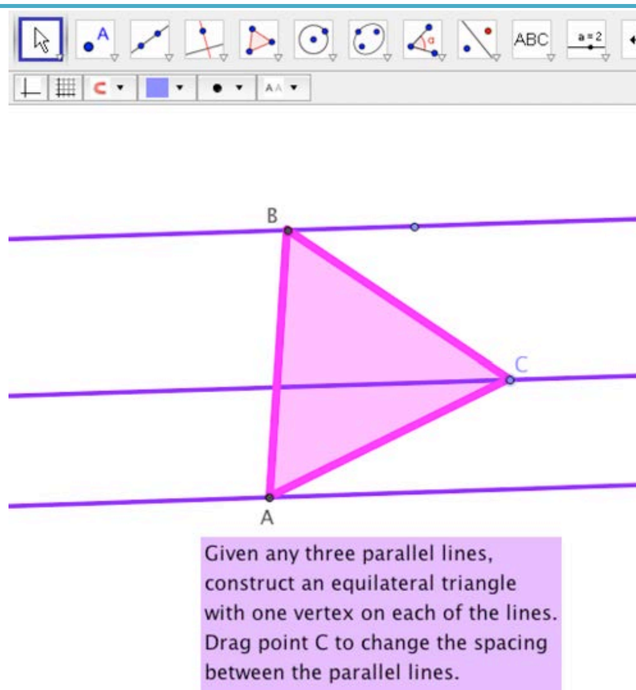
Can all polygons be inscribed the same way? Try it for a regular hexagon first. Do you see a general pattern between these three tabs: inscribed equilateral triangles, inscribed squares and inscribed regular hexagons? What is your conjecture about this? How might you prove it?



Section 11.04 Triangle on Parallel Lines

Here is a challenge problem if you enjoyed the inscribed polygon problems:

Given any three parallel lines, can you construct an equilateral triangle such that it has one vertex on each of the parallel lines? Drag the example triangle to discover the dependencies as the line through point C takes different positions between the other two parallel lines. Then construct your own figure such that points B and C are equidistant from A and from each other no matter what position the line through point C takes.



What we noticed:

What we wondered:

Topic 12: Building a Hierarchy of Triangles

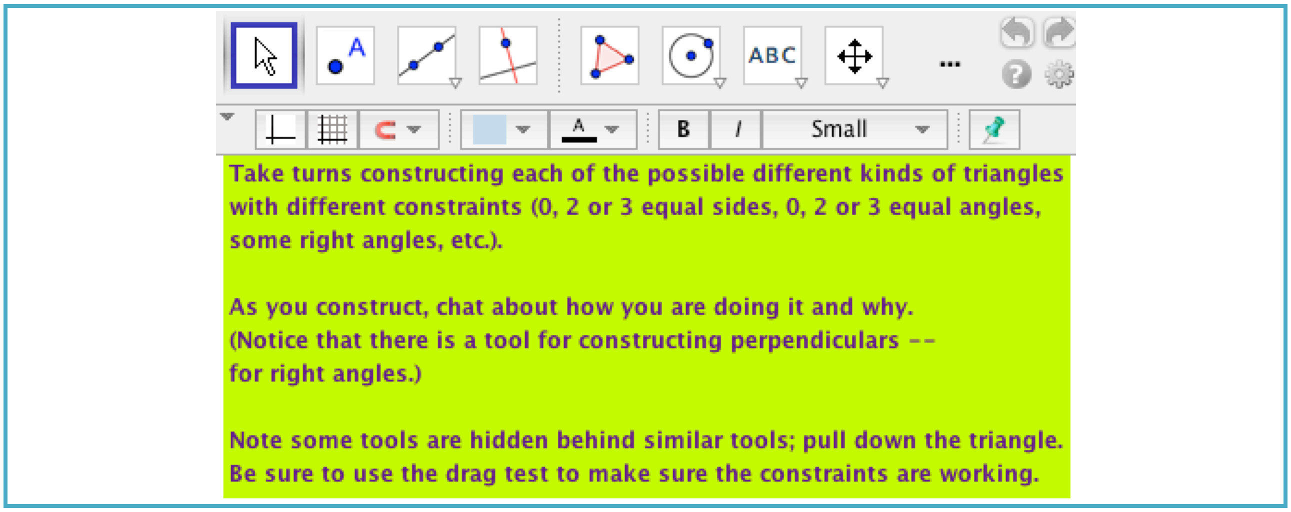
Now you can use GeoGebra tools (or your own custom tools) to construct triangles with different constraints. How are the different kinds of triangles related to each other? Which ones are special cases of other kinds?

Section 12.01 Constructing Other Triangles

How many different kinds of dynamic triangles can your team construct?

Challenge: Note that there is no tool for copying an angle; how can you create an angle equal to an existing angle and can you make a custom tool for it?

You might want to make a list or table of the different possible constraints on the sides and angles of triangles.



Section 12.02 The Hierarchy of Triangles

How are the different kinds of triangles related to each other?

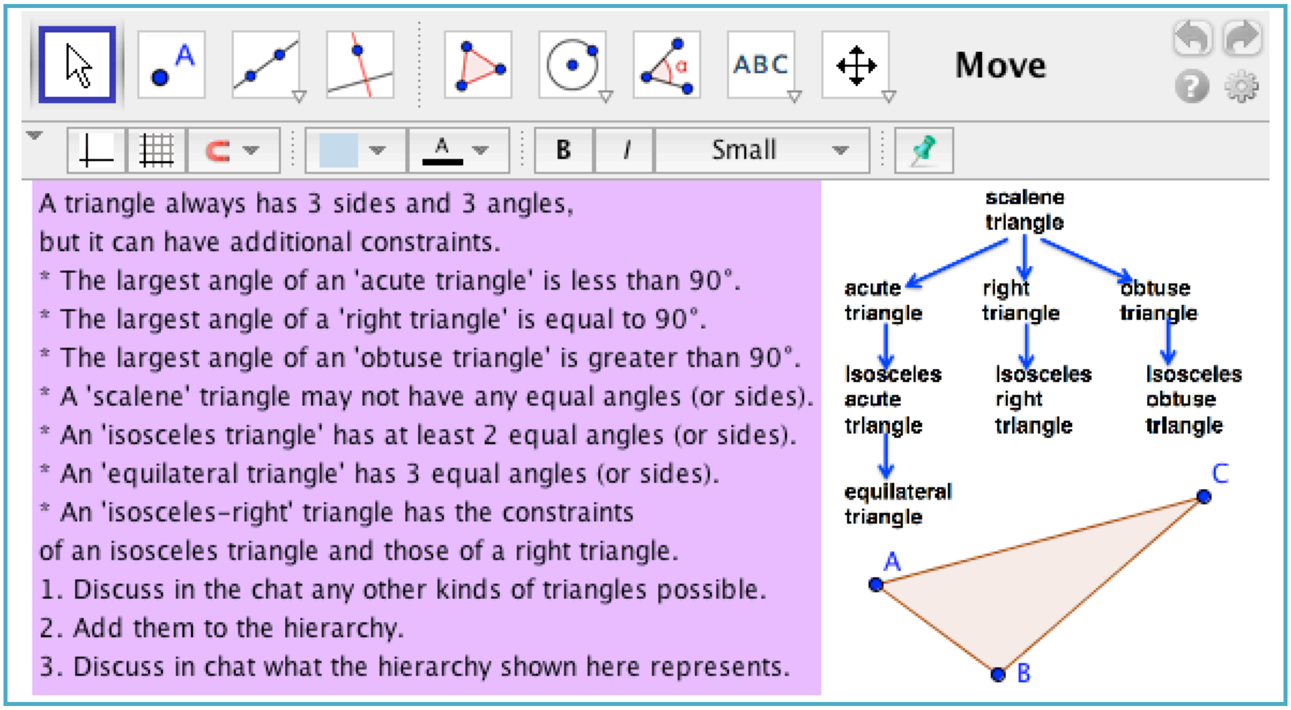
There are different ways of thinking about how triangles are related in dynamic geometry.

In dynamic geometry, a generic or "scalene" triangle with no special constraints on its sides or angles may be dragged into special cases, like a right triangle or an equilateral triangle. However, it does not have the constraints of a right angle vertex or equal sides built into it by its construction, so it will not necessarily retain the special-case characteristics when it is dragged again.

You can think of a hierarchy of kinds of triangles: an equilateral triangle can be viewed as a special case of an isosceles acute triangle, which can be viewed as a special case of an acute triangle, which can be viewed as a special case of a scalene triangle.

Can your team list all the distinct kinds of triangles?

Can your team connect them in a hierarchy diagram? Create a hierarchy diagram like the one shown in the tab. Add more kinds of triangles to it. You may want to reorganize the structure of the diagram.



What we noticed:

What we wondered:

Topic 13: Exploring Quadrilaterals

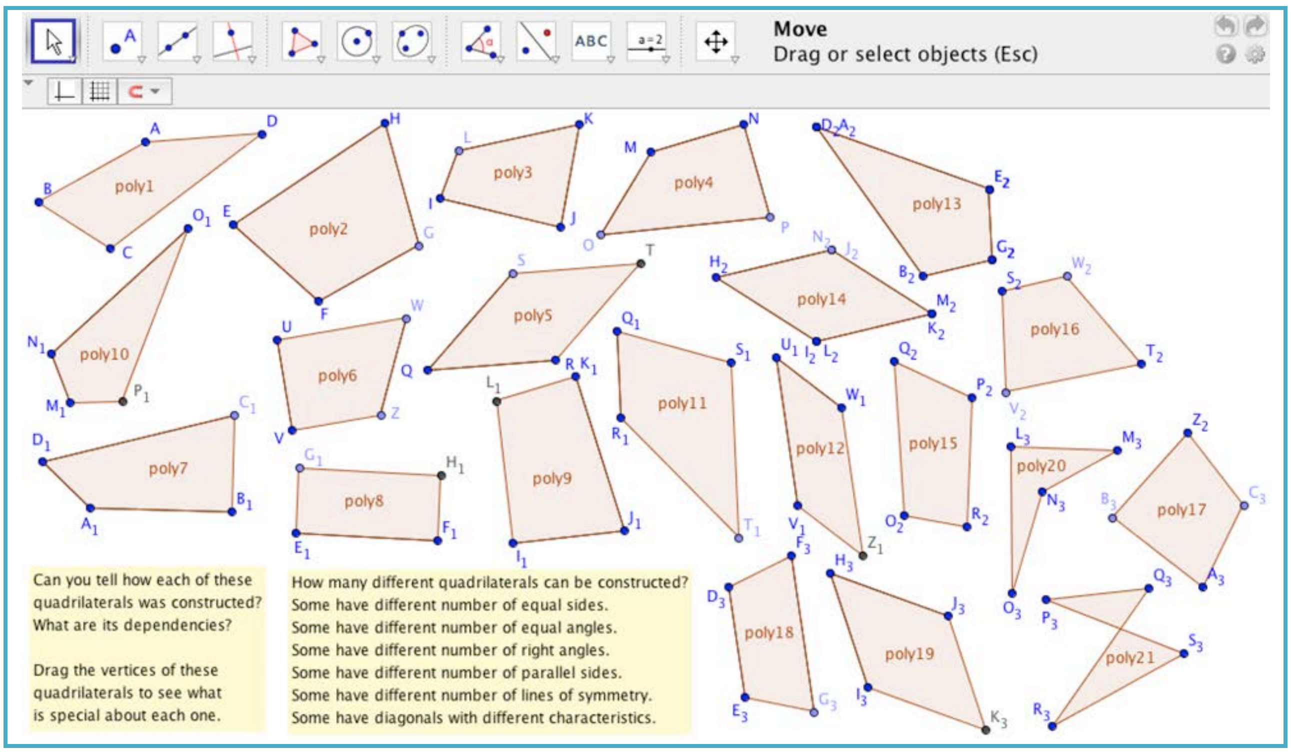
> Before working on this topic with your team, you may want to look at **Tour 6: "GeoGebra Videos and Resources"** near the end of this booklet. It lists some YouTube videos that you may enjoy watching.

The term "quadrilateral" means "four-sided" in Latin. Just join four segments at their endpoints. There are many different kinds of quadrilaterals and complex relationships among their related constructions.

Section 13.01 Dragging Different Quadrilaterals

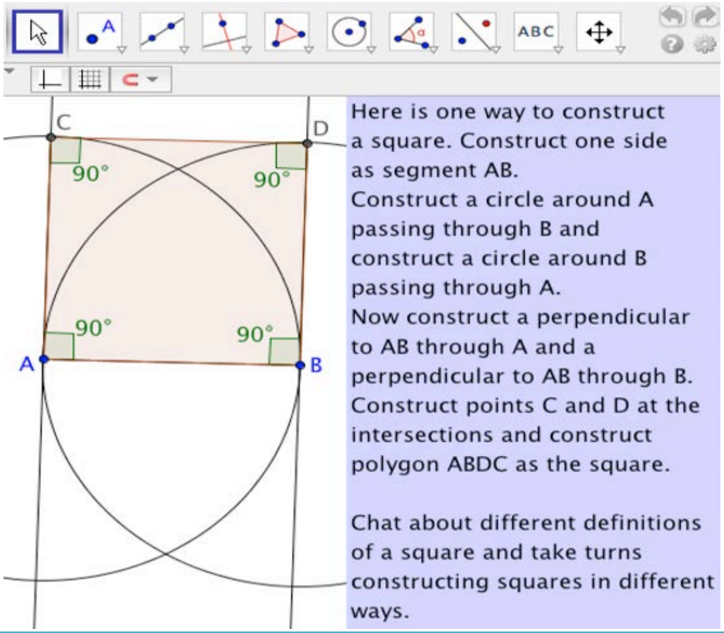
There are many different quadrilaterals with specific construction characteristics and dynamic behaviors.

Identify the constraints on each of the quadrilaterals in this tab. Can you think of any possible quadrilaterals that are not there? Does everyone on your team agree?



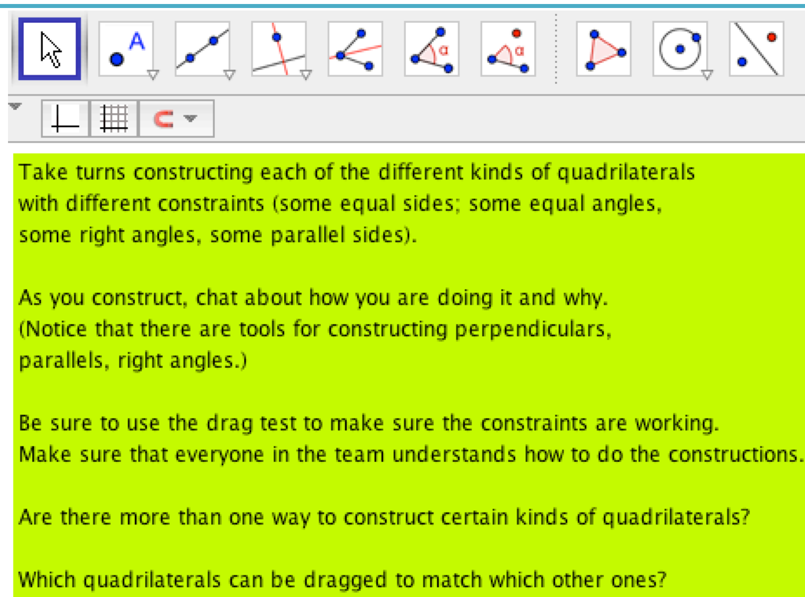
Section 13.02 Constructing a Square

There may be many ways to construct a quadrilateral with specific constraints. How many ways can you list to construct a square? Would you say that the construction in this tab defines the constraints as three equal sides and two right angles?



Section 13.03 Constructing Different Quadrilaterals

Can you construct the different kinds of quadrilaterals pictured in 13.1?



What we noticed:

What we wondered:

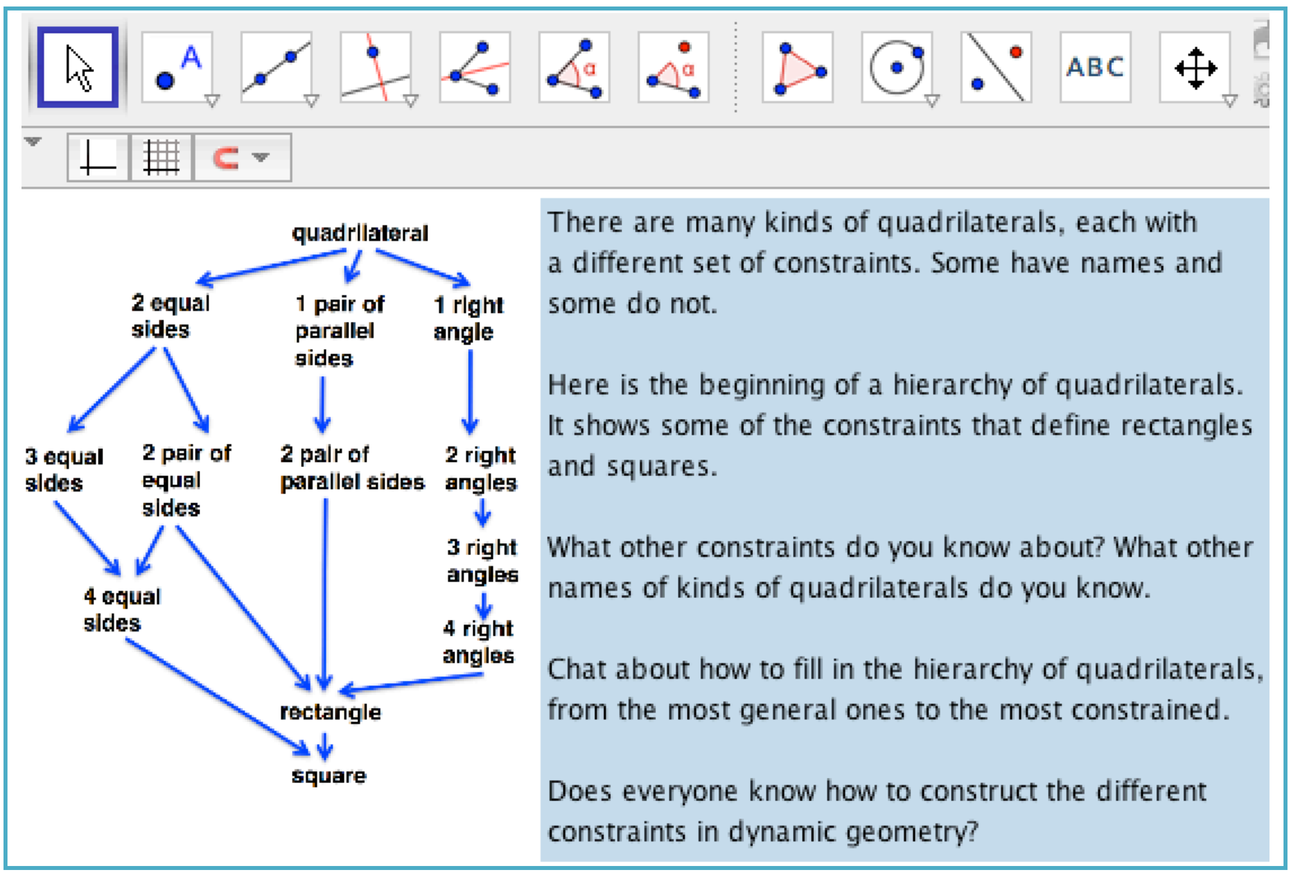
Topic 14: Building a Hierarchy of Quadrilaterals

(**Note**: *This is a long topic. Try to explore some of the tabs on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the final tabs after your team session*.)

Quadrilaterals have many special properties or relationships among their parts.

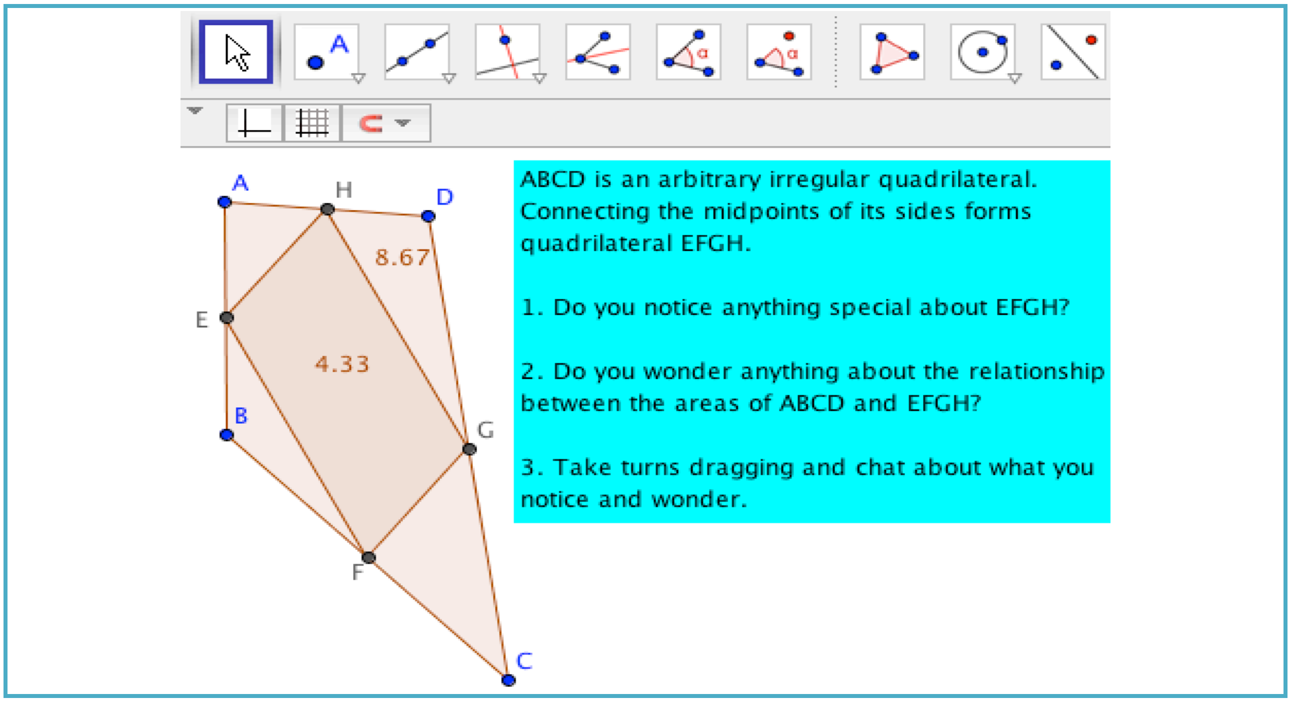
Section 14.01 The Hierarchy of Quadrilaterals

Can you construct a hierarchy of types of quadrilaterals? How many types can you include? Do you have square, rectangle, rhombus, trapezoid, parallelogram, kite and many types that do not have common names? Can your team connect the different types of quadrilaterals in a hierarchy diagram? Create a hierarchy diagram like the one shown in the tab. Add more kinds of triangles to it. You may want to reorganize the structure of the diagram. You can use the descriptions of constraints, use the common names like rhombus and kite or use a combination of these, like in the tab.



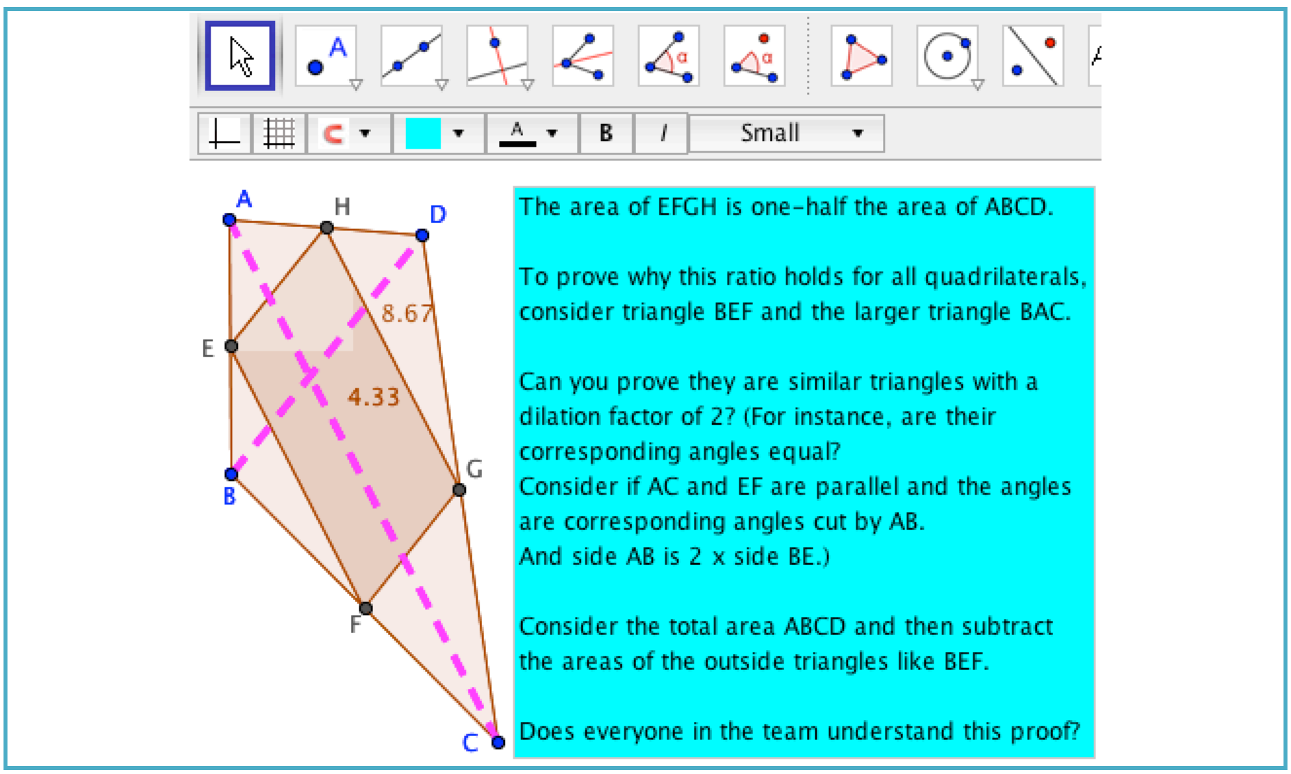
Section 14.02 Connecting the Midpoints of a Quadrilateral's Sides

You may be surprised by the quadrilateral formed by connecting an irregular quadrilateral's sides. Drag ABCD and see how EFGH changes. Why do you think it is like this?



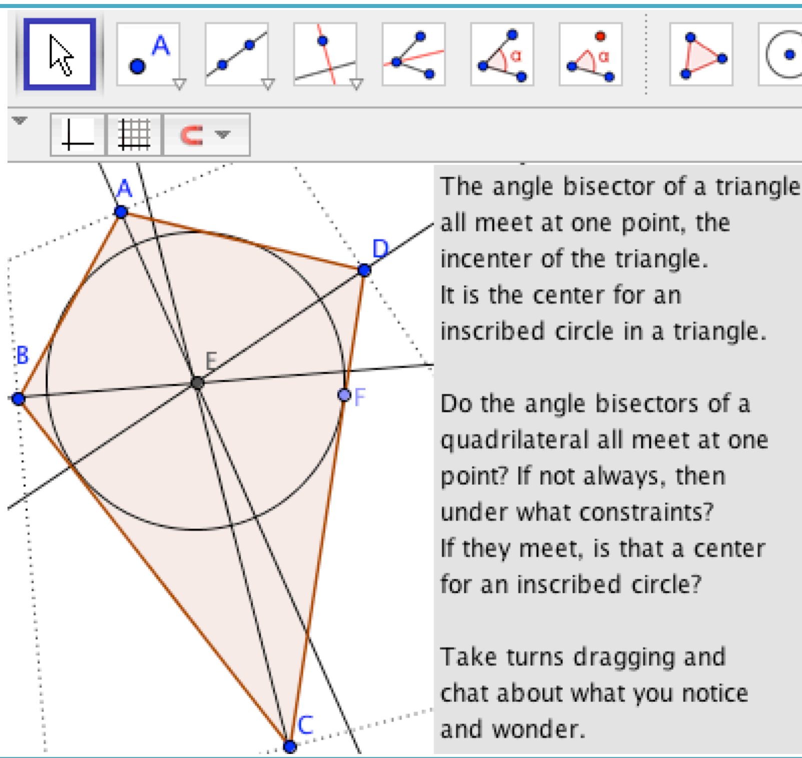
Section 14.03 A Proof about Areas of Quadrilaterals

Can you prove that the quadrilateral formed by connecting the midpoints is a parallelogram with half the area of the original quadrilateral?



Section 14.04 Angle Bisectors of Quadrilaterals

In certain cases – but not in all cases – the angle bisectors of a quadrilateral all meet in one point, which can serve as the center of an inscribed circle.



What we noticed:

What we wondered:

Topic 15: Individual Transition Activity

Before working on this topic with your team, you should read **Tour 7: "Creating VMT Chat Rooms"** near the end of this booklet. It will show you how to create your own VMT chat rooms so you can invite friends to work on dynamic-geometry topics you define.

(**Note**: *This is a topic for you to do on your own. It will help you to continue to use GeoGebra in your future mathematics studies*.)

In this topic, you will use the Theorem of Thales (see Section 3.01) to construct a tangent to a circle using the geometry tools of GeoGebra that you already know. Then, you will do the same thing in a very different way, using the algebra tools of GeoGebra. This will introduce you to GeoGebra's tools that go beyond geometry to algebra and other area of dynamic mathematics. GeoGebra integrates the different areas in interesting ways. You will see how geometry and algebra are integrated in this topic. As Descartes (the medieval French philosopher, scientist, and mathematician) did in his analytic geometry and calculus, the location of a point is now given (x, y) coordinates on a grid.

Section 15.01 Construct Tangents to a Circle Geometrically

Task: Given a circle and an arbitrary point outside the circle, construct the tangents to the circle going through the point.

A "tangent" to a circle touches the circle at one and only one point. The tangent is perpendicular to a radius from the center of the circle to the point of tangency (see Section 8.07).

You can use Thales Theorem to construct the tangent through a point C to a circle with center A if you construct another circle whose diameter is segment AC. According to Thales Theorem, the angle formed between line CE and a line from A to point E (at the intersection of the two circles) will be a right angle, making line CE a tangent to the circle centered at A.

Discuss in chat what tools to use and how to do the construction. Take turns doing the construction and checking the dependencies.

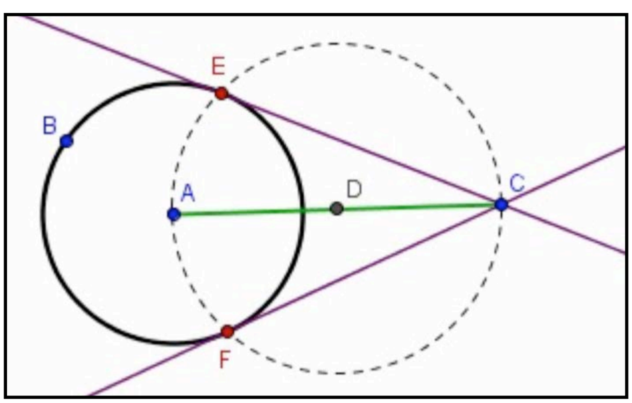


Figure 15-1. A geometric construction of tangents.

Construct a circle with center at point A, going through a point B. Also, construct a point C outside the circle.

Then, construct the tangents to the circle, going through point C, as indicated in the image above. Note that there are two tangents and that the diagram is symmetric along AC.

Hint: construct segment AC and its midpoint D. Then construct a circle centered on D and passing through C and A. Now the tangent goes from point C to point E or F.

Construct a supplementary segment AE and the angle AEC to check if the tangent is perpendicular to the radius. Drag point C to see if the relationships hold dynamically.

Explain in your summary what you observed in this activity. What is the Theorem of Thales and how did it help you to construct the tangent to the circle? State this in your own words and post it in the chat.

Section 15.02 Construct Tangents with the Algebra Interface

In this activity, you will use the Algebra interface of GeoGebra to do the same construction you did in the last activity with geometry tools. This will introduce you to the multiple representations of GeoGebra. (GeoGebra also supports statistics, spreadsheets, algebra, solid geometry, 3D geometry, trigonometry, calculus and other areas of mathematics—but that is all for you to explore on your own or with your friends later.)

GeoGebra has the ability to deal with algebra variables and equations as well as geometry points and lines. These two views are coordinated in GeoGebra: an expression in the algebra window corresponds to an object in the geometry window and vice versa.

GeoGebra's user interface consists of a graphics window and an algebra window. On the one hand, you can operate the provided geometry tools with the mouse in order to create geometric constructions in the graphics window. On the other hand, you can directly enter algebraic input, commands and functions into the input field (at the bottom of the tab) by using the keyboard. While the graphical representation of all objects is displayed in the graphics window, their algebraic numeric representation is shown in the algebra window.

GeoGebra offers algebraic input and commands in addition to the geometry tools. Every geometry tool has a matching algebra command. In fact, GeoGebra offers more algebra commands than geometry tools.

Tips and Tricks:

- Name a new object by typing in `name =` in front of its algebraic representation in the Input Field. Example: `P = (3, 2)` creates point P.
- Multiplication needs to be entered using an asterisk or space between the factors. Example: `a*x` or `a x`
- Raising to a power is entered using ^. Example: `f(x) = x^2 + 2*x + 1`
- GeoGebra is case sensitive! Thus, upper and lower case letters must not be mixed up. Note: Points are always named with upper case letters. Example: `A = (1, 2)`
- Segments, lines, circles, functions… are always named with lower case letters. Example: circle `c: (x – 2)^2 + (y – 1)^2 = 16`
- The variable x within a function and the variables x and y in the equation of a conic section always need to be lower case. Example: `f(x) = 3*x + 2`
- If you want to use an object within an algebraic expression or command, you need to create the object before using its name in the input field. Examples: `y = m x + b` creates a line whose parameters are already existing values m and b (e.g. numbers / sliders). `Line[A, B]` creates a line through existing points A and B.

- Confirm an expression you entered into the input field by pressing the Enter key.
- Open the "Input Help" panel for help using the input field and commands by clicking the "?" button next to the input field.
- Error messages: Always read the messages – they could possibly help to fix the problem!
- Commands can be typed in or selected from the list next to the input field. Hint: If you do not know which parameters are required within the brackets of a certain command, type in the full command name and press key F1. A pop-up window appears explaining the syntax and necessary parameters of the command.
- Automatic completion of commands: After typing the first two letters of a command into the input field, GeoGebra tries to complete the command. If GeoGebra suggests the desired command, hit the Enter key in order to place the cursor within the brackets. If the suggested command is not the one you wanted to enter, just keep typing until the suggestion matches.

Check out the list of textual algebraic commands next to the Input Help and look for commands corresponding to the geometry tools you have learned to use.

Preparation

Select the "Algebra View" and "Graphics View" from the View menu. Use the View menu and the Graphics View tool bar to make sure the Input Bar, the Algebra window and the Coordinate Axes are all displayed (as in Figure 15-2).

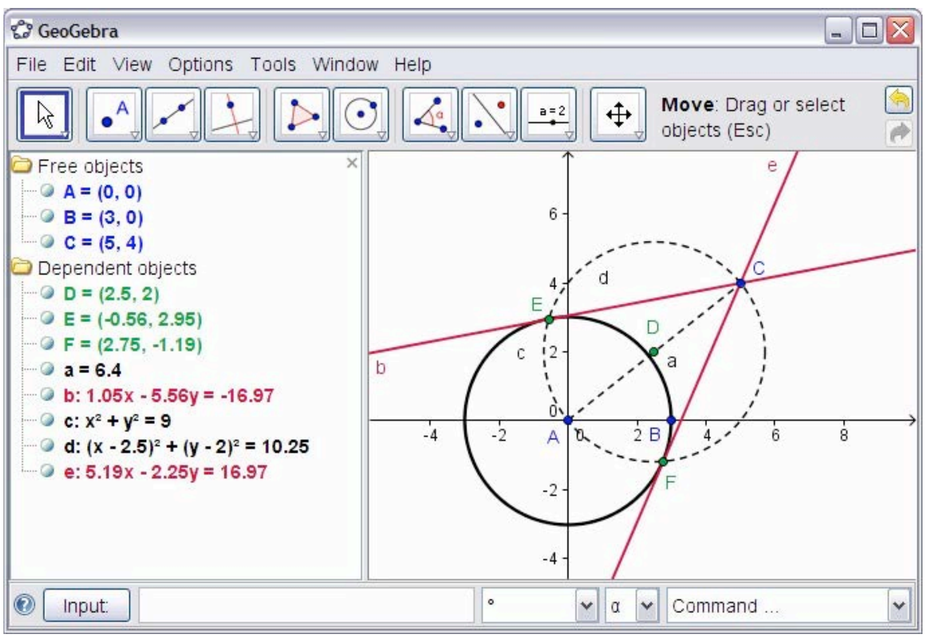


Figure 15-2. An algebraic construction of tangents.

Construction Process

Enter the following entries into the Algebra input field:

Step	Input Field entry	Object created
1	A = (0, 0)	Point A
Hint: Make sure to close the parenthesis.		
2	(3, 0)	Point B

Hint: If you do not specify a name objects are named in alphabetical order.		
3	c = Circle[A, B]	Circle with center A through point B
Hint: Circle is a dependent object		

Note: GeoGebra distinguishes between free and dependent objects. While free objects can be directly modified either using the mouse or the keyboard, dependent objects adapt to changes of their parent objects. It does not matter how an object was initially created (by mouse or keyboard)!

Hint 1: Activate Move mode and double click an object in the algebra window in order to change its algebraic representation using the keyboard. Hit the Enter key once you are done.

Hint 2: You can use the arrow keys to move free objects in a more controlled way. Activate move mode and select the object (e.g., a free point) in either window. Press the up / down or left / right arrow keys in order to move the object in the desired direction.

4	C = (5, 4)	Point C
5	s = Segment[A, C]	Segment AC
7	D = Midpoint[s]	Midpoint D of segment AC

8	d = Circle[D, C]	Circle with center D through point C
9	Intersect[c, d]	Intersection points E and F of the two circles
10	Line[C, E]	Tangent through points C and E
11	Line[C, F]	Tangent through points C and F

Checking and Enhancing the Construction

Perform the drag-test in order to check if the construction is correct.

Change properties of objects in order to improve the construction’s appearance (e.g., colors, line thickness, auxiliary objects dashed, etc.).

Discussion

Did any problems or difficulties occur during the construction process?

Which version of the construction (mouse or keyboard) do you prefer and why?

Why should we use keyboard input if we could also do it using tools?

Hint: There are algebra commands available that have no equivalent geometric tool.

Does it matter in which way an object was created? Can it be changed in the algebra window (using the keyboard) as well as in the graphics window (using the mouse)?

Congratulations

Your team has completed the core topics of "*Topics in Dynamic Geometry for Virtual Math Teams*"! You can now explore lots of mathematics—either in small groups or on your own—using GeoGebra. You can create your own VMT chat rooms and invite people to collaborate on your own topics. In addition, you can download GeoGebra from www.geogebra.org to use on your own. Enjoy, explore, create!

Following are some open-ended topics that invite you and your team to create new forms of mathematics using the skills you have just learned.

What we noticed:

What we wondered:

Topic 16: Proving with Dependencies

(**Note**: *This is a long topic. Explore this topic on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the topic after your team session*.)

This topic explores how identifying dependencies in a dynamic-geometry construction can help you prove a conjecture about that construction.

In Euclid's construction of an equilateral triangle, he made the lengths of the three sides of the triangle dependent on each other by constructing each of them as radii of congruent circles. Then to prove that the triangle was equilateral, all he had to do was to point out that the lengths of the three sides of the triangle were all radii of congruent circles and therefore they were all equal.

In this topic, you will look at a more complicated conjecture about triangles, namely relationships having to do with the incenter of a triangle. Remember that the "incenter" of a triangle is located at the intersection of the bisectors of the three vertex angles of the triangle (see Section 6.08). The conjecture has a number of parts:

1. The three bisectors of the vertex angles all meet at a single point. (It is unusual for three lines to meet at one point. For instance, do the angle bisectors of a quadrilateral always intersect at one point?)
2. The incenter of any triangle is located inside of the triangle. (Other kinds of centers of triangles are sometimes located outside of the triangle. For instance, can the circumcenter of a triangle be outside the triangle?)
3. Line segments that are perpendiculars to the three sides passing through the incenter are all of equal length.
4. A circle centered on the incenter is inscribed in the triangle if it passes through a point where a perpendicular from the incenter to a side intersects that side.
5. The inscribed circle is tangent to the three sides of the triangle.

These may seem to be surprising conjectures for a simple triangle. After all, a generic triangle just consists of three segments joined together at their endpoints. Why should a triangle always have these rather complicated relationships?

Construct the incenter of a general dynamic triangle and observe how the dependencies of the construction suggest a proof for these five parts of the conjecture about a triangle's incenter.

Section 16.01 Construct an incenter with a custom incenter tool

In a previous topic, Section 6.08, you programmed your own custom incenter tool. Open the .ggt file for it with the menu "File" | "Open." Then select your custom incenter tool or use the custom "my_incenter" tool that is already on the tool bar in this tab. Click on three points A, B and C to define the vertices of a triangle. The tool will automatically construct the triangle as a polygon ABC and a point D at the incenter of triangle ABC. You can then use a perpendicular tool to construct a line through point D and perpendicular to side AB of the triangle at point E. Next construct a circle centered on D and passing through E. That is the state shown in Figure 16-1.

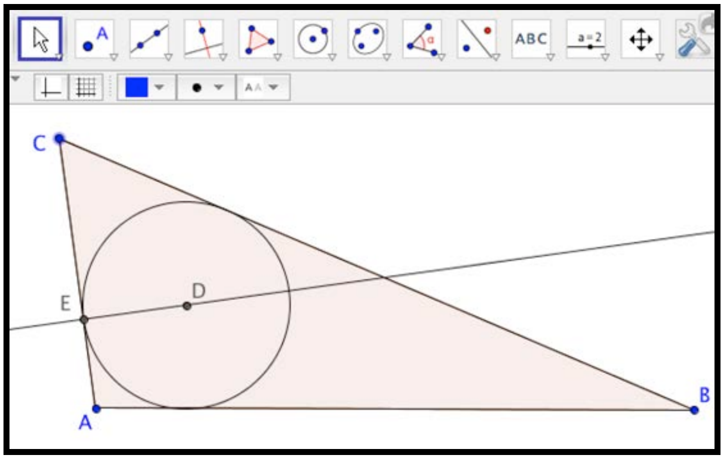


Figure 16-1: Given triangle ABC, its incenter D has been constructed with a custom tool.

Drag this figure around. Can you see why the five parts of the conjecture should always be true?

Add in the three angle bisectors and the other two perpendiculars through point D. You can change the properties of the perpendicular segments to show the value of their lengths. Drag the figure now. Do the three angle bisectors all meet at the same point? Is that point always inside the triangle? Are the three perpendicular segments between D and the triangle sides all equal? Is the circle through D always inscribed in the triangle? Is it always tangent to the three sides? Can you explain why these relationships are always true? Can you identify dependencies built into the construction that constrain the circle to move so it is always tangent to all three sides?

Section 16.02 Construct the incenter with standard GeoGebra tools

This time, construct the incenter without the custom tool, simply using the standard GeoGebra tools. Construct a simple triangle ABC. Use the angle-bisector tool (pull down from the perpendicular-line tool) to construct the three angle bisectors. They all meet at point D, which is always inside the circle. Now construct perpendiculars from D to the three sides, defining points E, F and G at the intersections with the sides. Segments DE, DF and DG are all the same length. Construct a circle centered on D and passing through E. The circle is tangent at E, F and G. That is the state shown in Figure 16-2.

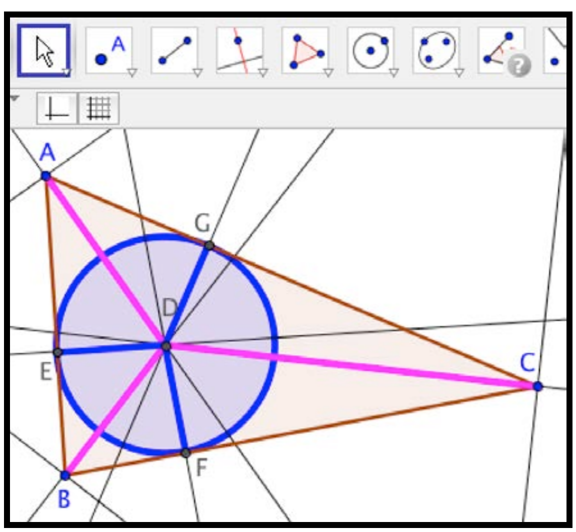


Figure 16-2: Given a triangle ABC, its incenter has been constructed with the GeoGebra angle-bisector tool.

Drag this figure around. Can you see why the five parts of the conjecture should always be true? Can you identify dependencies built into the construction that constrain the incenter to move in response to movements of A, B or C so that the five parts of the conjecture are always true?

Section 16.03 Construct the incenter with elementary line and circle tools

A formal deductive proof of the conjecture would normally start from a completed diagram like Figure 16-2. Rather than starting from this completed figure, instead proceed through the construction step by step using just elemental straightedge (line) and compass (circle) tools. Avoid using the angle-bisector tool, which hides the dependencies that make the produced line a bisector.

As a first step, construct the angle bisectors of vertex A of a general triangle ABC (see Figure 16-3). Construct the angle bisector by constructing a ray AF that goes from point A through some point F that lies between sides AB and AC and is equidistant from both these sides. This is the dependency that defines an angle bisector: that it is the locus of points equidistant from the two sides of the angle. The constraint that F is the same distance from sides AB and AC is constructed as follows: First construct a circle centered on A and intersecting AB and AC—call the points of intersection D and E. Construct perpendiculars to the sides at these points. The perpendiculars necessarily meet between the sides—call the point of intersection F. Construct ray AF.

AF bisects the angle at vertex A, as can be shown by congruent right triangles ADF and AEF. (Right triangles are congruent if any two sides are congruent because of the Pythagorean relationship, which guarantees that the third sides are also congruent.) This shows that angle BAF equals angle CAF, so that ray AF bisects the vertex angle CAC into two equal angles. By constructing perpendiculars from the angle sides to any point on ray AF, one can show by the corresponding congruent triangles that every point on AF is equidistant from the sides of the triangle.

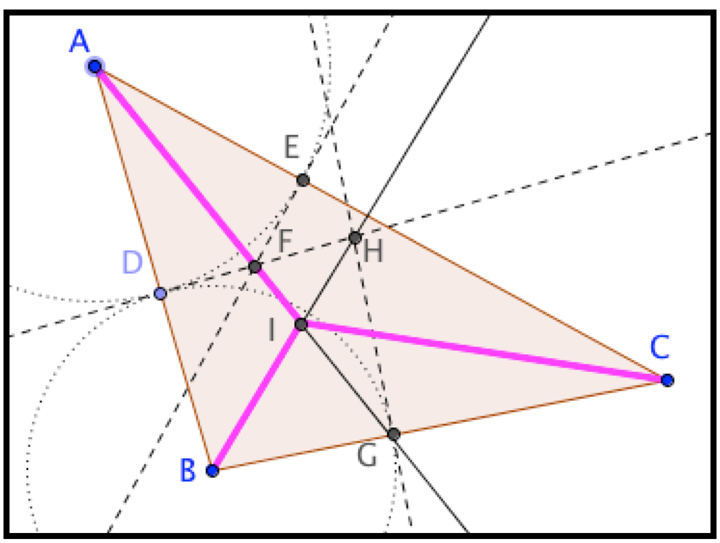


Figure 16-3: Given a triangle ABC, its incenter has been constructed with basic tools.

As the second step, construct the bisector of the angle at vertex B. First construct a circle centered on B and intersecting side AB at point D—call the circle's point of intersection with side BC point G. Construct perpendiculars to the sides at these points. The perpendiculars necessarily meet between the sides AB and BC—call the point of intersection H. H has been constructed to lie between AB and BC. Construct ray BH. BH bisects the angle at vertex B, as can be shown by congruent right triangles BDH and BGH, as before.

For the third step, mark the intersection of the two angle-bisector rays AF and BH as point I, the incenter of triangle ABC. Construct segment CI. You can see that CI is the angle bisector of the angle at the third vertex, C in Figure 16-4 as follows. Construct perpendiculars IJ, IK, IL from the incenter to the three sides. We know that I is on the bisector of angles A and B, so IJ=IK and IJ=IL. Therefore, IK=IL, which means that I is also on the bisector of angle C. This implies that triangles CKI and CLI are congruent, so that their angles at vertex C are equal and CI bisects angle ACB. You have now shown that point I is common to the three angle bisectors of an arbitrary triangle ABC. In other words, the three angle bisectors meet at one point. The fact that the bisectors of the three angles of a triangle are all concurrent is a direct consequence of the dependencies you imposed when constructing the bisectors.

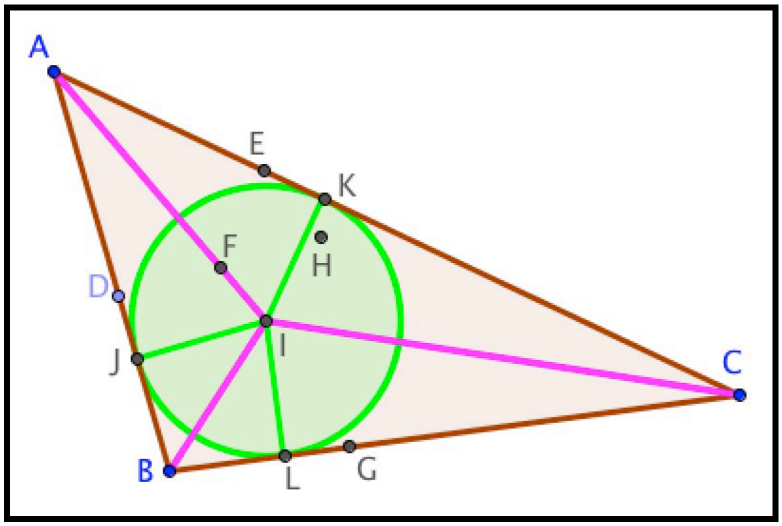


Figure 16-4: The incenter, I, of triangle ABC, with equal perpendiculars IJ, IK, and IL, which are radii of the inscribed circle.

Now construct a circle centered on the incenter, with radii IJ, IK, and IL. You have already shown that the lengths of IJ, IK and IL are all equal and you constructed them to be perpendicular to the triangle sides. The circle is inscribed in the triangle because it is tangent to each of the sides. (Remember, a circle is tangent to a line if its radius to the intersection point is perpendicular to the line.)

Drag the vertex points of the triangle to show that all the discussed relationships are retained dynamically.

Review the description of the construction. Can you see why all of the parts of the conjecture have been built into the dependencies of the figure? None of the parts seem surprising now. They were all built into the figure by the various detailed steps in the construction of the incenter.

When you used the custom incenter tool or even the GeoGebra angle-bisector tool, you could not notice that you were thereby imposing the constraint that DF=EF, etc. It was only by going step-by-step that you could see all the dependencies that were being designed into the figure by construction. The packaging of the detailed construction process in special tools obscured the imposition of dependencies. This is the useful process of "abstraction" in mathematics: While it allows you to build quickly upon past accomplishments, it has the unfortunate unintended consequence of hiding what is taking place in terms of imposing dependencies.

In Figure 16-5, only the elementary "straightedge and compass" tools of the point, line and circle have been used. The perpendiculars have been constructed without even using the perpendicular tool. All of the geometric relationships, constraints and dependencies that are at work in Figure 16-1 or Figure 16-4 are visible in Figure 16-5. This construction involved the creation of 63 objects (points, lines and circles). It is becoming visually confusing. That is why it is often useful to package all of this in a special tool, which hides the underlying complexity. It is wonderful to use these powerful tools, as long as you understand what dependencies are still active behind the visible drawing.

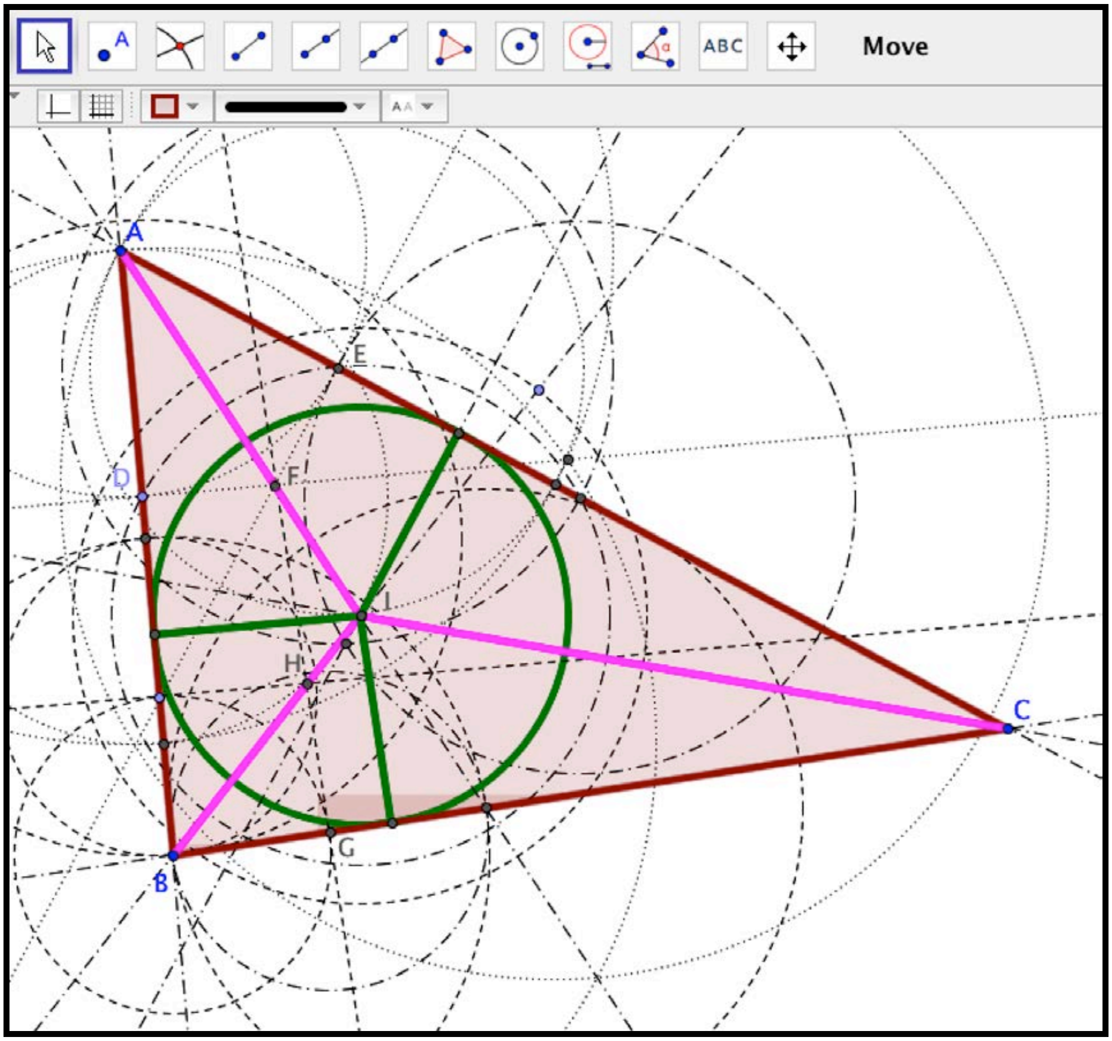


Figure 16-5: Given a triangle ABC, its incenter has been constructed with only elementary point, line and circle tools.

Section 16.04 Construct Euler's Segment and the Nine Point Circle

In Section 6.10, you explored Euler's Nine-Point Circle. The construction of this circle involves the orthocenter, the centroid and the circumcenter in addition to the incenter.

Construct the four different centers and the nine-point circle shown in Figure 16-6 using just elementary point, line and circle tools. You can start with the construction shown in Figure 16-5. When you are finished, drag the triangle to see that the relationships shown in Figure 16-6 remain dynamically. Using the dependencies that you have constructed, explain why the nine-point circle passes through those nine points.

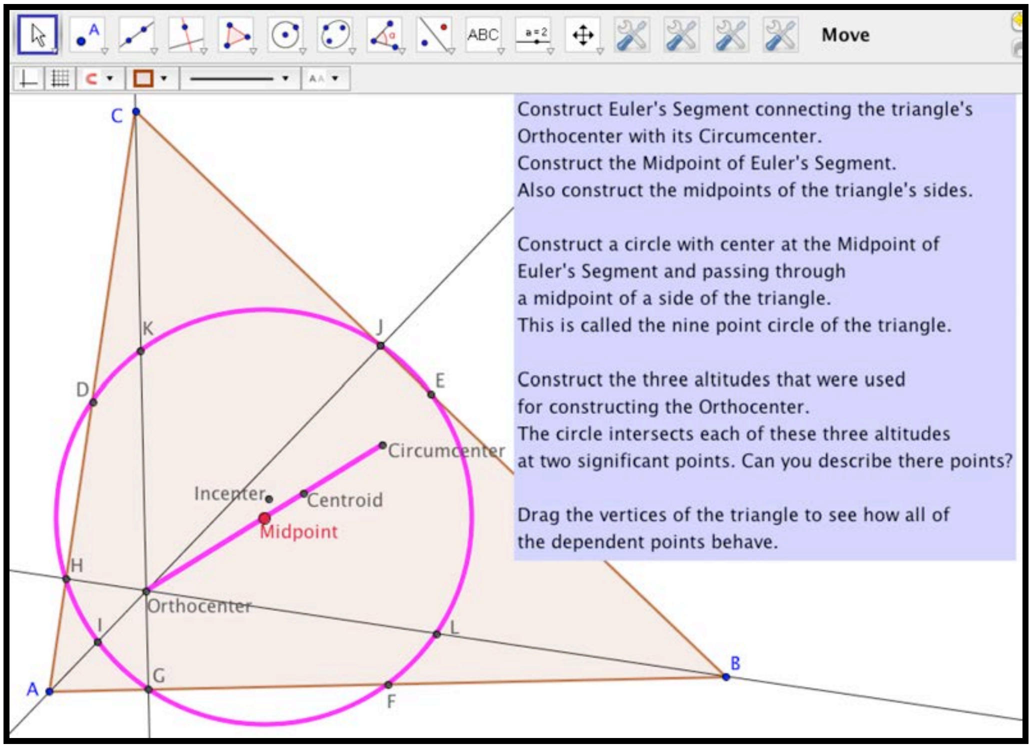


Figure 16-6: Euler's Segment and the Nine Point Circle of a triangle.

This construction is quite complicated. You will soon see why abstracting sequences of constructions into specialized tools is handy. Just as construction of dependencies is closely related to proving conjectures, the packaging of construction sequences in custom tools can be closely related to establishing theorems or propositions. In similar ways, the tools and the theorems help you to build up systems of complex mathematics.

While dragging figures that have already been constructed and even constructing with a large palette of construction tools can be extremely helpful for exploring geometric relationships and coming up with conjectures to investigate, such an approach can give the misimpression that the relationships are abstract truths to be accepted on authority and validated through routinized deduction. It is also important—at least when you want a deeper understanding of what is going on—to be able to construct figures for yourself, using the basic tools of line and circle (analogous to the classic tools of straightedge and compass). You should then understand how other tools are built up from the elementary construction methods and should know how to create your own custom tools, for which you understand the incorporated procedures.

Section 16.05 Prove Euler's Segment using its Construction Dependencies

Most discussions of Euler's segment and the nine-point circle talk about them as wonderful mysteries: how can a common, simple triangle contain such marvelous relationships and surprising but elegant complexities? By now, you should suspect that these relationships are not inherent in the essence of the plain triangle, but are built into the Euler segment by the construction of the various centers. We have just seen that there are not just the three segments that form the triangle and one additional segment in the middle, but a couple hundred inter-dependent points, lines and circles.

In fig, we see two diagrams highlighting a small number of the lines used in the construction of the centers and the segment that joins them. Accompanying the diagrams is a textual sketch of a proof of the **Euler Line Theorem**:

The orthocenter, the circumcenter and the centroid of a triangle are co-linear and the centroid is a third of the way between the orthocenter and the circumcenter.

The informal proof relies upon the dependencies built into the construction of the orthocenter, circumcenter and centroid by altitudes, perpendicular bisectors and medians of the triangle.

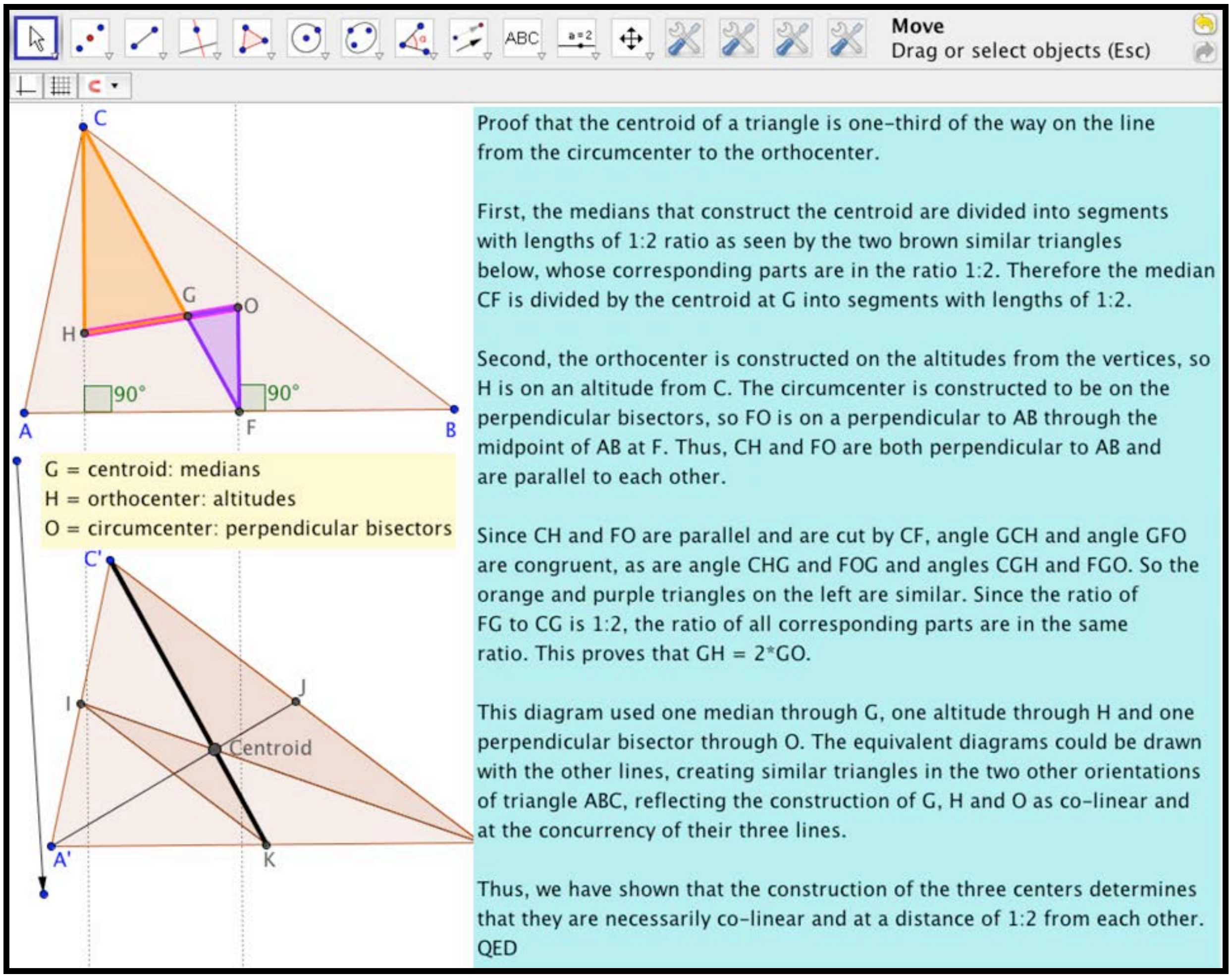


Figure 16-7: Sketch of a proof of Euler's segment.

Euler's theorems sparked renewed interest in geometry research, involving more centers of triangles, further axiomatization and the new field of topology. Hofstadter (1997)—whose popular books about math are always entertaining and thought provoking—extended the exploration of the Euler segment and circle in an interesting way using dynamic geometry. Venema (2013) provides a more systematic introduction to exploring advanced, post-Euclidean geometry using GeoGebra (see the "Further Readings" Tour).

What we noticed:

What we wondered:

Topic 17: Transforming a Factory

(**Note**: *This is a long topic. Explore this topic on your own before your team meets together. Do not spend too much time on any one section. Continue to work on the topic after your team session*.)

In this topic, you will conduct mathematical studies to help design a widget factory. The movement of polygon-shaped widgets, which the factory processes, can be modeled in terms of rigid transformations of polygons. You will explore physical models and GeoGebra simulations of different kinds of transformations of widgets. You will also compose multiple simple transformations to create transformations that are more complex, but might be more efficient. You will apply what you learned to the purchase of widget-moving machines in a factory.

Designing a Factory

Suppose you are the mathematician on a team of people designing a new factory to process widgets. In the factory, special machines will be used to move heavy widgets from location to location and to align them properly. There are different machines available for moving the widgets. One machine can flip a widget over; one can slide a widget in a straight line, one can rotate a widget. As the mathematician on the team, you are supposed to figure out the most efficient way to move the widgets from location to location and to align them properly. You are also supposed to figure out the least expensive set of machines to do the moving.

The factory will be built on one floor and the widgets that have to be moved are shaped like flat polygons, which can be laid on their top or bottom. Therefore, you can model the movement of widgets as rigid transformations of polygons on a two-dimensional surface. See what you can learn about such transformations.

Experiment with Physical Transformations

Before you get together with your team online, take a piece of cardboard and cut out an irregular polygon. This polygon represents a widget being processed at the factory. Imagine it is moved through the factory by a series of machines that flip it, slide it and rotate it to move it from one position to another on the factory floor.

Place the polygon on a piece of graph paper and trace its outline. Mark that as the “start state” of the polygon. Move the cardboard polygon around. Flip it over a number of times. What do you notice? Rotate it around its center or around another point. Slide it along the graph paper. Finally, trace its outline again and mark that as the “end state” of the transformation.

Place the polygon at its start state position. What is the simplest way to move it into its finish state position? What do you notice about different ways of doing this?

Now cut an equilateral triangle out of the cardboard and do the same thing. Is it easier to transform the equilateral triangle from its start state to its finish state than it was for the irregular polygon? What do you notice about flipping the triangle? What do you notice about rotating the triangle? What do you notice about sliding the triangle?

The other people in your group cannot see your cardboard polygon moving. Explain to them in the chat, what you did and what you noticed. Share what you are wondering about transformations of polygons and discuss these questions.

Transformational Geometry

In a previous activity with triangles, you saw that there were several kinds of rigid transformations of triangles that preserved the measures of the sides and the angles of the triangles. You also learned about GeoGebra tools that could transform objects in those ways, such as:

- Reflect Object about Line
- Rotate Object around Point by Angle
- Translate Object by Vector

These tools can transform any polygon in these ways and preserve the measures of their sides and angles. In other words, these geometric transformations can model the movement of widgets around the factory.

Composing Multiple Transformations

In addition to these three kinds of simple transformations, you can "compose" two or more of these to create a more complicated movement. For instance, a "glide reflection" consists of reflecting an object about a line and then translating the reflected object by a vector. Composing three transformations means taking an object in its start state, transforming it by the first transformation into a second state, then transforming it with the second transformation from its second state into a third state, and finally transforming it with the third transformation from its third state into its end state. You can conceive of this as a single complex transformation from the object's start state to its end state.

The study of these transformations is called "transformational geometry." There are some important theorems in transformational geometry. Maybe you can discover some of them and even find some of your own. These theorems can tell you what is possible or optimal in the widget factory's operation.

An Example of Transformations in GeoGebra

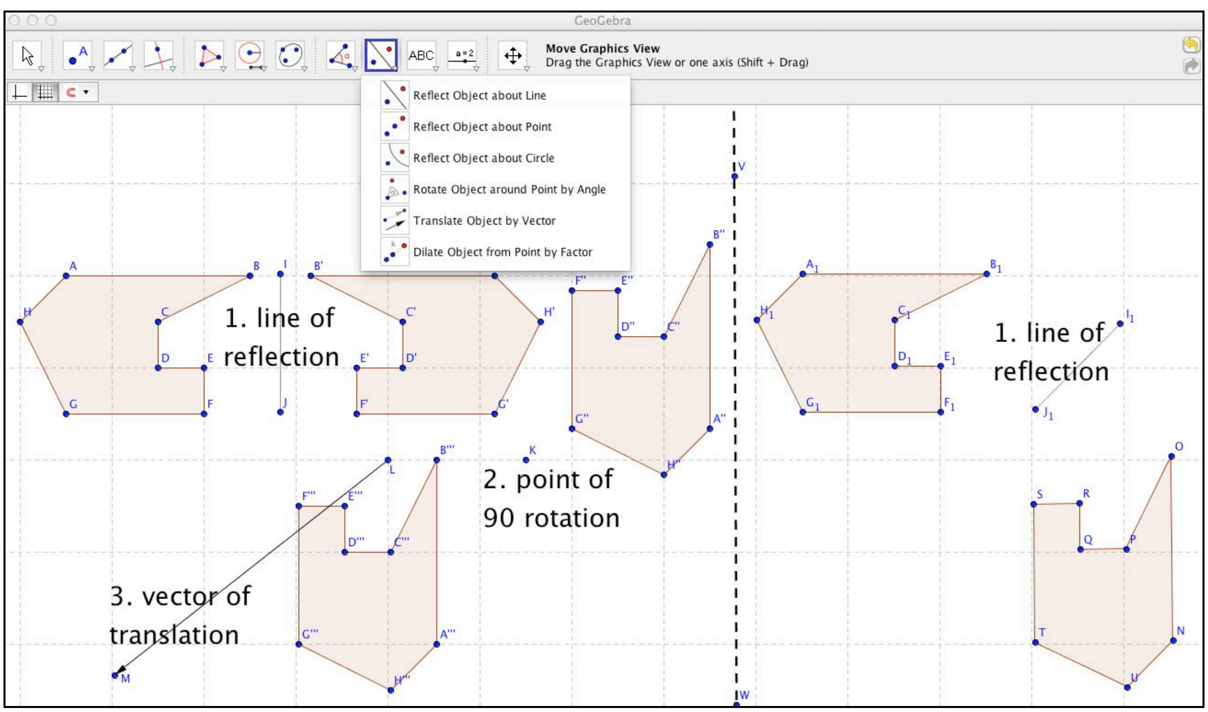


Figure 17-1. Transformations of a polygon.

In Figure 17-1, an irregular polygon ABCDEFGH has gone through 3 transformations: a reflection (about line IJ), a rotation (about point K), and a translation (by vector LM). A copy of the polygon has gone through just 1 transformation (a reflection about line I_1J_1) and ended in the same relative position and orientation. There are many sequences of different transformations to transform a polygon from a particular starting state (position and orientation) to an end state (position and orientation). Some possible alternative sequences are simpler than others.

Discuss with your group how you want to proceed with each of the following explorations. Do each one together with your group, sharing GeoGebra constructions. Save a construction view for each exploration to include in your summary. Discuss what you are doing, what you notice, what you wonder, how you are constructing and transforming polygons, and what conjectures you are considering.

Section 17.01 Exploration 1

Consider the transformations in Figure 17-1. Drag the line of reflection (line IJ), the point of rotation (point K), the translation vector (vector LM) and the alternative line of reflection (line NO). How does this affect your ability to substitute the one reflection for the sequence on three transformations? What ideas does this give you for the lay-out of work-flow in a factory?

Section 17.02 Exploration 2

Consider just simple rotations of an irregular polygon. Suppose you perform a sequence of several rotations of the polygon widget around different points. Would it be possible to get from the start state to the end state in a fewer number of rotations? In other words, can the factory be made more efficient?

Consider the same question for translations of widgets.

Consider the same question for reflections of widgets.

Section 17.03 Exploration 3

Perhaps instead of having a machine in the factory to flip widgets and a different machine to move the widgets, there should be a machine that does both at the same time. Consider a composite transformation, like a glide reflection composed of a reflection followed by a translation. Suppose you perform a sequence of glide reflections on an irregular polygon. Does it matter what order you perform the glide reflections? Would it be possible to get from the start state to the end state in a fewer number of glide reflections?

Does it matter if a glide reflection does the translation before or after the reflection?

Consider the same questions for glide rotations.

Section 17.04 Exploration 4

Factory managers always want to accomplish tasks as efficiently as possible. What is the minimum number of simple transformation actions needed to get from any start state of the irregular polygon in the figure to any end state? For instance, can you accomplish any transformation with three (or fewer) simple actions: one reflection, one rotation and one translation (as in the left side of the Figure 17-1)? Is it always possible to achieve the transformation with fewer than three simple actions (as in the right side of the figure)?

Section 17.05 Exploration 5

Factory managers always want to save costs. If they can just buy one kind of machine instead of three kinds, that could save money. Is it always possible to transform a given polygon from a given start state to a specified end state with just one *kind* of simple transformation – e.g., just reflections, just rotations or just translations? How about with a certain composition of two simple kinds, such as a rotation composed with a translation or a reflection composed with a rotation?

Section 17.06 Exploration 6

Help the factory planners to find the most direct way to transform their widgets. Connect the corresponding vertices of the start state and the end state of a transformed polygon. Find the midpoints of the connecting segments. Do the midpoints line up in a straight line? Under what conditions (what kinds of simple transformations) do the midpoints line up in a straight line? Can you prove why the midpoints line up for some of these conditions?

If you are given the start state and the end state of a transformed polygon, can you calculate a transformation (or a set of transforms) that will achieve this transformation? This is called "reverse engineering" the transformation. *Hint*: constructing the perpendicular bisectors of the connecting segments between corresponding vertices may help in some conditions (with some kinds of simple transformations).

Section 17.07 Exploration 7

Different factories process differently shaped widgets. How would the findings or conjectures from Explorations 1 to 5 be different for a widget which is an equilateral triangle than they were for an irregular polygon? How about for a square? How about for a hexagon? How about for other regular polygons?

Section 17.08 Exploration 8

So far, you have only explored rigid transformations – which keep the corresponding angles and sides congruent from the start state to the end state. What if you now add dilation transformations, which keep corresponding angles congruent but change corresponding sides proportionately? Use the Dilate-Object-from-Point-by-Factor tool and compose it with other transformations. How does this affect your findings or conjectures from Explorations 1 to 5? Does it affect your factory design if the widgets produced in the factory can be uniformly stretched or shrunk?

Factory Design

Consider the factory equipment now. Suppose the factory needs machines for three different complicated transformations and the machines have the following costs: a reflector machine $20,000; a rotator machine $10,000; a translator machine $5,000. How many of each machine would you recommend buying for the factory?

What if instead they each cost $10,000?

Summarize

Summarize your trials with the cardboard polygons and your work on each of the explorations in your chat discussion. What did you notice that was interesting or surprising? State your conjectures or findings. Can you make some recommendations for the design of the factory? If you did not reach a conclusion, what do you think you would have to do to reach one? Do you think you could develop a formal proof for any of your conjectures in the explorations?

What we noticed:

What we wondered:

Topic 18: Navigating Taxicab Geometry

(**Note**: *This is a long topic. Explore this topic on your own before your team meets together. Do not spend too much time on any one tab. Continue to work on the topic after your team session*.)

In this topic, you will explore an invented transformational geometry that has probably never been analyzed before (except by other teams who did this topic). Taxicab geometry is considered a "non-Euclidean" form of geometry, because in taxicab geometry the shortest distance between two points is not necessarily a straight line. Although it was originally considered by the mathematician Minkowski (who helped Einstein figure out the non-Euclidean geometry of the universe), taxicab geometry can be fun for amateurs to explore. Krause (1986) wrote a nice introductory book on it that uses an inquiry approach, mainly posing thought-provoking problems for the reader. Gardner devoted his column on mathematical games in *Scientific American* to clever extensions of it in November 1980.

Section 18.01 An Invented Taxicab Geometry

There is an intriguing form of geometry that is called "taxicab geometry" because all lines, objects and movements are confined to a grid. It is like a grid of streets in a city where all the streets either run north and south or they run east and west. For a taxicab to go from one point to another in the city, the shortest route involves movements along the grid. Taxicab geometry provides a model of urban life and navigation.

In taxicab geometry as we will define it for this topic, all points are at grid intersections, all segments are confined to the grid lines and their lengths are confined to integer multiples of the grid spacing. The only angles that exist are multiples of 90° — like 0°, 90°, 180°, 270° and 360°. Polygons consist of segments connected at right angles to each other.

How would you define the rigid transformations of a polygon in taxicab geometry? Discuss this with your team and decide on definitions of rotation, translation and reflection for this geometry. (See Section 17.01 for an example.)

Use GeoGebra with the grid showing. Use the grid icon on the lower toolbar to display the grid; the pull-down menu from the little triangle on the right lets you activate "Snap to Grid" or "Fixed to Grid. The menu "Options" | "Advance" | "Graphics" | "Grid" lets you modify the grid spacing when you have Control. Only place points on the grid intersections.

Construct several taxicab polygons. Can you use GeoGebra's transformation tools (rotation, translation and reflection)? Or do you need to define custom transformation tools for taxicab geometry? Or do you have to manually construct the results of taxicab transformations? Rotate (by 90° or 180°), translate (along grid lines to new grid intersections) and reflect (across segments on grid lines) your polygons.

Section 18.02 Explore Taxicab Transformational Geometry

Now consider the question that you explored for classical transformational geometry in Topic 17: Can all complex transformations be accomplished by just one kind of transformation, such as reflection on the grid? What is the minimum number of simple transformations required to accomplish any change that can be accomplished by a series of legal taxicab transformations?

In Euclidean geometry, if a right triangle has sides of length 3 and 4, the hypotenuse is 5, forming a right triangle with integer lengths. In taxicab geometry, it seems to have a hypotenuse of 7, which can be drawn along several different paths. In the grid shown (Figure 18-1), a 3-4-7 right triangle ABC (green) has been reflected about segment IJ (blue), then translated by vector KL (blue), and then rotated 180° clockwise about point C'' (brown). Equivalently, ABC (green) has been reflected about segment BC (red), then reflected about the segment going down from C'_1 (red), and then reflected about segment A'''M''' (brown). Thus, in this case, the composition of a reflection, a translation and a rotation can be replicated by the composition of just reflections, three of them.

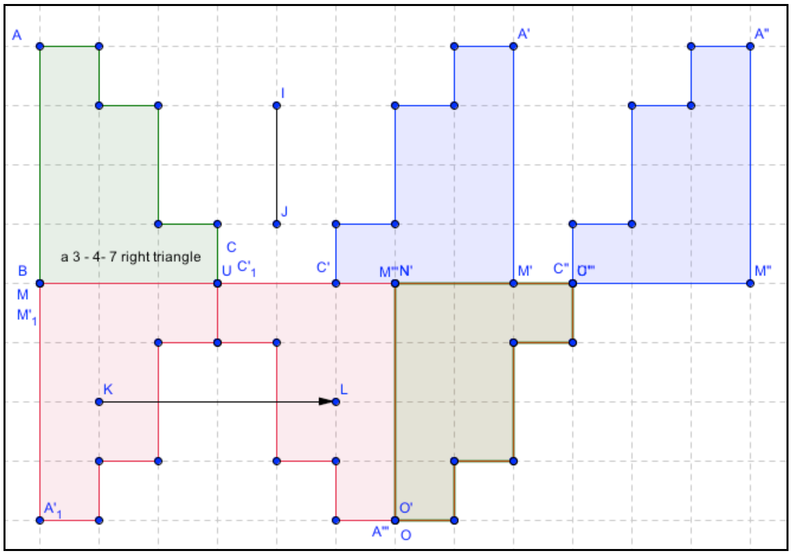


Figure 18-1. Transformations in taxicab geometry.

Section 18.03 Explore Kinds of Polygons and their Symmetries

What distinct kinds of "polygons" are possible in taxicab geometry? Can you work out the hierarchy of different kinds of "taxicab polygons" with each number of sides? E.g., are there right or equilateral taxicab triangles? Are there square or parallelogram taxicab quadrilaterals?

Discuss and Summarize

What has your group noticed about taxicab transformational geometry? What have you wondered about and investigated? Do you have conjectures? Did you prove any theorems in this new geometry? What questions do you still have?

Be sure to list your findings in the chat, as well as wonderings that you would like to investigate in the future.

Congratulations!

You have now completed the topics in this series. You are ready to explore dynamic geometry and GeoGebra on your own or to propose further investigations for your team. You can also create new VMTwG chat rooms with your own topics and invite people to work together in them. If you are a teacher, you can set up rooms with topics from this series for groups of your students.

What we noticed and what we wondered:

TOURS: Tutorials on VMT and GeoGebra

Tour 1: Joining a Virtual Math Team

In this tour, you will explore the VMT-with-GeoGebra environment and learn how to use it. You will learn about many special features of the VMT system, which you will need to use in work on the topics.

The Virtual Math Teams (VMT) Environment

The VMT system has been developed to support small groups of people to discuss mathematics online. It has tabs and tools to help individuals, small groups (about 2-6 people) and larger groups (like classes) to explore math collaboratively.

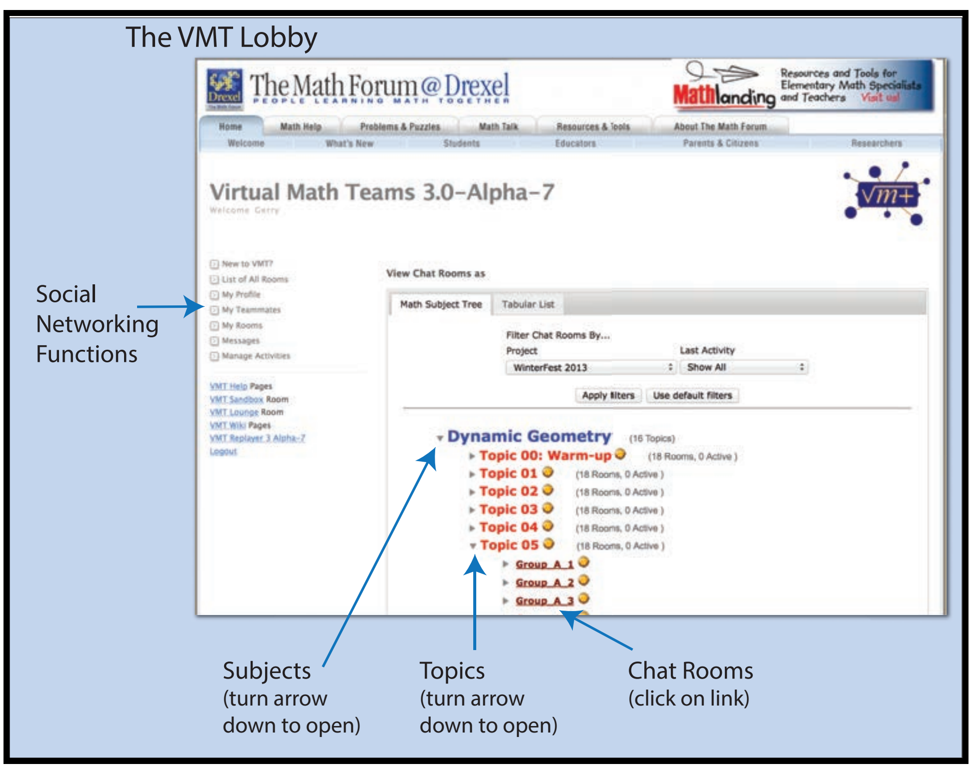
Register and Login to VMT

Go to the VMT Lobby at http://vmt.mathforum.org

Log in. If you do not have a VMT login, then first register. If you are using VMT in a class, your instructor may have already registered you and assigned your username and password. If not, then choose a username that you want to be known by online in VMT. To protect your privacy, you should select a username that is different from your real name. (Work in VMT is publicly available and anyone can see your username and what you say in VMT chats.)

Be sure to choose the project that is defined for your class or group.

Look Around the VMT Lobby



Interface of the VMT Lobby.

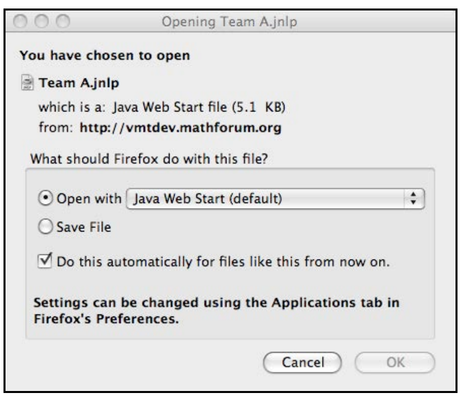
In the center of the VMT Lobby is a list of math subjects. For each subject, if you click on the little triangle in front of the subject name, you can view activity topics related to that subject. For each topic, if you click on the little triangle in front of the activity name, you can view the links to chat rooms for discussing that topic. Find the room where you are supposed to meet with your group. Click on the link (the name is a live link) for that room to open a window with the chat room.

On the left of the Lobby is a list of links to other functions. The link, "List of All Rooms," displays the list of math subjects, like you see in the figure above. The link, "My Profile," allows you to change your login name, password, or information about you. The link, "My Rooms," lets you see links to chat rooms that you have been invited to by your teacher or a friend, as well as rooms that you have been in before.

You can use the "VMT Sandbox" link to open a practice chat room. However, it is better to meet with the members of your group in a chat room that has been created for your group to do an activity. You should be able to find it in the "List of all Rooms" under your project, the subject "Dynamic Geometry," the topic (start with "Topic 01"), and the name of your group. It may also be listed under "My Rooms" or you might have been given a direct link to the room.

Enter a VMT Chat Room

When you click on a chat room link to open it, your computer will download VMT files. This may take a couple minutes, especially the first time it is done on your computer. You will see a dialog box window asking if you want to open the file with Java Web Start. Just select "Open with Java Web Start" and press the OK button. (See "Tour 10: Technical Problems" at the end of this document if you have problems at this point.)



Dialog box for Java Web Start.

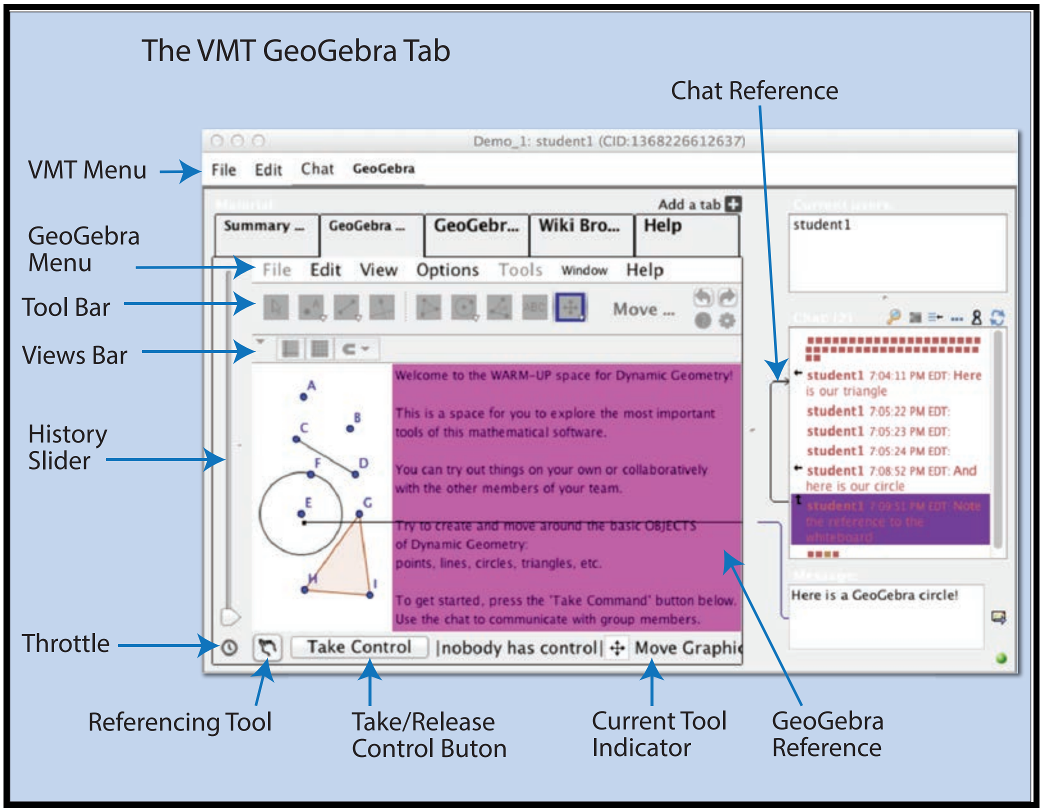
It is important to try to log into VMT using the computer you will be using with your team before you are going to meet with your team online the first time. Teachers having their students use a school computer lab for VMT should meet with their school technical support personnel well in advance, and test each of the computers to make sure they can access VMT on the Internet and have all the necessary software and permissions.

You will learn more about how to use the VMT tools in future tours. For now, just click on the tab for GeoGebra and proceed with the topic.

Tour 2: GeoGebra for Dynamic Math

Go to the VMT Chat Room and Open the GeoGebra Tab

Open the GeoGebra tab in your VMT chat room and identify the parts listed in the figure below. You will be using this GeoGebra tab most of the time in the topic activities.



The GeoGebra tab interface in VMT.

Take Turns

This is a multi-user version of GeoGebra. What you see in the team's GeoGebra tab is the same as what everyone in the VMT chat room with you also sees in their GeoGebra tab (except that they may have their view options set differently, like having the tab opened wider or smaller than you do).

Two people cannot be creating and manipulating objects at the same time in GeoGebra, so you have to take turns. While someone else is constructing or dragging, you can be watching and chatting.

Use the chat to let people know when you want to "take control" of the GeoGebra construction. Use the chat to tell people what you notice and what you are wondering about the construction.

Decide in the chat who will go first. That person should press the "Take Control" button and do some dragging and constructing. Then "Release Control" and let the others take a turn.

Before you start to drag or construct, say in the chat what you plan to do. After you release control, say in the chat what you discovered if anything surprised you. You can also ask other people in your group questions about what they constructed and how they did it.

There is a history slider on the left side of the GeoGebra tab. You can only use the history slider in the GeoGebra tab when you are *not* "in control." Sliding the history slider shows you previous versions of constructions in the GeoGebra tab, so you can review how your group did its work.

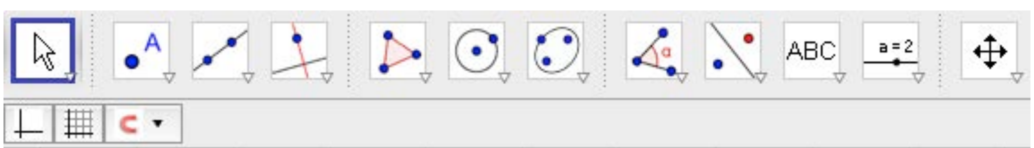
Create a Practice Tab

To create a new GeoGebra tab for yourself in the VMT chat room, use the "+" button in the upper-right corner above the other tabs.

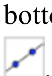
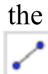
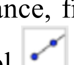
This way, you can create your own GeoGebra tab, where you can practice doing things in GeoGebra before you get together with your team in the team's GeoGebra tab. You can use your own tab to try out the construction tools described below. At the beginning of each activity, there may be tasks for you to try yourself in your own tab; then you will discuss them in chat and share your figures in the team GeoGebra tab. Anyone can view any tab, so you can post a chat invitation to other people to go to your GeoGebra tab and see what you have done. You can even let someone else "take control" in your tab to help you construct something or to explore your construction. After your group constructs something in the group GeoGebra tab, you should make sure that you can do it yourself by doing the construction in your own tab.

Some Drawing Tools in GeoGebra

When you open a GeoGebra tab, the Tool Bar may look something like this:

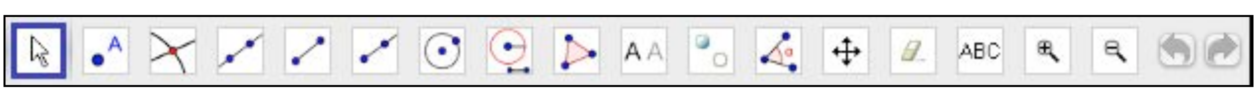


For some topics, the Tool Bar has been simplified, so only the tools needed for that topic are available.

Notice that you may be able to "pull down" many different tools by clicking on the small arrow at the bottom of each icon in the tool bar. For instance, from the third icon , you can select the Line Tool , the Segment Tool , and the Ray Tool . If your tool bar does not look like this, then change the perspective to the "Geometry" perspective from the "View" menu or the pop-up menu on the right edge of the tab. If there are grid lines, you can remove them with the Grid button below the tool bar. If there are coordinate axes, you can remove them with the coordinates button below the tool bar. You can change the color or thickness of a selected line with the other buttons there.

Make sure that the menu "Options" | "Labeling" | "New Points Only" is checked so that new points you create will have their names showing.

Here are some of the first tools you will be using in GeoGebra:



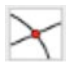
These tools correspond to the traditional Euclidean geometry construction tools of straightedge and compass. *The first several tools let you construct dynamic points and lines (including lines, segments,* *rays and circles), much as you would with a pencil and paper using a straightedge for the lines, segments and rays or a compass for the circles.*

Check out this video for an overview and some tips on the use of these tools: http://www.youtube.com/watch?v=2NqblDIPl38

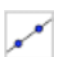
Here is how to use these tool buttons. Try each one out in the construction area of your own GeoGebra tab. First click on the button for the tool in the tool bar, and then click in the construction area to use the tool. The tool will remain selected in the tool bar until you select another one:

Use the Move Tool  to select a point that already exists (or segment or circle) and drag it to a new position. Everyone will see the object being dragged.

Use the Point Tool  to create some points. Each place you click with the Point Tool will leave a point. These points will appear in the GeoGebra tab of everyone in your chat room. By convention, points are named with capital letters.

Use the Intersection Tool  to mark the intersection of two objects—like a line and a circle—with a new point. When you click on the intersection of two objects, both objects should get thicker to show they have been selected. You can also select the two objects separately, one after another and the new point will be on their intersection. If you click at a location where three objects meet, you will get a pull-down menu to select the two objects that you want.

Note on intersections: a circle and a line or two circles can intersect in two points. If the two intersecting objects move apart, a point defined at their intersection will disappear, but it will come back it they move together and intersect again. Sometimes, they move to a configuration in which the two intersecting objects are just tangent to each other and then they will continue to intersect in two points. After they pass through the point of tangency, the two points of intersection (which are sometimes labeled with a 1 and a 2) may switch. This is called the "continuity problem" (see Kortenkamp, 1999 in the "Further Readings" Tour). If you use the Intersection Tool and click on one intersecting object at a time, this may place points at both intersections. You can turn on "Continuity" in the menu Options | Advanced (Preferences) | Gears tab (Advanced) | checkbox for Continuity On. This may help a point move continuously through a tangent configuration. By playing with the Intersection Tool and Continuity, you can often get complicated situations to work the way you want—but not always. This is a foundational mathematical limitation of all dynamic geometry implementations.

Use the Line Tool  to create a line with no endpoints. A line has to pass through two points. You can either select two existing points or click with the Line tool to create the points while you are constructing the line. By convention, lines (as well as segments, rays, circles and polygons) are named with lowercase letters.

Use the Segment Tool  to connect two points with a line segment. You can also create points as you click for the ends of the segment. See what happens when two segments use the same point for one of their endpoints.

Use the Ray Tool  to connect two points with a ray. First click for the starting point of the ray and then click for a point along the ray. You can also select existing points for the endpoint and the other point.

Use the Circle Tool to draw a circle. You must click to place a point where you want the center to be and then click again for a point on the circumference of the circle. You can also use existing points for the center and the other point.

Use the Compass to draw a circle whose radius is equal to the distance between two points and whose center is at a third point. First, click on two points to define the length of the radius. Then without releasing the cursor, drag the circle to the point where you want its center to be. This tool is like a mechanical compass, where you first set the size of the opening and then fix one end at a center and draw a circle around it. The Compass tool is very handy for copying a length from one part of a construction to another in a way that will be preserved through any dragging; if you change the original length, the copied length will change automatically to still be equal to the original one.

The following tools can be used for modifying the display of a construction to make it easier to see what is going on with the dependencies of the construction.

The Polygon Tool is used to display a two-dimensional polygon. For instance, if three segments connecting three points form a triangle, then you can use the Polygon tool to display a filled-in triangle. Click on the vertex points in order around the polygon and then complete the figure by clicking on the first point again.

Show/Hide Label . Select this tool. Then click on an object to hide its label (or display it if it was hidden).

Show/Hide Object . Select this tool. Then click on an object to hide it (or to display it if it was hidden).

Use the Angle tool to display an angle with its measurement. Click on the three points that form an angle *in clockwise order*—if you do it in counterclockwise order it will display the exterior angle, which you probably do not want. You can also click on the two lines that form the angle in clockwise order.

Use the Move Graphic tool to drag the whole construction area.

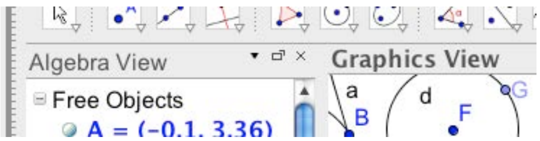
To delete a point, either use the Delete tool or select the object and press the "delete" key on your keyboard. *Hint*: Before you delete something that someone else created, be sure to ask in the chat if everyone agrees that it should be deleted. *Warning*: When you delete an object, anything dependent on that object will also be deleted; it may be better to just *hide* the object.

Insert Text . This tool can be used to place text on the drawing surface. You can add a title, a comment, etc. First, an input box will open for you to type your text; when you select OK, it appears in a text box.

You can use the Zoom in and Zoom out tools to change the scale of your view of the construction area. On a Mac computer, you can also use two-finger gestures for zooming; on a Windows computer, you can use a mouse scroll wheel or right button. Changing your view with the zoom tools will not affect what others see in their views.

The Algebra View

A good way to view the locations, lengths, areas or other values of all the GeoGebra objects is to open the Algebra View from the GeoGebra "View" | "Algebra" menu. This opens a window listing all the free and dependent objects that you have constructed. You can un-attach this window with the little window icon that is above the Algebra View:



Top of the Algebra View and the Graphics View in GeoGebra.

The "Drag Test"

This is where dynamic geometry gets especially interesting. Select an object in the construction area with the Move tool. Drag the object by holding down the Move Tool on the object and moving it. Observe how other parts move with the selected object. That is because the other parts are "dependent" on the part you are dragging. For instance, a segment depends on its end-points; when the points move, the segment must also move. If two segments both depend on the same point, then they will always move together; if you drag one of the two segments, it will drag the common end-point, which will drag the other segment. Dragging is an important way to check that parts have the correct connections or "dependencies" on other parts. GeoGebra lets you construct objects that have dependencies that are important in geometry and in other branches of mathematics.

A thorough explanation of a simple construction with a dependency is given in a YouTube video using GeoGebra tools that are equivalent to straightedge and compass:

http://www.youtube.com/watch?v=AdBNfEOEVco

Explore!

Construct some lines that share the same points. Think about how the figures are connected. State what you think will happen if certain objects are dragged. Then try it out. Take control and drag part of a figure. Discuss the dependencies in chat.

Hint

If two elements share a point – for instance, if a line segment starts at a point on a circle, then we say there is a "dependency" between the segment and the circle. That is, the position of the segment depends on the position of the circle, and when you move one, the other also moves. *Geometry is all about such dependencies*. A dynamic-math environment lets you see how the dependencies work and lets you explore them. Check out these videos of complicated dependencies:

http://www.youtube.com/watch?v=Oyj64QnZIe4&NR=1

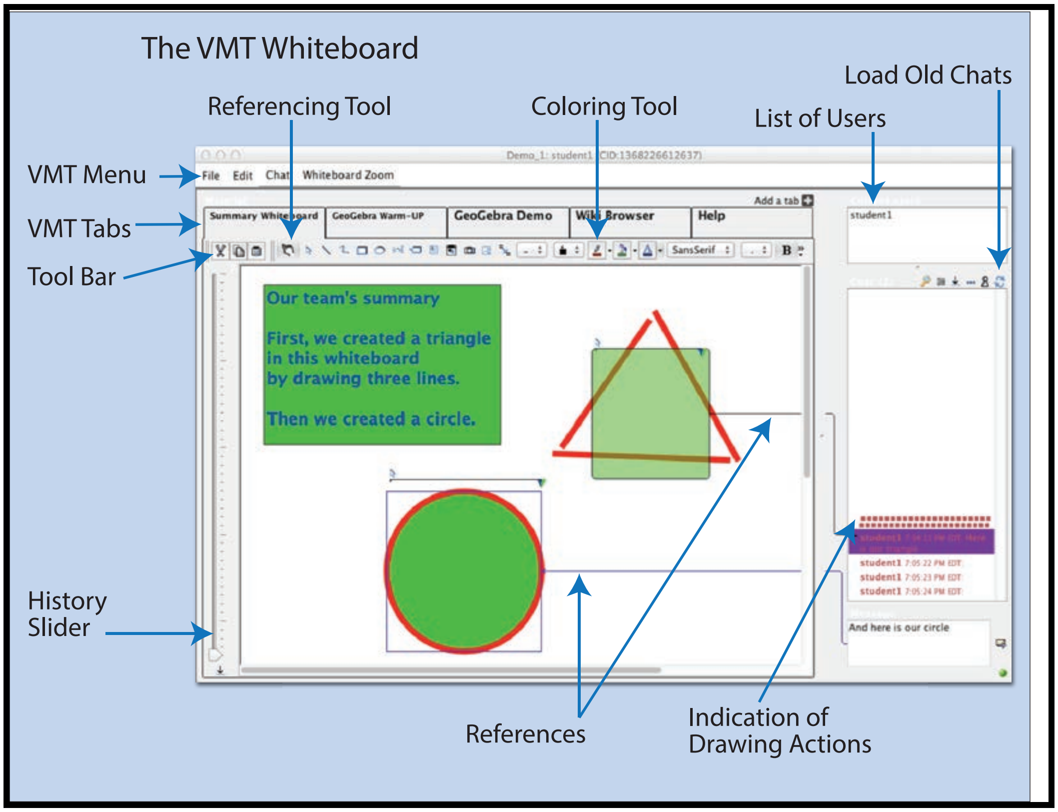
http://www.youtube.com/watch?v=-GgOn66knqA&NR=1

Tour 3: VMT to Learn Together Online

In this tour, you will explore the VMT-with-GeoGebra environment and learn how to use it to collaborate. You will learn about many special features of the VMT system, which you will need to use in the topic activities.

Enter a VMT Whiteboard Tab

When the VMT chat room is open, it may look something like this:



Interface of a VMT chat room with a whiteboard tab.

Note that this screen image shows a VMT shared whiteboard tab, *not* a GeoGebra tab. The whiteboard is useful for free-hand sketches and textboxes. You may want to use a whiteboard tab to collaboratively write a summary of your team's work in GeoGebra tabs. You can always add a shared whiteboard tab if there is none in your VMT chat room by using the "Add a tab +" button above the tab names.

See What is Going On

See the list of users present in the upper right. It shows all the people who are currently logged into this chat room

Awareness messages near the bottom of the window state who is currently typing a chat message or drawing in the shared whiteboard. You should see all the messages that anyone posted in the chat room and all the drawings that anyone did in the whiteboard as soon as they finish typing (after they post the message by pressing the Return key on their keyboard) or drawing (and after they click on the whiteboard background).

Post a Greeting Message

Type in the chat input box. Press the Return or Enter key on your keyboard to post your message for others to read. Your message should appear above in the chat messages area with your login name and the current time. Other people in the same chat room will also see your message.

Look Back in Chat and Whiteboard History

Load old messages if you are joining a room where people have already been chatting. Use the reload icon  (two curved arrows). You can scroll back in the chat if there are too many messages to be displayed at once.

The whiteboard also has a history slider so you can see how the images in the whiteboard tab changed over time.

Reference a Previous Chat Message

Point to a previous chat message by double clicking on the previous message when you type a new chat message. This will create an arrow from your new chat message to the previous chat message. Everyone will see this arrow when your message is posted or if they click on your message later.

If a reference arrow exists and you want to delete it, then press the ESC key on your keyboard before you post the message. (You cannot delete a reference arrow after you post the message.)

Leave a Message on a Shared Whiteboard

Click on the different tabs to see the different work areas. A Summary Tab is just a Whiteboard Tab. Your group can use a Summary Tab to summarize your work on an activity.

Go back to the Whiteboard Tab. Open a textbox (the icon for this  is in the middle of the Whiteboard tool menu; it has an “A” in it; if you roll your cursor over it, it says “Add a textbox”.) Type a message in the textbox. Double-click on someone else’s textbox to edit and add to what they wrote.

You can draw a square or circle and change its color, outline, etc.

Reference an Object in the Whiteboard

You can also create an arrow from your new chat message to an object or an area on the whiteboard, just like you did from a new chat message to a previous one.

Point to the square or circle with the Reference Tool. First, click on the Referencing Tool (the pointing hand  in the whiteboard tool bar — see screenshot of “VMT chat room with Whiteboard tab” above). Then select the square or circle — or else drag the cursor to select a rectangular area around the square or circle. Finally, type a chat message and post it. You should see a line connecting your chat posting to the object in the whiteboard. This is handy to use when you want to make a comment or ask a question about an object in the Whiteboard or in GeoGebra.

You can also use the Referencing Tool to point to an object in a GeoGebra tab.

Draw Two Triangles

Draw an equilateral triangle (where all three sides are of equal length) on the shared whiteboard.

Or draw a right triangle (where one angle is a 90-degree right angle) on the shared whiteboard.

Try to move these triangles around.

What do you notice about them? Is it hard to rotate or move the triangle around? What would you like to be able to do?

If you drag one end of a line to change the lengths of the sides, are the triangles still equilateral or right triangles?

GeoGebra has other ways of constructing triangles. You will be doing a lot of that in the topics.

Open Extra Tabs

Use the "+" button (above the upper right corner of the tab) to create a new tab. This is handy if a whiteboard tab or a GeoGebra tab becomes filled up and you want to open a new one without erasing everything.

Get Some Help on Math Notation

Go to the Help tab to learn more about VMT. For instance, look up how to enter Mathematical Equations/Expressions in the chat, in Whiteboard textboxes and in the VMT wiki. They use the $ to indicate math notation. You can cut and paste these expressions between the chat, Whiteboard textboxes, and the VMT wiki.

Review Your Team's Work

Use the history sliders on the left side of the whiteboard and on the left side of the chat to get an overview of what your group has done and discussed if you come in late or return on another day. Discuss a summary of your work with your group. You can put a textbox with this summary in your Summary Tab. You can even start with an outline or a first draft of a summary in the Summary Tab and then have everyone discuss it in the chat and edit it in the Summary Tab.

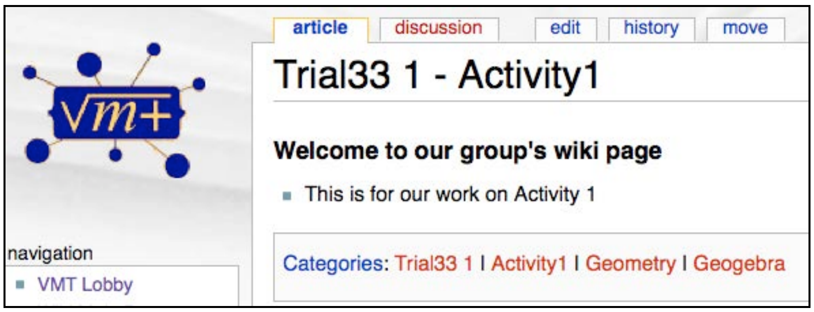
Try to create a reference from a chat posting about an idea in the summary to the sentence in the Summary tab.

What have you learned in this activity? What do you wonder about it? What did you not understand or what do you want to know about? Ask the other people in your group—they may have some answers for you or be able to help you find the answers.

Tour 4: The VMT Wiki for Sharing

There is a special wiki page for your chat room. You can find it from the VMT Lobby. Click on the wiki icon that is displayed after the link to your chat room in the VMT Lobby. A wiki page is a page on the Internet that anyone can easily add to or edit.

Go to the wiki page for your chat group and copy the summary of your group's work from the Summary tab of your chat room to this wiki page. Copy and paste text from your Summary tab into the "edit" tab of the wiki page and then format it for the wiki. Instructions for editing are available from the "Help" link in the "navigation" panel on the left side of the wiki page. (The VMT wiki is edited the same way as Wikipedia).



A wiki page for chat room "Trial33_1" on topic "Activity1," subject "Geometry," community "GeoGebra."

The wiki page for your chat room is automatically linked to all the wiki pages for its activity and also to all the wiki pages for its subject. Finally, there is a wiki page that links to all the groups or teams in your whole project. This way, you can compare your group's findings with everyone else's.

For instance, if your chat room has the topic "Activity1", then go to the wiki page for Activity1 by clicking on the Category link for "Activity1" at the bottom of your group's wiki page and browse to see summaries of other groups. Now return to your group's wiki page and comment on how your work compares to that of other groups.

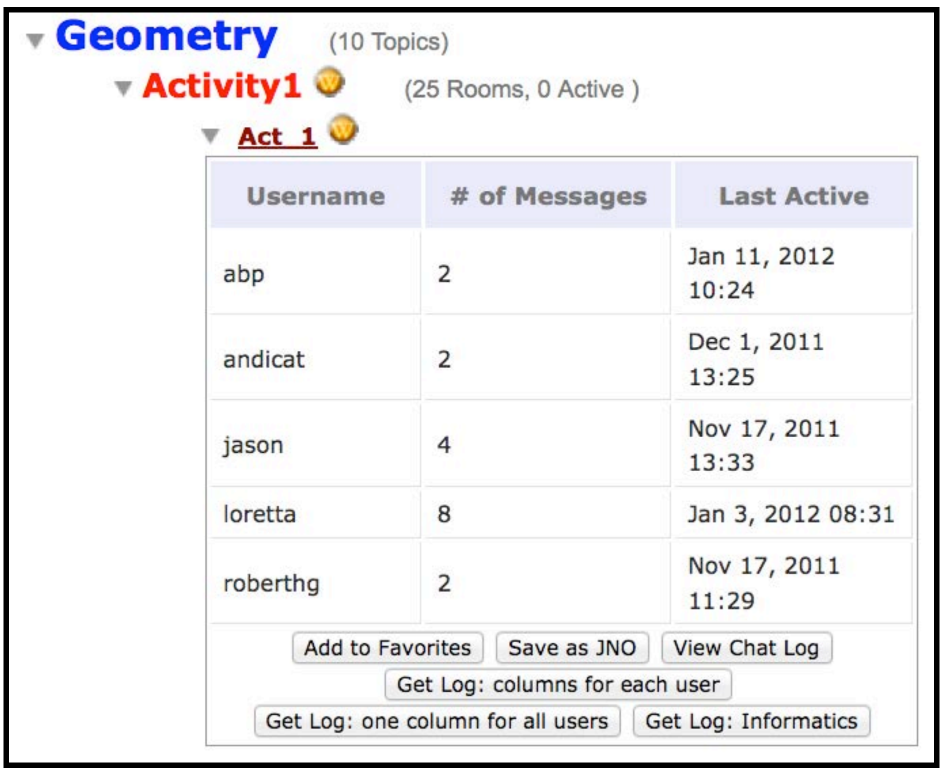
Tour 5: VMT Logs & Replayer for Reflection

Get a Log of Your Group's Work

You can get a spreadsheet containing a log of all the chat postings in a VMT chat room. You can use this log for documentation of your group's work by pasting excerpts from the log into a report. This can also be useful for reflection on the work of your group or for analysis of the interaction and knowledge building that took place.

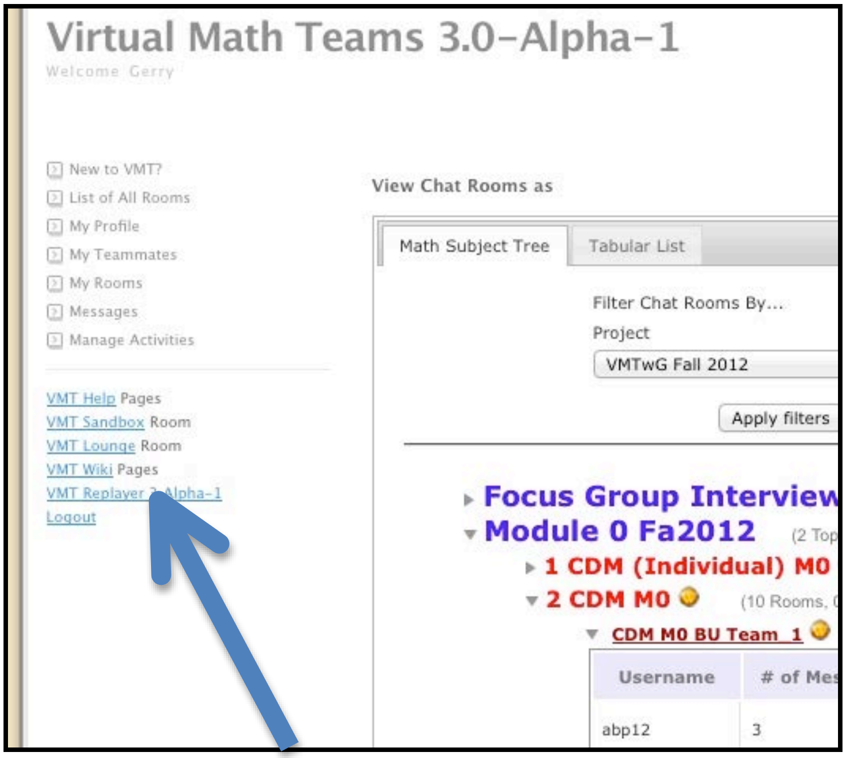
To view the log, go to the VMT Lobby and find the chat room. In front of the link to the room is an arrow, which you can click on to turn down. This will pop open a dashboard window as shown below. This gives a quick view of who has been in the chat room, how many chat posts they had and when they were last active.

At the bottom of the dashboard is a button that says "View Chat Log." This will immediately display a spreadsheet of the chat log. You can cut and paste from the display window to your report document. There are also three "Get Log" options for downloading the chat log as a spreadsheet file: with each participant's posting in a different column, with all postings in one chronological column or in a special format for automated analysis. The spreadsheets can be filtered by event type to display a subset of events.



Replay Your Group's Work

You can replay all the chat and constructions in a VMT chat room with the VMT Replayer. Then you can save a screenshot of any stage of your session to include in a report. This can also be useful for reflection on the work of your group or for analysis of the interaction and knowledge building that took place. Save a complete history of a VMT chat room with the "Save as JNO" button; this will download your room's .JNO file to your desktop.



Click on the "VMT Replayer" link in the Lobby to download "vmtPlayer.jnlp." Start the Replayer. Select menu item "File" | "Open Session" and browse to your room's .JNO file. It may take a few minutes for the Replayer to open with the chat room history, depending upon how much activity took place in the room. When the room is opened in the Replayer, it will look just like the original VMT room, except that at the bottom it will have a history slider and some buttons to replay the entire session at a selected speed or to step through the interaction one action at a time with your keyboard's arrow keys. Scroll the timeline back to the start of the session.

Try It on Your Own

When you are finished working with your group on the next topic, download the Replayer and the JNO file for your room. Compose a brief report on your group experience and include excerpts from the chat and screen shots from the Replayer on your group's wiki page. You may want to download a free screen-capture application like Grab to make images of the Replayer on your computer screen.

Tour 6: GeoGebra Videos & Resources

GeoGebra was created to harness the power of personal computers to help people learn about how exciting geometry can be as an interactive and creative world of exploration and expression. The original developer of GeoGebra discusses his vision and the worldwide response to it in this YouTube video:

http://www.youtube.com/watch?v=w7lgMx8-1c0

Another video shows students engrossed in artistic, evolving and three-dimensional images of mathematical phenomena constructed in the GeoGebra environment:

http://www.youtube.com/watch?v=9IrZAYHpGfk

A third video provides a sampling of advanced GeoGebra constructions, showing the boundless possibilities of the system for representing mathematical objects:

www.youtube.com/watch?v=rZnKMwicW_M

Check out this video for an overview and some tips on the use of the GeoGebra tools that are equivalent to traditional straightedge and compass:

http://www.youtube.com/watch?v=2NqblDIPl38

A thorough explanation of a simple construction with a dependency is given in a YouTube video using GeoGebra tools that are equivalent to straightedge and compass:

http://www.youtube.com/watch?v=AdBNfEOEVco

Here is a video showing how to construct an equilateral triangle with those tools in GeoGebra: http://www.youtube.com/watch?v=ORIaWNQSM_E

Check out these videos of complicated dependencies:

http://www.youtube.com/watch?v=Oyj64QnZIe4&NR=1

http://www.youtube.com/watch?v=-GgOn66knqA&NR=1

There are a large number of YouTube tutorials for GeoGebra. Some of them are collected on the GeoGebra channel:

http://www.youtube.com/geogebrachannel

A good place to begin these videos is:

www.youtube.com/watch?v=2NqblDIPl38

There is a GeoGebra wiki site with resources for students and teachers:

www.geogebra.org

http://www.geogebratube.org

GeoGebra becomes even more powerful in its multi-user version, as part of the VMT (Virtual Math Teams) software environment. Here are some YouTube demos of important aspects of the VMT-with-GeoGebra system:

The multi-user version of GeoGebra—each person sees the actions of the others as they happen:

http://youtube.googleapis.com/v/4oBBynYVrY0

GeoGebra's history slider—you can go back and forth to see how a diagram evolved step by step in a GeoGebra or Whiteboard tab of VMT:

http://youtube.googleapis.com/v/DRlDnadcfRE

The VMT Replayer—you can replay an entire session, including all the tabs. The chat is coordinated with the drawings as you scroll or replay. You can speed up the replaying at multiple speeds. You can stop and step through, action-by-action, forward or backward to analyze the group interaction in detail:

http://youtube.googleapis.com/v/3IzkcVSyYjM

The three videos on VMTwG are integrated in a PowerPoint slide show introduction to VMTwG, available at: http://GerryStahl.net/pub/vmtdemo.pptx

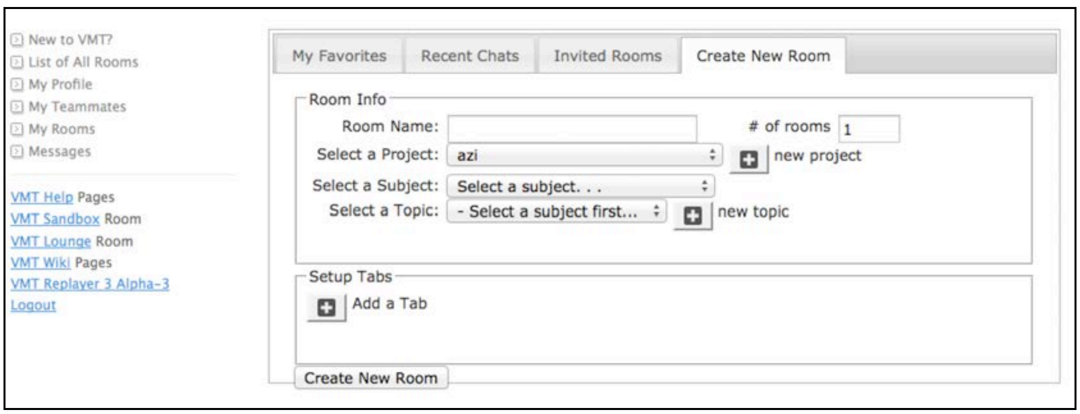
Tour 7: Creating VMT Chat Rooms

VMT-with-GeoGebra is freely available worldwide for people to create rooms and invite others to collaborate in them. The Lobby includes tools for students, teachers and other people to define their own topics and create VMT chat rooms for exploring and discussing those topics.

Anyone Can Create New Chat Rooms

Anyone who is logged into VMT can create new rooms. There is an expanded interface for teachers. First, we will see how people who are not registered as teachers can create chat rooms.

Enter the VMT Lobby and click on the link "My Rooms" on the left side of the Lobby, as shown in the figure. Select the tab "Create New Room."



The Lobby interface to create a new chat room.

The Interface to Create New Chat Rooms

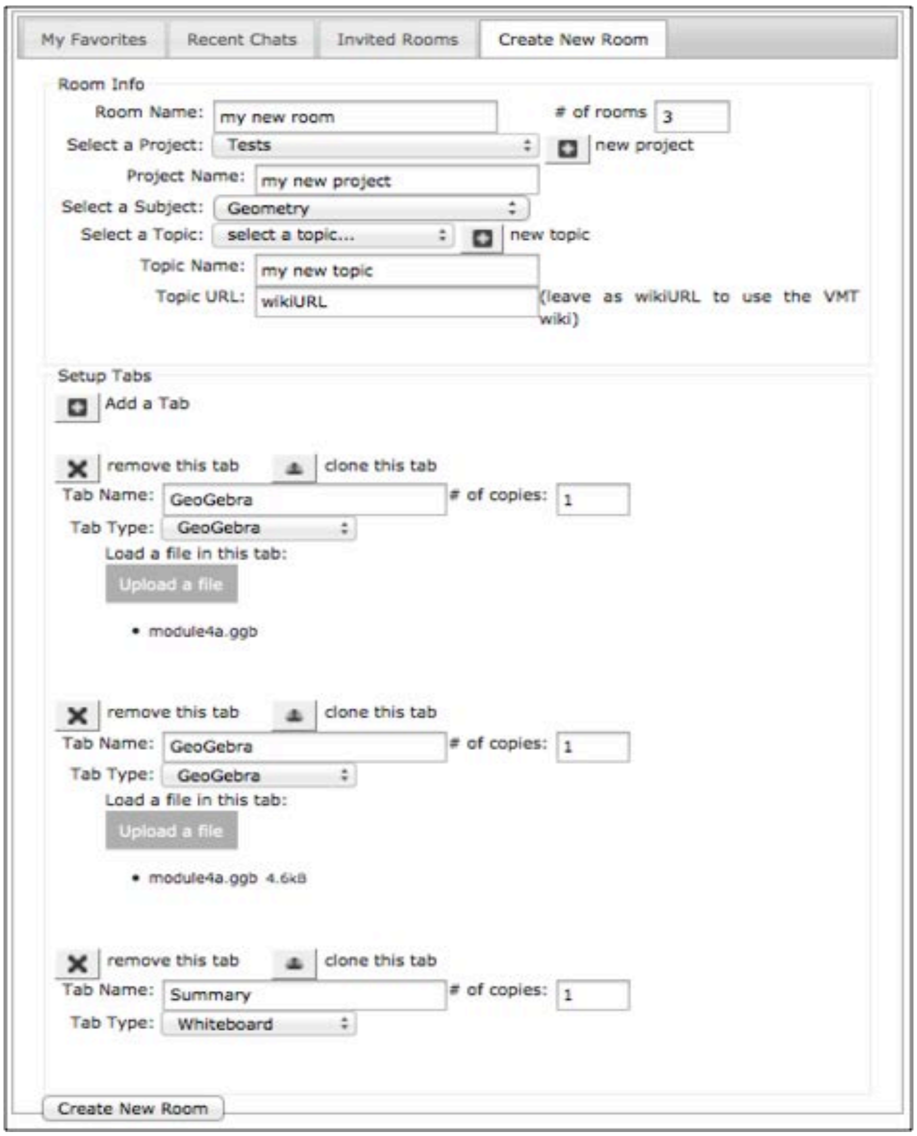
The first decision is to define a name for the new room that you are creating. You can create up to nine (9) identical rooms at once. They will be numbered: name_1, name_2, etc. So if you want 3 rooms for 3 teams, you can name the room with a name ending in "team" and then the room names will end in "team_1," "team_2" and "team_3."

Rooms are organized by Projects. Within each Project is a hierarchy of Subjects within the Project, Topics within the Subjects, and Rooms within the Topics. So if you are setting up rooms for a course that will have many topics (e.g., one each week for a term), you might want to create a new project for that course, like "Ms. Taylor Spring 2013." For our example, we will use the existing Project "Tests" for creating test rooms. It is always good to create a test room, then open it and make sure it works the way you intend. Once rooms are created, they cannot be deleted and their names cannot be reused in the same Subject.

Next, select a Subject, like "Geometry." Only people who are registered in VMT as teachers or administrators can create new Subjects. There is already a list of Subjects covering most areas of mathematics.

Now select a Topic. A Topic can have a description associated with it, although this is not necessary. If you create a new Topic, you will have an option to give a URL pointing to a description. This can be an html page on any web server. The description for the topic will appear if someone clicks on the link for that Topic in the Lobby listing of rooms. You can also leave the Topic URL set to its default, "wikiURL." Then the topic description is defined on a wiki page on the VMT wiki associated with the new chat room, and you can go there later to edit that description.

The new chat room has now been defined and it is time to define the tabs that will appear in the room when it is first opened. Press the "+ Add a Tab" icon to add each tab. You can add Whiteboard tabs for textboxes and simple shapes or GeoGebra tabs for constructing dynamic-mathematics figures. You can also add Web Browser tabs for displaying the topic description, wiki pages or websites—however this is a relatively primitive browser and it is generally better to use browsers like IE, Safari, Firefox or Chrome outside of VMT, especially for editing a wiki page.



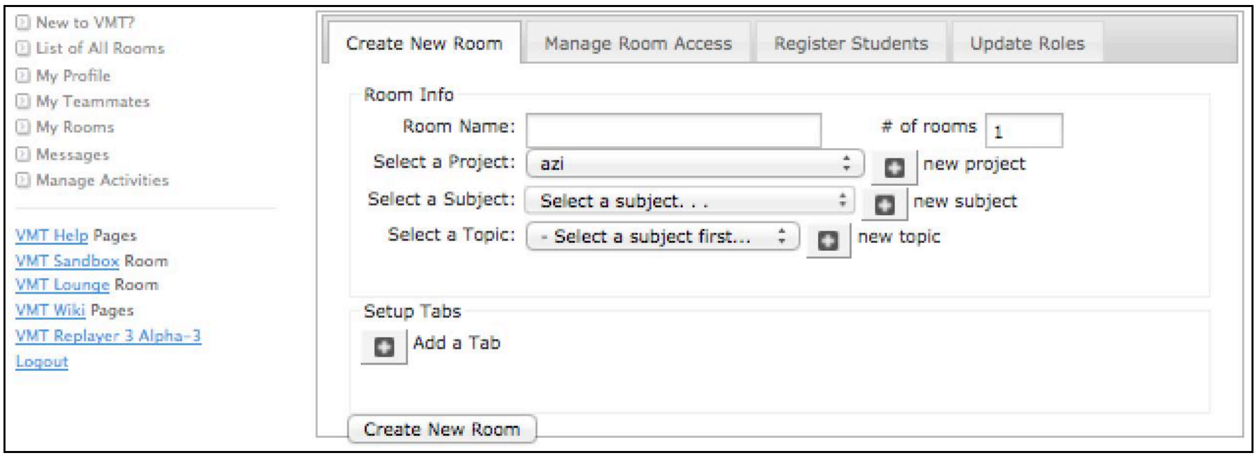
A form for creating a new chat room.

For a GeoGebra tab, you can upload a .ggb file with a figure already constructed. For instance, you might want four students using the new room to each explore a figure that you have already constructed or that you downloaded from GeoGebraTube. Then you would first construct the figure and save it on your computer desktop or download it from GeoGebraTube to your desktop. Then, when you are creating the GeoGebra tab for the new room, upload the file using the "Upload a file" button. Then you can "clone" that tab so each student will have their own copy to explore. Alternatively, you can specify a number of copies of the tab to have in the room.

When you have defined all the tabs you want, check your entries and press the "Create New Room" button at the bottom. Wait a minute while the rooms are being created. Eventually, you will see a pop-up message that the rooms have been created. Go to the Lobby to find and try your new rooms

The Interface for Teachers

People who are registered in VMT as teachers or administrators have some extra tools for creating new rooms and registering students. The figure shows the interface for teachers. Click on the link in the Lobby labeled "Manage Activities."



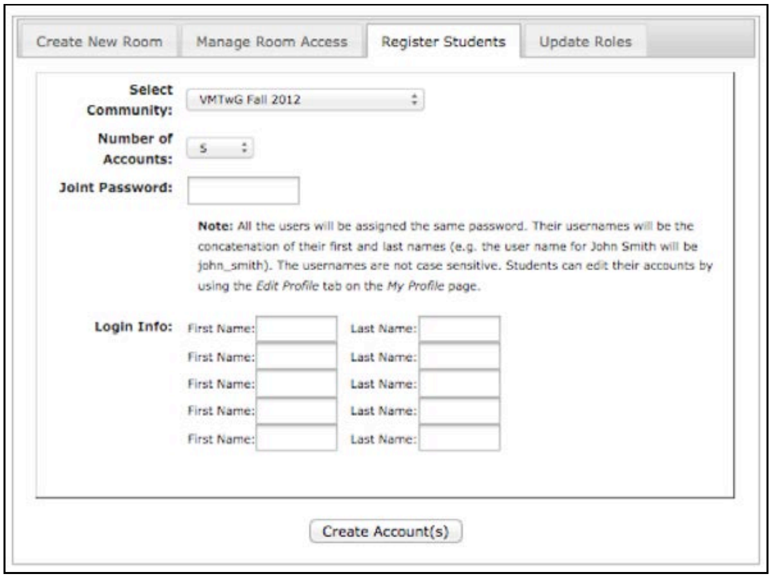
A form for teachers to create new rooms.

The tab to "Create New Room" is the same for teachers as for everyone else, except that they can define a new Subject as part of the process.

The tab to "Manage Room Access" allows a teacher to ban specific students from entering a particular chat room.

The tab to "Register Students" is shown in the next figure. It allows a teacher to quickly register up to 5 students at a time by just listing their names. When this registration procedure is used, the teacher's email is associated with each student login and all the students have the same password. As soon as each student logs in to VMT, they should click on the "My Profile" link in the Lobby and change their username, email, and password. For privacy reasons, it is highly recommended that students do not use their regular names as usernames. The teacher might want to keep track of the new usernames, email, and password for each student in case the students forget these and in order to track the work of each student in VMT logs.

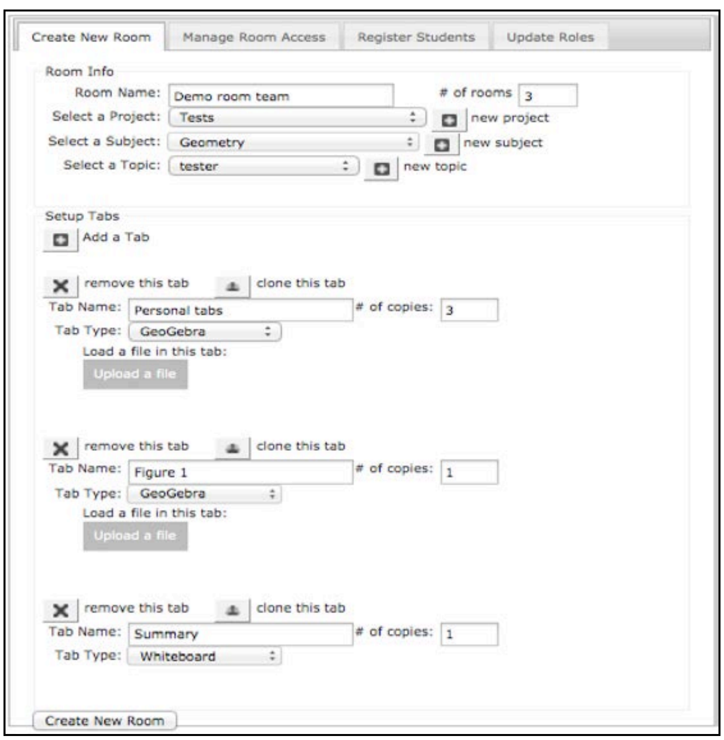
The tab to "Update Roles" allows someone who is already registered with the role of "teacher" or "administrator" to change the roles of other people. The names of people in a given Project are listed with their assigned role.



The interface for a teacher to register students.

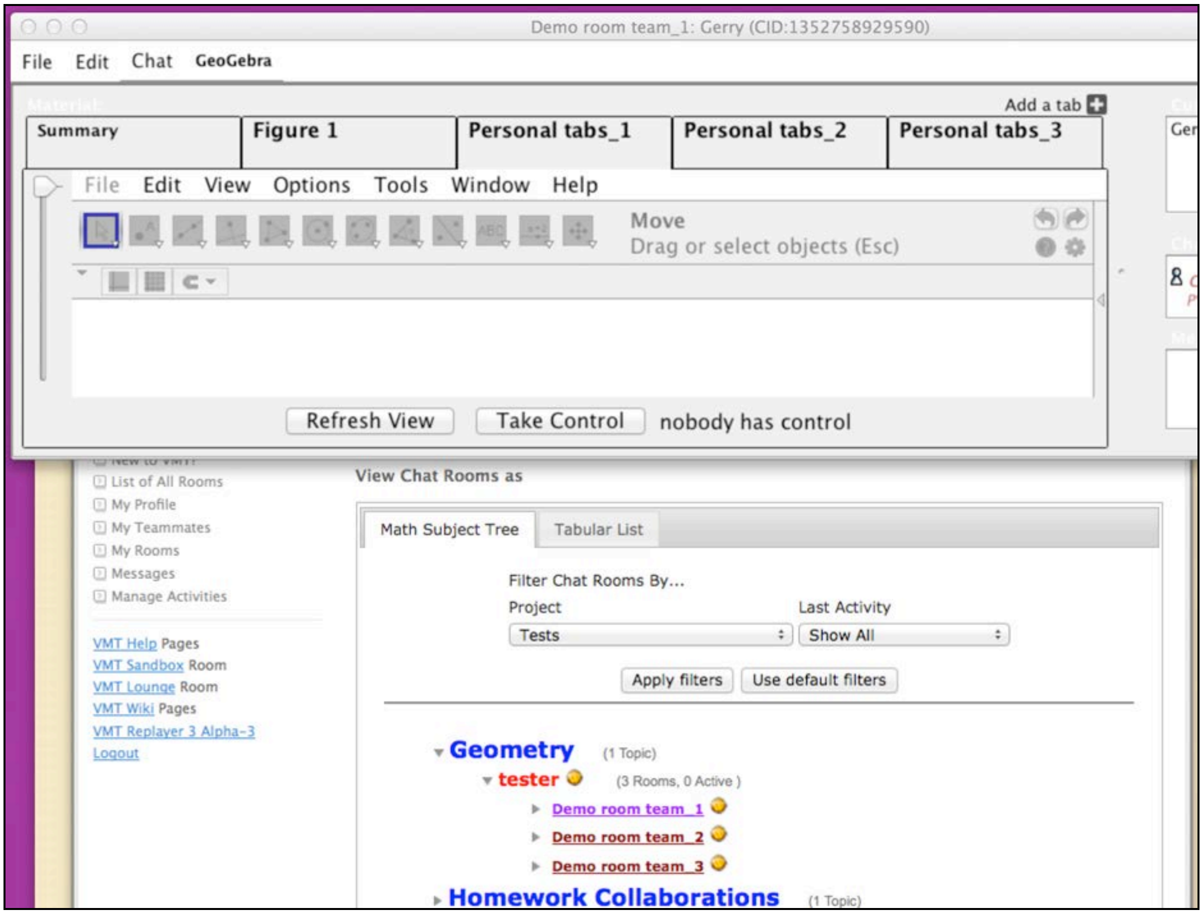
An Example of Creating Test Chat Rooms

The next figure shows the interface filled out to create 3 chat rooms, each having 5 tabs.



An example of the form for a new room filled in.

The final figure shows one of the 3 chat rooms, with its 5 tabs. Below the room is a view of the Lobby listing the 3 rooms in Project "Tests," Subject "Geometry," Topic "Tester" and rooms "Demo_room_team_1" to "team_3."



A room created in the example and the Lobby listing it.

Tour 8: Design of the Topics

The topics have been designed to promote collaborative learning, particularly as exhibited in significant mathematical discourse about geometry. Collaborative learning involves a subtle interplay of processes at the individual-student, small-group and whole-classroom levels of engagement, cognition and reflection. Accordingly, the activities are structured with sections for individual-student work, small-group collaboration and whole-class discussion. It is hoped that this mixture will enhance motivation, extend attention and spread understanding.

Goals

The goal of this set of activities is to improve the following skills in math teachers and students:

1. To engage in significant mathematical *discourse*; to collaborate on and discuss mathematical activities in supportive small online groups
2. To collaboratively *explore* mathematical phenomena and dependencies; to make mathematical phenomena visual in multiple representations; and to vary their parameters
3. To *construct* mathematical diagrams – understanding, exploring and designing their structural dependencies
4. To notice, wonder about and form conjectures about mathematical relationships; to justify, explain and *prove* mathematical findings
5. To understand core concepts, relationships, theorems and constructions of basic high-school *geometry*

The working hypothesis of the activities is that these goals can be furthered through an effective combination of:

1. Collaborative experiences in mathematical activities with guidance in collaborative, mathematical and accountable geometric *discourse*
2. *Exploring* dynamic-mathematical diagrams and multiple representations
3. Designing dependencies in dynamic-mathematical *constructions*
4. Explaining conjectures, justifications and *proofs*
5. Engagement in well-designed activities around basic high-school *geometry* content

In other words, the activities seek a productive synthesis of *collaboration*, *discourse, visualization, construction, and argumentation* skills applied in the domain of beginning geometry.

Development of Skills

The set of activities should gradually increase student skill levels in each of these dimensions. The design starts out assuming relatively low skill levels and gradually increases the level of skill expected. There is a theoretical basis for gradually increasing skill levels in terms of both understanding and proof in geometry. Here "understanding" and "proof" are taken in rather broad senses. The van Hiele theory (see deVilliers, 2003, p. 11) specifies several levels in the development of students' understanding of geometry, including:

1. *Recognition*: visual recognition of general appearance (something looks like a triangle)
2. *Analysis*: initial analysis of properties of figures and terminology for describing them
3. *Ordering*: logical ordering of figures (a square is a kind of rectangle in the quadrilateral hierarchy)
4. *Deduction*: longer sequences of deduction; understanding of the role of axioms, theorems, proof

The implication of van Hiele's theory is that students who are at a given level cannot properly grasp ideas presented at a higher level until they reach that level. Thus, a developmental series of activities pegged to the increasing sequence of levels is necessary to effectively present the content and concepts of geometry, such as, eventually, formal proof. Failure to lead students through this developmental process is likely to cause student feelings of inadequacy and consequent negative attitudes toward geometry.

Citing various mathematicians, deVilliers (2003) lists several roles and functions of proof, particularly when using dynamic-geometry environments:

1. *Communication*: proof as the transmission of mathematical knowledge
2. *Explanation*: proof as providing insight into why something is true
3. *Discovery*: proof as the discovery or invention of new results
4. *Verification*: proof as concerned with the truth of a statement
5. *Intellectual challenge*: proof as the personal self-realization or sense of fulfillment derived from constructing a proof
6. *Systematization*: proof as the organization of various results into a deductive system of axioms, major concepts and theorems

In his book, deVilliers suggests that students be introduced to proof by gradually going through this sequence of levels of successively more advanced roles of proof through a series of well-designed activities. In particular, the use of a dynamic-geometry environment can aid in moving students from the early stages of these sequences (recognition and communication) to the advanced levels (deduction and systematization). The use of dragging geometric objects to explore, analyze, and support explanation can begin the developmental process. The design and construction of geometric objects with dependencies to help discover, order and verify relationships can further the process. The construction can initially be highly scaffolded by instructions and collaboration; then students can be guided to reflect upon and discuss the constructed dependencies; finally, they can practice constructing objects with gradually reduced scaffolding. This can bring students to a stage where they are ready for deduction and systematization that builds on their exploratory experiences.

Practices for Significant Mathematical Discourse in Collaborative Dynamic Geometry

The following set of practices state the main skills that these activities are designed to instill. They integrate math and discourse skills. They are specifically oriented to dynamic geometry and its unique strengths:

a. Visualize: View and analyze constructions of geometric objects and relationships
b. Drag: Explore constructions of geometric objects through manipulation
c. Discourse: Notice, wonder, conjecture, strategize about relationships in constructions and how to investigate them further
d. Dependencies: Discover and name dependencies among geometric objects in constructions
e. Construction: Construct dependencies among objects, and define custom tools for doing so
f. Argumentation: Build deductive arguments, explain and prove them in terms of the dependencies
g. Math Accountability: Listen to what others say, solicit their reactions, re-voice their statements, re-state in math terminology and representations
h. Collaboration: Preserve discourse, reflect on it and organize findings; refine the statement of math knowledge; build knowledge together by building on each other's ideas

These practices can be placed in rough isomorphism with the Common Core math practices:

1. Make sense of problems and persevere in solving them: (b)
2. Reason abstractly and quantitatively: (c)
3. Construct viable arguments and critique the reasoning of others: (g)
4. Model with mathematics: (a)
5. Use appropriate tools strategically: (e)
6. Attend to precision: (f)
7. Look for and make use of structure: (d)

8. Look for and express regularity in repeated reasoning: (h)

It may be possible to organize, present, and motivate course activities in terms of these practices. Then pedagogy could be discussed in terms of how to promote and scaffold each of these; formative assessment (including student/team portfolio construction) could also be structured according to these practices.

Tour 9: Further Reading and Browsing

GeoGebra

www.geogebra.org -- main website for downloading stand-alone GeoGebra

www.geogebratube.org -- website for sharing and downloading GeoGebra constructions and applets

wiki.geogebra.org -- website with GeoGebra help pages, tutorials and user forum for questions

Bu, L., Schoen, R. (Eds.) (2011). *Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra*. Sense Publishers.

Kortenkamp, U. (1999). *Foundations of dynamic geometry.* Unpublished Dissertation, Ph. D., Department of Mathematics, Swiss Federal Institute of Technology. Zurich, Switzerland. Web: http://www.geogebra.org/forum/viewtopic.php?f=20&t=21895

Venema, G. (2013) *Exploring Advanced Euclidean Geometry with GeoGebra*. Mathematical Association of America.

Geometer's Sketchpad

Bennett, D. (2002). *Exploring Geometry with the Geometer's Sketchpad*. Key Curriculum Press.

deVilliers, M. (2003). *Rethinking Proof with the Geometer's Sketchpad*. Key Curriculum Press.

Hofstadter, D. (1997). Discovery and dissection of a geometric gem. In J. King & D. Schattschneider (Eds.), *Geometry turned on!: Dynamic software in learning, teaching and research.* (pp. 3-14): The Mathematics Association of America.

Scher, D. (2002). *Students' conceptions of geometry in a dynamic geometry software environment.* Unpublished Dissertation, Ph.D., School of Education, New York University. New York, NY. Web: http://GerryStahl.net/pub/GSP_Scher_Dissertation.pdf

Geometry

McCrone, S., King. J., Orihuela, Y., Robinson, E. (2010). *Focus in High School Mathematics: Reasoning and Sense Making: Geometry*. National Council of Teachers of Mathematics.

Serra, M. (2008). *Discovering Geometry: An investigative Approach*. Key Curriculum Press.

Common Core State Standards Initiative (2011). *Common Core State Standards for Mathematics*. "High School – Geometry" pp. 74-78. Web: www.corestandards.org/assets/CCSSI_Math%20Standards.pdf .

Krause, E. (1986). *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*. New York, NY: Dover.

Euclid

Euclid (c. 300 BCE/2002). *Euclid's Elements*. Thomas L. Heath, translator, Dana Densmore, editor. Green Lion Press.

Virtual Math Teams Research Project

Stahl, G. (2009). *Studying Virtual Math Teams*. New York, NY: Springer. Web: www.GerryStahl.net/elibrary/svmt .

Stahl, G. (2013). *Translating Euclid: Creating a Human-Centered Mathematics*: Morgan & Claypool Publishers. Web: www.GerryStahl.net/elibrary/euclid .

The Math Forum
www.mathforum.org.

Creative Commons Share Alike Copyright

In addition, this document has drawn many ideas from the other sources listed in this section.

Tour 10: Technical Problems

Some Common Problems while Starting Up VMT

- **VMT Cannot Find Java**

Look in your applications library. If you do not have the latest version of Java, download it from the Internet.

- **VMT Cannot Find Java Web Start**

Look in your applications library. If you do not have the latest version of Java WebStart, download it from the Internet.

Some Common Problems While Using VMT

- **If Your View of VMT or the Shared GeoGebra Construction Becomes Dysfunctional**

If your view of the shared GeoGebra construction becomes dysfunctional or you do not think you are receiving and displaying chat messages, then ***close the VMT chat room window. Log in to the VMT Lobby again and enter the chat room again.*** Hopefully, everything will be perfect now. If not, press the Reload button if there is one. If all else fails, read the Help manual, which is available from the links on the left side of the VMT Lobby.

- **After Adding/Removing the Algebra View or Changing the Perspective, Part of the GeoGebra Tab is Blank**

Press the REFRESH button at the bottom left of the VMT window.

- **Unable to Take Control Even Though Nobody Else Has Control**

Make sure the history slider (on the left) is at the current event (all the way down). You cannot take control while scrolling through the history. On rare occasions, the control mechanism breaks and that tab can no longer be used.

- **When Trying to Open a VMT Chat Room, the Password Field is Blank and the Logon Fails**

This can happen when your VMT-Lobby session has expired. Go to the VMT Lobby and logout. Then log back in and try again to enter your room. If that does not work, try closing your browser, then logging back into the lobby. As a last resort, reboot your computer.

- **Your Username is Refused When You Try to Enter a Chat Room That You Recently Left**

Sometimes when a chat room crashes, your username is still logged in and you cannot use that username to enter the room again. After a few minutes, that username will be automatically logged out and you will be able to enter the room again with that username. Alternatively, you can register a new username and enter the chat room with the new username.

Technical Requirements to Start a VMT Chat Room

- VMT is a Java WebStart application, so Java WebStart must be enabled. Note, on Macs you may need to go to the Java Control Panel (or Preferences depending on the version) and explicitly enable Java WebStart.
- VMT downloads a .jnlp Java WebStart file, so .jnlp must be an allowed file type to download.
- When VMT starts it will download the needed Java jars from the VMT server. So downloading jars must be allowed.

- If VMT does not start when a .jnlp file is downloaded, then the .jnlp file extension needs to be associated with the Java WebStart program (javaws).
- It should also be possible to start VMT by finding the .jnlp file in the browser downloads folder and double clicking it.
- The firewall (for instance at a school) must allow vmt.mathforum.org
- The firewall (for instance at a school) may need to open port 8080
- To use the VMT Lobby, JavaScript must be enabled
- It might be helpful to list vmt.mathforum.org as a trusted site for java downloads

Contact Us

Problems or questions? Email us at: vmthelp@mathforum.org.

Notes & Sketches

This space is for your notes. Paste in views of your constructions. List files of constructions or custom tools that you have saved. Jot down interesting things you have noticed, questions you have wondered about or conjectures you might want to explore in the future. Collect more dynamic-math activities here.